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CATEGORIES ENRICHED OVER A QUANTALOID: ALGEBRAS

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ABSTRACT. Given a small quantaloid $\mathcal{Q}$ with a set of objects $\mathcal{Q}_0$, it is proved that complete skeletal $\mathcal{Q}$-categories, completely distributive skeletal $\mathcal{Q}$-categories, and $\mathcal{Q}$-powersets of $\mathcal{Q}$-typed sets are all monadic over the slice category of **Set** over $\mathcal{Q}_0$.

1. Introduction

A quantaloid [Ros1996] is a category enriched over the symmetric monoidal closed category **Sup** consisting of complete lattices and suprema-preserving functions. Since a quantaloid $\mathcal{Q}$ is a bicategory [Ben1967] (a 2-category indeed), following [BC1982, BCSW1983, Str1981, Wal1981], a theory of categories enriched over $\mathcal{Q}$ (or $\mathcal{Q}$-categories for short) has been developed, see e.g. [Stu2005, Stu2006, Stu2007].

Given a small quantaloid $\mathcal{Q}$, with $\mathcal{Q}_0$ its set of objects, objects in the slice category $\mathbf{Set} \downarrow \mathcal{Q}_0$ are called $\mathcal{Q}$-typed sets. Then $\mathcal{Q}$-categories can be treated as structured $\mathcal{Q}$-typed sets. In this paper, we emphasize this aspect of $\mathcal{Q}$-categories. That is to say, we treat the theory of $\mathcal{Q}$-categories as one on the topos $\mathbf{Set} \downarrow \mathcal{Q}_0$. It should be stressed that this theory is not developed within the topos $\mathbf{Set} \downarrow \mathcal{Q}_0$, but rather, it depends heavily on the structure of $\mathcal{Q}$ which is formed outside of that topos. The role of $\mathcal{Q}$ is something like a "dynamic table of truth values" (c.f. [Stu2007]). The purpose of this paper is to show that some interesting classes of $\mathcal{Q}$-categories are exactly the Eilenberg-Moore algebras corresponding to certain monads on the topos $\mathbf{Set} \downarrow \mathcal{Q}_0$. These results show that the relationship between $\mathcal{Q}$-categories and $\mathcal{Q}$-typed sets are analogous to that between preordered sets and sets, exemplifying a benefit of treating $\mathcal{Q}$-categories as structured $\mathcal{Q}$-typed sets (instead of structured sets).

First, both the category $\mathcal{Q}$-**Sup** consisting of complete skeletal $\mathcal{Q}$-categories and cocontinuous $\mathcal{Q}$-functors and the category $\mathcal{Q}$-**CD** consisting of completely distributive skeletal $\mathcal{Q}$-categories and bicontinuous $\mathcal{Q}$-functors are monadic over $\mathbf{Set} \downarrow \mathcal{Q}_0$. These conclusions extend the classical results that both the category **Sup** of complete lattices and join-preserving maps, and the category **CD** of completely distributive lattices and complete lattice homomorphisms, are monadic over **Set**.

Second, the correspondence that sends each object A in $\mathbf{Set} \downarrow \mathcal{Q}_0$ to its $\mathcal{Q}$-powerset

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$|\mathcal{P}A|$ (defined below) yields a monadic functor $(\mathbf{Set}\downarrow\mathcal{Q}_0)^{\rm op}\longrightarrow\mathbf{Set}\downarrow\mathcal{Q}_0$. We hasten to remark that the monadicity of $(\mathbf{Set}\downarrow\mathcal{Q}_0)^{\rm op}$ over $\mathbf{Set}\downarrow\mathcal{Q}_0$ is a special case of a general result in topos theory [MM1992] that states that for each topos $\mathbf{E}$, the opposite category $\mathbf{E}^{\rm op}$ is monadic over $\mathbf{E}$. The point of the result presented here is that for each non-empty set X, there exist many monadic functors from $(\mathbf{Set}\downarrow X)^{\rm op}$ to $\mathbf{Set}\downarrow X$.

The contents are arranged as follows. In Section 2 we recall some basic concepts and results about $\mathcal{Q}$-categories and fix notations for later use. Section 3 proves that both $\mathcal{Q}$-**Sup** and $\mathcal{Q}$-**CD** are monadic over $\mathbf{Set}\downarrow\mathcal{Q}_0$. Section 4 proves the monadicity of the functor $(\mathbf{Set}\downarrow\mathcal{Q}_0)^{\rm op}\longrightarrow\mathbf{Set}\downarrow\mathcal{Q}_0$ that sends each object A in $\mathbf{Set}\downarrow\mathcal{Q}_0$ to its $\mathcal{Q}$-powerset.

2. Categories enriched over a quantaloid

We refer to [Stu2005, Stu2006] for an overview of the theory of quantaloid-enriched categories. In this preliminary section, we recall some basic concepts and fix some notations for later use. It should be noted that the theory of quantaloid-enriched categories is a special case of that of $\mathcal{W}$-categories; and that some of the results in this section are also known to be valid for $\mathcal{W}$-categories, for example, the construction of $\mathcal{P}\mathbb{A}$ and the Yoneda lemma. The reader is referred to [BC1982, BCSW1983, Str1981, Str1983, Wal1981, Wal1982] for more on these categories.

$\mathcal{Q}$-CATEGORIES, $\mathcal{Q}$-FUNCTORS, AND $\mathcal{Q}$-DISTRIBUTORS. A quantaloid $\mathcal{Q}$ is a category such that $\mathcal{Q}(X,Y)$ is a complete lattice for any objects X,Y in $\mathcal{Q}$ and that the composition $\circ$ of arrows preserves suprema in both variables, i.e.

$$g\circ\bigvee_i f_i=\bigvee_i g\circ f_i \quad\text{and}\quad \bigvee_i g_i\circ f=\bigvee_i g_i\circ f$$

whenever the operations are defined. The identity arrow on an object X is written 1_X. The top and bottom elements in $\mathcal{Q}(X,Y)$ are denoted by $\top_{X,Y}$ and $\bot_{X,Y}$ respectively. The identity 1_X is required to be different from the bottom element $\bot_{X,X}$ for all objects X in $\mathcal{Q}$. However, for different objects X and Y, it may happen that $\top_{X,Y}=\bot_{X,Y}$. The class of objects in $\mathcal{Q}$ is denoted by $\mathcal{Q}_0$ as usual.

For any arrow $f:X\longrightarrow Y$ and any object Z in a quantaloid $\mathcal{Q}$, both of the maps

$$-\circ f:\mathcal{Q}(Y,Z)\longrightarrow\mathcal{Q}(X,Z),\quad f\circ -:\mathcal{Q}(Z,X)\longrightarrow\mathcal{Q}(Z,Y)$$

have respective right adjoints

$$-\swarrow f:\mathcal{Q}(X,Z)\longrightarrow\mathcal{Q}(Y,Z),\quad f\searrow -:\mathcal{Q}(Z,Y)\longrightarrow\mathcal{Q}(Z,X).$$

The operators $\searrow$ and $\swarrow$ are called the right and left implication respectively.

In this paper, $\mathcal{Q}$ is assumed to be a small quantaloid. This means that $\mathcal{Q}_0$ is a set.

A $\mathcal{Q}$-typed set A is a pair (A_0,t) with A_0 being a set and t a function $A_0\longrightarrow\mathcal{Q}_0$. The function t is called the type function of A with the value tx the type of x. Type functions of $\mathcal{Q}$-typed sets are all denoted by "t", as usual. A type-preserving map $F : A \longrightarrow B$ between $\mathcal{Q}$-typed sets is a function $F : A_0 \longrightarrow B_0$ such that $t(Fx) = tx$ for all $x \in A_0$. The category of $\mathcal{Q}$-typed sets and type-preserving maps is exactly the slice category $\mathbf{Set} \downarrow \mathcal{Q}_0$.

For each $X \in \mathcal{Q}_0$, we write $*_X$ for the $\mathcal{Q}$-typed set with exactly one element $*$ that is of type X.

For a $\mathcal{Q}$-typed set $A = (A_0, t)$, the underlying set A_0 is often written A for simplicity if no confusion would arise.

A $\mathcal{Q}$-matrix $\phi : A \to\!\!\!\!\circ\; B$ between $\mathcal{Q}$-typed sets is a function that assigns to each pair $(x, y) \in A \times B$ an arrow $\phi(x, y) \in \mathcal{Q}(tx, ty)$. In particular, if A (resp. B) is of the form $*_X$, then we write $\phi(x)$ for $\phi(*, x)$ (resp. $\phi(x, *)$).

$\mathcal{Q}$-typed sets and $\mathcal{Q}$-matrices constitute a quantaloid $\mathcal{Q}$-$\mathbf{Mat}$ in which

- The composition $\psi \circ \phi : A \to\!\!\!\!\circ\; C$ of $\phi : A \to\!\!\!\!\circ\; B$ and $\psi : B \to\!\!\!\!\circ\; C$ is given by

$$(\psi \circ \phi)(x, z) = \bigvee_{y \in B} \psi(y, z) \circ \phi(x, y).$$

- The identity $\mathcal{Q}$-matrix $\mathrm{id}_A : A \to\!\!\!\!\circ\; A$ on a $\mathcal{Q}$-typed set A is given by

$$\mathrm{id}_A(x, y) = \begin{cases} 1_{tx}, & x = y; \\ \bot_{tx,ty}, & \text{otherwise.} \end{cases}$$

- The local order is defined pointwise, that is,

$$\phi_1 \leq \phi_2 : A \to\!\!\!\!\circ\; B \text{ if and ony if } \phi_1(x, y) \leq \phi_2(x, y) \text{ for all } (x, y) \in A \times B.$$

- For any $\mathcal{Q}$-matrices $\phi : A \to\!\!\!\!\circ\; B, \psi : B \to\!\!\!\!\circ\; C$ and $\lambda : A \to\!\!\!\!\circ\; C$, $\lambda \swarrow \phi : B \to\!\!\!\!\circ\; C$ and $\psi \searrow \lambda : A \to\!\!\!\!\circ\; B$ are respectively given by

$$(\lambda \swarrow \phi)(y, z) = \bigwedge_{x \in A} \lambda(x, z) \swarrow \phi(x, y), \ (\psi \searrow \lambda)(x, y) = \bigwedge_{z \in C_0} \psi(y, z) \searrow \lambda(x, z).$$

A $\mathcal{Q}$-category $\mathbb{A}$ is a monad in the 2-category $\mathcal{Q}$-$\mathbf{Mat}$. Explicitly, a $\mathcal{Q}$-category is a pair $(A, \mathbb{A})$ where A is a $\mathcal{Q}$-typed set and $\mathbb{A} : A \to\!\!\!\!\circ\; A$ is a $\mathcal{Q}$-matrix such that $\mathrm{id}_A \leq \mathbb{A}$ and $\mathbb{A} \circ \mathbb{A} \leq \mathbb{A}$.

In the following we write $\mathbb{A}$ for a $\mathcal{Q}$-category, $|\mathbb{A}|$ for its underlying $\mathcal{Q}$-typed set and $\mathbb{A}_0$ for the underlying set of $|\mathbb{A}|$.

A $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories is a type-preserving map $F : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ such that $\mathbb{A}(x, y) \leq \mathbb{B}(Fx, Fy)$ for all objects x, y in $\mathbb{A}$. The category of $\mathcal{Q}$-categories and $\mathcal{Q}$-functors is denoted by $\mathcal{Q}$-$\mathbf{Cat}$.

The correspondence $\mathbb{A} \mapsto |\mathbb{A}|$ defines a (forgetful) functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$. Conversely, each $\mathcal{Q}$-typed set A together with the identity $\mathcal{Q}$-matrix on A is a $\mathcal{Q}$-category. Such $\mathcal{Q}$-categories are said to be discrete. In this paper, we do not distinguish $\mathcal{Q}$-typed sets and discrete $\mathcal{Q}$-categories.

For a $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories, we write F instead of $|F|$ for the underlying type-preserving map $|\mathbb{A}| \longrightarrow |\mathbb{B}|$.

The underlying order of a $\mathcal{Q}$-category $\mathbb{A}$ [Stu2005] refers to the preorder on the set of objects in $\mathbb{A}$ defined by

$$x \leq y \iff tx = ty \text{ and } 1_{tx} \leq \mathbb{A}(x, y).$$

It is trivial that $\mathcal{Q}$-functors preserve underlying orders of $\mathcal{Q}$-categories. Two objects x, y of $\mathbb{A}$ are isomorphic, in symbols $x \cong y$, if $x \leq y$ and $y \leq x$. A $\mathcal{Q}$-category $\mathbb{A}$ is skeletal if its underlying order is antisymmetric.

The underlying order of a $\mathcal{Q}$-category $\mathbb{B}$ induces a preorder on the set of all $\mathcal{Q}$-functors from a $\mathcal{Q}$-category $\mathbb{A}$ to $\mathbb{B}$:

$$F \leq G \iff \forall x \in \mathbb{A}, Fx \leq Gx.$$

Thus, $\mathcal{Q}$-**Cat** is indeed a locally ordered category. Two $\mathcal{Q}$-functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$ are isomorphic, in symbols $F \cong G$, if $F \leq G$ and $G \leq F$.

A pair of $\mathcal{Q}$-functors $F : \mathbb{A} \longrightarrow \mathbb{B}$ and $G : \mathbb{B} \longrightarrow \mathbb{A}$ is said to form an adjunction, written $F \dashv G : \mathbb{A} \rightharpoonup \mathbb{B}$, if $1_{\mathbb{A}} \leq G \circ F$ and $F \circ G \leq 1_{\mathbb{B}}$. In this case, F is called a left adjoint of G and G a right adjoint of F.

A $\mathcal{Q}$-distributor $\phi : \mathbb{A} \looparrowright \mathbb{B}$ between $\mathcal{Q}$-categories is a $\mathcal{Q}$-matrix $\phi : |\mathbb{A}| \looparrowright |\mathbb{B}|$ that is compatible with the structures on $\mathbb{A}$ and $\mathbb{B}$ in the sense that

$$\mathbb{B}(y, y') \circ \phi(x, y) \leq \phi(x, y') \quad \text{and} \quad \phi(x, y) \circ \mathbb{A}(x', x) \leq \phi(x', y)$$

for any objects x, x' in $\mathbb{A}$ and y, y' in $\mathbb{B}$; or equivalently, $\phi \circ \mathbb{A} = \phi = \mathbb{B} \circ \phi$ in $\mathcal{Q}$-**Mat**. $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors constitute a quantaloid $\mathcal{Q}$-**Dist** in which compositions, the left and right implications are calculated as in $\mathcal{Q}$-**Mat**.

Following [Lack2010], for a 2-category $\mathbf{C}$, we denote by $\mathbf{C}^{\mathrm{op}}$ ($\mathbf{C}^{\mathrm{co}}$, resp.) the 2-category obtained by reversing the 1-arrows (the 2-arrows, resp.) in $\mathbf{C}$. For each quantaloid $\mathcal{Q}$, $\mathcal{Q}^{\mathrm{op}}$ is also a quantaloid, but $\mathcal{Q}^{\mathrm{co}}$ is not in general. Given a $\mathcal{Q}$-category $\mathbb{A}$, there is a corresponding $\mathcal{Q}^{\mathrm{op}}$-category $\mathbb{A}^{\mathrm{op}}$ with the same underlying $\mathcal{Q}$-typed set as that of $\mathbb{A}$ and with $\mathbb{A}^{\mathrm{op}}(x, y) = \mathbb{A}(y, x)$.[1] For each $\mathcal{Q}$-distributor $\phi : \mathbb{A} \looparrowright \mathbb{B}$, the assignment $\phi^{\mathrm{op}}(y, x) = \phi(x, y)$ defines a $\mathcal{Q}^{\mathrm{op}}$-distributor $\mathbb{B}^{\mathrm{op}} \looparrowright \mathbb{A}^{\mathrm{op}}$. If $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a $\mathcal{Q}$-functor, then

$$F^{\mathrm{op}} : \mathbb{A}^{\mathrm{op}} \longrightarrow \mathbb{B}^{\mathrm{op}}, \quad x \mapsto Fx$$

is a $\mathcal{Q}^{\mathrm{op}}$-functor. Furthermore, $F \leq G$ in $\mathcal{Q}$-**Cat** if and only if $G^{\mathrm{op}} \leq F^{\mathrm{op}}$ in $\mathcal{Q}^{\mathrm{op}}$-**Cat**. Therefore, $(\mathcal{Q}\text{-}\mathbf{Cat})^{\mathrm{co}}$ is isomorphic to $\mathcal{Q}^{\mathrm{op}}$-**Cat** [Stu2005].

[1] We would like to point out that the terminologies adopted here are not exactly the same as in our main references, [Stu2005, Stu2006], on quantaloid-enriched categories. Our $\mathcal{Q}$-categories and $\mathcal{Q}$-distributors are exactly the $\mathcal{Q}^{\mathrm{op}}$-categories and $\mathcal{Q}^{\mathrm{op}}$-distributors in the sense of Stubbe. The difference arises in the interpretations of $\mathbb{A}(x, y)$ for a $\mathcal{Q}$-category $\mathbb{A}$: it is interpreted as the hom-arrow from y to x in [Stu2005, Stu2006], but from x to y here. Note that this difference also leads to the swap of presheaves and co-presheaves.

The graph and cograph of a $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ refer to the $\mathcal{Q}$-distributors $F_\natural = \mathbb{B}(F-,-) : \mathbb{A} \looparrowright \mathbb{B}$ and $F^\natural = \mathbb{B}(-,F-) : \mathbb{B} \looparrowright \mathbb{A}$ respectively. $F_\natural$ is a left adjoint of $F^\natural$ in $\mathcal{Q}$-**Dist**, i.e., $\mathbb{A} \leq F^\natural \circ F_\natural$ and $F_\natural \circ F^\natural \leq \mathbb{B}$.

The following proposition is a special case of an observation in [BCSW1983] about modules (= distributors) between $\mathcal{W}$-categories. We record it here because of its usefulness.

2.1. Proposition. *Let $F : \mathbb{A} \longrightarrow \mathbb{B}$ be a $\mathcal{Q}$-functor.*

(1) *F is fully faithful in the sense that $\mathbb{A}(x,y) = \mathbb{B}(Fx,Fy)$ for all $x,y \in \mathbb{A}$ if and only if $F^\natural \circ F_\natural = \mathbb{A}$.*

(2) *If F is essentially surjective in the sense that there is some $x \in \mathbb{A}$ such that $Fx \cong y$ in $\mathbb{B}$ for all $y \in \mathbb{B}$, then $F_\natural \circ F^\natural = \mathbb{B}$.*

A presheaf [Stu2005] on a $\mathcal{Q}$-category $\mathbb{A}$ is a $\mathcal{Q}$-distributor of the form $\phi : \mathbb{A} \looparrowright *_X$. All presheaves on $\mathbb{A}$ constitute a skeletal $\mathcal{Q}$-category $\mathcal{P}\mathbb{A}$ with

$$t\phi = X \text{ and } \mathcal{P}\mathbb{A}(\phi,\phi') = \phi' \swarrow \phi$$

for any $\phi : \mathbb{A} \looparrowright *_X$ and $\phi' : \mathbb{A} \looparrowright *_Y$.

Dually, a co-presheaf on $\mathbb{A}$ is a $\mathcal{Q}$-distributor of the form $\psi : *_X \looparrowright \mathbb{A}$. All co-presheaves on $\mathbb{A}$ constitute a skeletal $\mathcal{Q}$-category $\mathcal{P}^\dagger\mathbb{A}$ with

$$t\psi = X \text{ and } \mathcal{P}^\dagger\mathbb{A}(\psi,\psi') = \psi' \searrow \psi$$

for any $\psi : *_X \looparrowright \mathbb{A}$ and $\psi' : *_Y \looparrowright \mathbb{A}$.

It should be stressed that the underlying order of $\mathcal{P}\mathbb{A}$ coincides with the local order in $\mathcal{Q}$-**Dist** while the underlying order of $\mathcal{P}^\dagger\mathbb{A}$ is the reverse local order in $\mathcal{Q}$-**Dist**.

The correspondences

$$x \mapsto \mathbb{A}(-,x) : \mathbb{A} \looparrowright *_{tx}$$

and

$$x \mapsto \mathbb{A}(x,-) : *_{tx} \looparrowright \mathbb{A}$$

define two $\mathcal{Q}$-functors

$$\mathsf{Y}_\mathbb{A} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$$

and

$$\mathsf{Y}^\dagger_\mathbb{A} : \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{A}$$

which are called respectively the Yoneda and the co-Yoneda embedding due to the following:

2.2. Lemma. *(Yoneda lemma, [Stu2005])* $\mathcal{P}\mathbb{A}(\mathsf{Y}_{\mathbb{A}}(x), \phi) = \phi(x)$ *and* $\mathcal{P}^\dagger\mathbb{A}(\psi, \mathsf{Y}^\dagger_{\mathbb{A}}(x)) = \psi(x)$ *for any* $x \in \mathbb{A}$, $\phi \in \mathcal{P}\mathbb{A}$, *and* $\psi \in \mathcal{P}^\dagger\mathbb{A}$.

The correspondence $\mathbb{A} \mapsto \mathcal{P}\mathbb{A}$ gives a contravariant functor

$$\mathcal{P} : (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$$

that sends a $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ to

$$\mathcal{P}F : \mathcal{P}\mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}, \quad \mathcal{P}F(\psi) = \psi \circ F_\natural.$$

Dually, the correspondence $\mathbb{A} \mapsto \mathcal{P}^\dagger\mathbb{A}$ gives a contravariant functor

$$\mathcal{P}^\dagger : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op}$$

that sends a $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ to

$$\mathcal{P}^\dagger F : \mathcal{P}^\dagger\mathbb{B} \longrightarrow \mathcal{P}^\dagger\mathbb{A}, \quad \mathcal{P}^\dagger F(\psi) = F^\natural \circ \psi.$$

2.3. Theorem. *[Hoh2014, Stu2005] The functor* $\mathcal{P}^\dagger : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op}$ *is left adjoint to* $\mathcal{P} : (\mathcal{Q}\text{-}\mathbf{Cat})^{\rm op} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$.

2.4. Proposition. *[Stu2013] Let* $F : \mathbb{A} \longrightarrow \mathbb{B}$ *be a* $\mathcal{Q}$*-functor.*

(1) *The* $\mathcal{Q}$*-functor* $\mathcal{P}F$ *has a left adjoint* $\exists_F : \mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$ *and a right adjoint* $\forall_F : \mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$ *given by* $\exists_F(\phi) = \phi \circ F^\natural$ *and* $\forall_F(\phi) = \phi \swarrow F_\natural$, *respectively.*

(2) *The* $\mathcal{Q}$*-functor* $\mathcal{P}^\dagger F : \mathcal{P}^\dagger\mathbb{B} \longrightarrow \mathcal{P}^\dagger\mathbb{A}$ *has a left adjoint* $\forall^\dagger_F : \mathcal{P}^\dagger\mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{B}$ *and a right adjoint* $\exists^\dagger_F : \mathcal{P}^\dagger\mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{B}$ *given by* $\forall^\dagger_F(\psi) = F^\natural \searrow \psi$ *and* $\exists^\dagger_F(\psi) = F_\natural \circ \psi$, *respectively.*

Different notations have been used for the $\mathcal{Q}$-functors $\exists_F, \forall_F, \exists^\dagger_F$ and $\forall^\dagger_F$ in [SZ2013a, Stu2013]. The notations adopted here originate from topos theory [MM1992].

2.5. Proposition. *Given a pair of* $\mathcal{Q}$*-functors* $F : \mathbb{A} \longrightarrow \mathbb{B}$ *and* $G : \mathbb{B} \longrightarrow \mathbb{A}$, *the following are equivalent:*

(1) $F \dashv G : \mathbb{A} \rightharpoonup \mathbb{B}$.

(2) $\exists_F \dashv \exists_G : \mathcal{P}\mathbb{A} \rightharpoonup \mathcal{P}\mathbb{B}$.

(3) $\mathcal{P}F \dashv \mathcal{P}G : \mathcal{P}\mathbb{B} \rightharpoonup \mathcal{P}\mathbb{A}$.

(4) $\exists^\dagger_F \dashv \exists^\dagger_G : \mathcal{P}^\dagger\mathbb{A} \rightharpoonup \mathcal{P}^\dagger\mathbb{B}$.

(5) $\mathcal{P}^\dagger F \dashv \mathcal{P}^\dagger G : \mathcal{P}^\dagger\mathbb{B} \rightharpoonup \mathcal{P}^\dagger\mathbb{A}$.

Proof. We prove the equivalence of (1) and (2) for example.

(1) $\Rightarrow$ (2) This follows from the fact that a 2-functor preserves adjunctions [Lack2010].

(2) $\Rightarrow$ (1) For any object x in $\mathbb{A}$,

$$\mathsf{Y}_{\mathbb{A}}(x) \leq \exists_G \circ \exists_F(\mathsf{Y}_{\mathbb{A}}(x)) = \mathsf{Y}_{\mathbb{A}}(x) \circ (G \circ F)^\natural = \mathsf{Y}_{\mathbb{A}}(GFx)$$

showing that $x \leq GFx$. Thus $1_{\mathbb{A}} \leq G \circ F$. Similarly it can be verified that $F \circ G \leq 1_{\mathbb{B}}$. Hence $F \dashv G : \mathbb{A} \rightharpoonup \mathbb{B}$. ∎

It is clear that the assignments $F \mapsto \exists_F$ and $F \mapsto \exists_F^\dagger$ give rise to two functors:

$$\mathcal{P}_\exists : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat} \quad \text{and} \quad \mathcal{P}_\exists^\dagger : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}.$$

Both $\mathcal{P}_\exists$ and $\mathcal{P}_\exists^\dagger$ preserve the local order in $\mathcal{Q}$-**Cat**, hence both of them are 2-functorial $\mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$. Both of the contravariant functors $\mathcal{P}$ and $\mathcal{P}^\dagger$ reverse the local order, so, both of them are 2-functorial from $\mathcal{Q}\text{-}\mathbf{Cat}^{\mathrm{coop}}$ to $\mathcal{Q}$-**Cat**.

For any $\mathcal{Q}$-functor F, it follows from Proposition 2.4 that $\mathcal{P}^\dagger F \dashv \mathcal{P}_\exists^\dagger F$. Thus, $\mathcal{P}_\exists(\mathcal{P}^\dagger F) \dashv \mathcal{P}_\exists(\mathcal{P}_\exists^\dagger F)$ by Proposition 2.5(2). Also by Proposition 2.4 one has $\mathcal{P}_\exists(\mathcal{P}^\dagger F) \dashv \mathcal{P}(\mathcal{P}^\dagger F)$. Thus, $\mathcal{P}(\mathcal{P}^\dagger F) = \mathcal{P}_\exists(\mathcal{P}_\exists^\dagger F)$. This proves the following:

2.6. Corollary. *[Hoh2014, Stu2013]* $\mathcal{P} \circ \mathcal{P}^\dagger = \mathcal{P}_\exists \circ \mathcal{P}_\exists^\dagger$.

$$\begin{array}{ccc} \mathcal{Q}\text{-}\mathbf{Cat} & \xrightarrow{\mathcal{P}_\exists^\dagger} & \mathcal{Q}\text{-}\mathbf{Cat} \\ {\scriptstyle \mathcal{P}^\dagger}\downarrow & & \downarrow{\scriptstyle \mathcal{P}_\exists} \\ \mathcal{Q}\text{-}\mathbf{Cat}^{\mathrm{op}} & \xrightarrow{\mathcal{P}} & \mathcal{Q}\text{-}\mathbf{Cat} \end{array}$$

The following conclusion is a direct consequence of Proposition 2.1, it will be useful in the last section.

2.7. Proposition. *Let $F : \mathbb{A} \longrightarrow \mathbb{B}$ be a $\mathcal{Q}$-functor.*

(1) *F is fully faithful if and only if $\mathcal{P}F \circ \exists_F = 1_{\mathcal{P}\mathbb{A}}$ if and only if $\mathcal{P}^\dagger F \circ \exists_F^\dagger = 1_{\mathcal{P}^\dagger\mathbb{A}}$.*

(2) *If F is essentially surjective, then $\exists_F \circ \mathcal{P}F = 1_{\mathcal{P}\mathbb{B}}$ and $\exists_F^\dagger \circ \mathcal{P}^\dagger F = 1_{\mathcal{P}^\dagger\mathbb{B}}$.*

Complete and completely distributive $\mathcal{Q}$-categories. Let $\mathbb{A}$ be a $\mathcal{Q}$-category and $\phi : \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} *_X$ a presheaf on $\mathbb{A}$. A supremum of ϕ is an object $\sup\phi$ in $\mathbb{A}$ of type X such that for any x in $\mathbb{A}$,

$$\mathbb{A}(\sup\phi, x) = \mathcal{P}\mathbb{A}(\phi, \mathsf{Y}_\mathbb{A}(x));$$

or equivalently, $\mathbb{A}(\sup\phi, -) = \mathbb{A} \swarrow \phi$. It is clear that the supremum of a presheaf $\mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} *_X$, if exists, is unique up to isomorphism. Dually, the infimum of a co-presheaf $\psi : *_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ is an object $\inf\psi$ in $\mathbb{A}$ of type X such that for any x in $\mathbb{A}$,

$$\mathbb{A}(x, \inf\psi) = \mathcal{P}^\dagger\mathbb{A}(\mathsf{Y}_\mathbb{A}^\dagger(x), \psi);$$

or equivalently, $\mathbb{A}(-, \inf\psi) = \psi \searrow \mathbb{A}$.

2.8. Definition. [Stu2005] A $\mathcal{Q}$-category $\mathbb{A}$ is cocomplete if every presheaf on $\mathbb{A}$ has a supremum; $\mathbb{A}$ is complete if every co-presheaf on $\mathbb{A}$ has an infimum.

It is known that (i) $\mathbb{A}$ is cocomplete if and only if the Yoneda embedding $\mathsf{Y}_\mathbb{A} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ has a left adjoint $\sup_\mathbb{A} : \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$; (ii) $\mathbb{A}$ is complete if the co-Yoneda embedding $\mathsf{Y}_\mathbb{A}^\dagger : \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{A}$ has a right adjoint $\inf_\mathbb{A} : \mathcal{P}^\dagger\mathbb{A} \longrightarrow \mathbb{A}$; and (iii) $\mathbb{A}$ is complete if and only if it is cocomplete.

2.9. Example. [Stu2005] Let $\mathbb{A}$ be a $\mathcal{Q}$-category. Then both $\mathcal{P}\mathbb{A}$ and $\mathcal{P}^{\dagger}\mathbb{A}$ are complete, hence cocomplete. Explicitly, for any $\Phi \in \mathcal{P}(\mathcal{P}\mathbb{A})$ and $\Psi \in \mathcal{P}^{\dagger}(\mathcal{P}\mathbb{A})$,

$$\sup \Phi = \Phi \circ (\mathsf{Y}_{\mathbb{A}})_{\natural}, \quad \text{and} \quad \inf \Psi = \Psi \searrow (\mathsf{Y}_{\mathbb{A}})_{\natural};$$

for any $\Phi \in \mathcal{P}(\mathcal{P}^{\dagger}\mathbb{A})$ and $\Psi \in \mathcal{P}^{\dagger}(\mathcal{P}^{\dagger}\mathbb{A})$,

$$\sup \Phi = (\mathsf{Y}^{\dagger}_{\mathbb{A}})^{\natural} \swarrow \Phi, \quad \text{and} \quad \inf \Psi = (\mathsf{Y}^{\dagger}_{\mathbb{A}})^{\natural} \circ \Psi.$$

In particular, $\sup_{\mathcal{P}\mathbb{A}} = \mathcal{P}\mathsf{Y}_{\mathbb{A}}$ and $\inf_{\mathcal{P}^{\dagger}\mathbb{A}} = \mathcal{P}^{\dagger}\mathsf{Y}^{\dagger}_{\mathbb{A}}$.

A $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be cocontinuous if it preserves suprema in the sense that $F(\sup_{\mathbb{A}}\phi)$ is a supremum of $\exists_F(\phi)$ whenever $\sup_{\mathbb{A}}\phi$ exists. Dually, $F : \mathbb{A} \longrightarrow \mathbb{B}$ is continuous if it preserves infima in the sense that $F(\inf_{\mathbb{A}}\phi)$ is an infimum of $\exists^{\dagger}_F(\psi)$ whenever $\inf_{\mathbb{A}}\psi$ exists. $F : \mathbb{A} \longrightarrow \mathbb{B}$ is bicontinuous if it is both cocontinuous and continuous.

It is known [Stu2005] that a $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between complete $\mathcal{Q}$-categories is a left adjoint (resp. right adjoint) if and only if F is cocontinuous (resp. continuous). In particular, for each $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$, $\mathcal{P}F : \mathcal{P}\mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}$ is bicontinuous; $\exists_F : \mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$ is cocontinuous; and $\forall_F : \mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$ is continuous.

2.10. Definition. [Stu2007] A $\mathcal{Q}$-category $\mathbb{A}$ is completely distributive if it is cocomplete and the left adjoint $\sup_{\mathbb{A}} : \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$ of the Yoneda embedding $\mathsf{Y}_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ has a left adjoint $\Downarrow_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$.

Note that completely distributive $\mathcal{Q}$-categories are said to be *totally continuous* in [Stu2007]. Here we call them completely distributive following the practice in lattice theory, e.g. [Joh1982, Ran1952, Wood2004].

2.11. Example. [Stu2007] For a $\mathcal{Q}$-category $\mathbb{A}$, it follows from Example 2.9 that $\sup_{\mathcal{P}\mathbb{A}} = \mathcal{P}\mathsf{Y}_{\mathbb{A}}$. Thus, $\sup_{\mathcal{P}\mathbb{A}}$ is a right adjoint by Proposition 2.4. This shows that $\mathcal{P}\mathbb{A}$ is completely distributive.

2.12. Proposition. *Let $\mathbb{A}, \mathbb{B}$ be skeletal $\mathcal{Q}$-categories, $F : \mathbb{A} \longrightarrow \mathbb{B}$ a left and right adjoint $\mathcal{Q}$-functor.*

(1) *If F is an epimorphism in $\mathcal{Q}$-**Cat** and $\mathbb{A}$ is completely distributive, then so is $\mathbb{B}$.*

(2) *If F is a monomorphism in $\mathcal{Q}$-**Cat** and $\mathbb{B}$ is completely distributive, then so is $\mathbb{A}$.*

Proof. (1) Suppose that $H \dashv F \dashv G$. Then $F \circ G \circ F = F$, hence $F \circ G = 1_{\mathbb{B}}$ since F is an epimorphism. It follows that for any $y \in \mathbb{B}$,

$$(\mathcal{P}G \circ \mathsf{Y}_{\mathbb{A}} \circ G)(y) = \mathcal{P}G(\mathsf{Y}_{\mathbb{A}}(Gy)) = \mathbb{A}(G-, Gy) = \mathbb{B}(F \circ G-, y) = \mathbb{B}(-, y) = \mathsf{Y}_{\mathbb{B}}(y),$$

showing that $\mathcal{P}G \circ \mathsf{Y}_{\mathbb{A}} \circ G = \mathsf{Y}_{\mathbb{B}}$.

By assumption, the Yoneda embedding $\mathsf{Y}_{\mathbb{A}}$ has a left adjoint $\sup_{\mathbb{A}}$ that also has a left adjoint $\Downarrow_{\mathbb{A}}$. By virtue of Proposition 2.5 it holds that $\mathcal{P}H \dashv \mathcal{P}F \dashv \mathcal{P}G$, hence

$$(\mathcal{P}H \circ \Downarrow_{\mathbb{A}} \circ H) \dashv (F \circ \sup_{\mathbb{A}} \circ \mathcal{P}F) \dashv \mathcal{P}G \circ \mathsf{Y}_{\mathbb{A}} \circ G = \mathsf{Y}_{\mathbb{B}}.$$

Therefore, $\mathbb{B}$ is completely distributive with $\sup_{\mathbb{B}} = F \circ \sup_{\mathbb{A}} \circ \mathcal{P}F$.

(2) Suppose that $H \dashv F \dashv G$. Then $F \circ H \circ F = F \circ G \circ F = F$, and thus $H \circ F = G \circ F = 1_{\mathbb{A}}$ since F is a monomorphism. Hence, for each x in $\mathbb{A}$,

$$(\mathcal{P}F \circ \mathsf{Y}_{\mathbb{B}} \circ F)(x) = \mathcal{P}F(\mathsf{Y}_{\mathbb{B}}(Fx)) = \mathbb{B}(F-, Fx) = \mathbb{A}(H \circ F-, x) = \mathbb{A}(-, x) = \mathsf{Y}_{\mathbb{A}}(x).$$

That means $\mathcal{P}F \circ \mathsf{Y}_{\mathbb{B}} \circ F = \mathsf{Y}_{\mathbb{A}}$. Since $\exists_F \dashv \mathcal{P}F$, $\sup_{\mathbb{B}} \dashv \mathsf{Y}_{\mathbb{B}}$, and $H \dashv F$, it follows that $H \circ \sup_{\mathbb{B}} \circ \exists_F$ is a left adjoint of $\mathsf{Y}_{\mathbb{A}} = \mathcal{P}F \circ \mathsf{Y}_{\mathbb{B}} \circ F$. Therefore, $\mathbb{A}$ is complete with $\sup_{\mathbb{A}} = H \circ \sup_{\mathbb{B}} \circ \exists_F$. Since F is cocontinuous (being a left adjoint), we have that $F \circ \sup_{\mathbb{A}} = \sup_{\mathbb{B}} \circ \exists_F$. Hence

$$\sup_{\mathbb{A}} = G \circ F \circ \sup_{\mathbb{A}} = G \circ \sup_{\mathbb{B}} \circ \exists_F.$$

This shows that $\sup_{\mathbb{A}} : \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$ is a composite of right adjoints ($\exists_F$ is a right adjoint by Proposition 2.5, $\sup_{\mathbb{B}}$ is a right adjoint by complete distributivity of $\mathbb{B}$), hence it is itself a right adjoint. The conclusion thus follows. ∎

Now we form the following categories:

- $\mathcal{Q}$-**Sup**, the category of skeletal cocomplete $\mathcal{Q}$-categories and cocontinuous $\mathcal{Q}$-functors.
- $\mathcal{Q}$-**Inf**, the category of skeletal complete $\mathcal{Q}$-categories and continuous $\mathcal{Q}$-functors.
- $\mathcal{Q}$-**CD**, the category of skeletal completely distributive $\mathcal{Q}$-categories and bicontinuous $\mathcal{Q}$-functors.

The categories $\mathcal{Q}$-**Sup** and $\mathcal{Q}$-**Inf** are dually isomorphic. For each cocontinuous $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between complete $\mathcal{Q}$-categories, let $F^{\dashv}$ denote its right adjoint. Dually, for each continuous $\mathcal{Q}$-functor $G : \mathbb{B} \longrightarrow \mathbb{A}$ between complete $\mathcal{Q}$-categories, let $G^{\vdash}$ denote its left adjoint. Then we obtain a pair of functors

$$\mathcal{Q}\text{-}\mathbf{Inf} \underset{(\text{-})^{\vdash}}{\overset{(\text{-})^{\dashv}}{\leftrightarrows}} \mathcal{Q}\text{-}\mathbf{Sup}^{\mathrm{op}}$$

that are inverse to each other.

THE QUESTIONS. The categories $\mathcal{Q}$-**Sup**, $\mathcal{Q}$-**CD** are respectively the $\mathcal{Q}$-analogue of the category **Sup** of complete lattices and join-preserving maps, and the category **CD** of completely distributive lattices and complete lattice homomorphisms. Since both **Sup** and **CD** are monadic over **Set** [Joh1982], our first question is whether the categories $\mathcal{Q}$-**Sup** and $\mathcal{Q}$-**CD** are monadic over $\mathbf{Set} \downarrow \mathcal{Q}_0$?

The forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ has a left adjoint $\mathcal{I} : \mathbf{Set} \downarrow \mathcal{Q}_0 \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$, given by identifying $\mathcal{Q}$-typed sets with discrete $\mathcal{Q}$-categories. Consider the adjunction $|\mathcal{P}^{\dagger}| \dashv |\mathcal{P}|$ obtained by composing the following

$$\mathbf{Set} \downarrow \mathcal{Q}_0 \underset{\mathcal{I}}{\overset{|\text{-}|}{\leftrightarrows}} \mathcal{Q}\text{-}\mathbf{Cat} \underset{\mathcal{P}^{\dagger}}{\overset{\mathcal{P}}{\leftrightarrows}} \mathcal{Q}\text{-}\mathbf{Cat}^{\mathrm{op}} \underset{|\text{-}|^{\mathrm{op}}}{\overset{\mathcal{I}^{\mathrm{op}}}{\leftrightarrows}} (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}}.$$

It is clear that $|\mathcal{P}^\dagger| \dashv |\mathcal{P}|$ is the $\mathcal{Q}$-version of the adjunction $\mathcal{P}^{\rm op} \dashv \mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}^{\rm op}$. It is known that $\mathcal{P} : \mathbf{Set}^{\rm op} \longrightarrow \mathbf{Set}$ is monadic and the corresponding algebras are the complete atomic Boolean algebras [Joh1982]. So, it is natural to ask what are the algebras of the monad determined by the adjunction $|\mathcal{P}^\dagger| \dashv |\mathcal{P}|$? This is the second question we'll consider in this paper.

Before proceeding, we list below some known facts about $\mathcal{Q}$-**Cat**, $\mathcal{Q}$-**Sup**, and $\mathcal{Q}$-**CD**.

(a) The monad corresponding to the adjunction $\mathcal{I} \dashv |\text{-}| : \mathbf{Set} \downarrow \mathcal{Q}_0 \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ is the identity monad on $\mathbf{Set} \downarrow \mathcal{Q}_0$, hence its Eilenberg-Moore category is $\mathbf{Set} \downarrow \mathcal{Q}_0$. Therefore, the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ is not monadic. This is an extension of the well-known fact that the forgetful functor from preordered sets to sets is not monadic.

(b) The composite of the adjunctions

$$\mathcal{Q}\text{-}\mathbf{Cat} \underset{\mathcal{P}_\exists^\dagger}{\overset{\mathcal{E}^\dagger}{\rightleftarrows}} \mathcal{Q}\text{-}\mathbf{Inf} \underset{(\text{-})^{\vdash}}{\overset{(\text{-})^{\dashv}}{\rightleftarrows}} \mathcal{Q}\text{-}\mathbf{Sup}^{\rm op} \underset{\mathcal{E}^{\rm op}}{\overset{\mathcal{P}_\exists^{\rm op}}{\rightleftarrows}} \mathcal{Q}\text{-}\mathbf{Cat}^{\rm op}$$

is exactly the adjunction $\mathcal{P}^\dagger \dashv \mathcal{P} : \mathcal{Q}\text{-}\mathbf{Cat} \rightharpoonup \mathcal{Q}\text{-}\mathbf{Cat}^{\rm op}$ in Theorem 2.3. It is proved in [Stu2013] that the algebras of the monad corresponding to the adjunction $\mathcal{P}^\dagger \dashv \mathcal{P}$ are the completely distributive $\mathcal{Q}$-categories with bicontinuous $\mathcal{Q}$-functors as morphisms. Hence, the category $\mathcal{Q}$-**CD** is monadic over $\mathcal{Q}$-**Cat**.

(c) Restricting the codomain of the 2-functor $\mathcal{P}_\exists : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ to $\mathcal{Q}$-**Sup** gives a left adjoint, also written $\mathcal{P}_\exists$, to the forgetful functor $\mathcal{E} : \mathcal{Q}\text{-}\mathbf{Sup} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$. It is proved in [Stu2013] that the forgetful functor $\mathcal{Q}\text{-}\mathbf{Sup} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ is lax-idempotent monadic (see Theorem 3.16 below).

3. $\mathcal{Q}$-Sup and $\mathcal{Q}$-CD are monadic over $\mathbf{Set} \downarrow \mathcal{Q}_0$

The aim of this section is to show that both $\mathcal{Q}$-**Sup** and $\mathcal{Q}$-**CD** are strictly monadic over $\mathbf{Set} \downarrow \mathcal{Q}_0$. Recall that a right adjoint functor $G : \mathbf{D} \longrightarrow \mathbf{C}$ is monadic (resp. strictly monadic) [Mac1998, MM1992] if the comparison functor $K : \mathbf{D} \longrightarrow \mathbf{C}^{\mathbf{T}}$ is an equivalence (resp. isomorphism) of categories, where $\mathbf{T}$ is the corresponding monad and $\mathbf{C}^{\mathbf{T}}$ is the Eilenberg-Moore category of $\mathbf{T}$-algebras and homomorphisms. A category $\mathbf{D}$ is (strictly) monadic over a category $\mathbf{C}$ if there exists a (strictly) monadic functor $G : \mathbf{D} \longrightarrow \mathbf{C}$.

For an object x in a $\mathcal{Q}$-category $\mathbb{A}$ and an arrow $f : tx \longrightarrow Y$ in $\mathcal{Q}$, the tensor of f and x, denoted by $f \otimes x$, is an object in $\mathbb{A}$ of type Y such that $\mathbb{A}(f \otimes x, -) = \mathbb{A}(x, -) \swarrow f$. Dually, for an arrow $g : Y \longrightarrow tx$, the cotensor of g and x, denote by $g \rightarrowtail x$, is an object in $\mathbb{A}$ of type Y such that $\mathbb{A}(-, g \rightarrowtail x) = g \searrow \mathbb{A}(-, x)$. A $\mathcal{Q}$-category $\mathbb{A}$ is tensored if the tensor $f \otimes x$ exists for all objects x in $\mathbb{A}$ and all arrows f in $\mathcal{Q}$ with codomain tx [Stu2006]. The dual notion is *cotensored*.

It is easy to see that the tensor $f \otimes x$ is the supremum of the presheaf $f \circ \mathbb{A}(-, x)$; the cotensor $g \rightarrowtail x$ is the infimum of the co-presheaf $\mathbb{A}(x, -) \circ g$. So, a complete $\mathcal{Q}$-category is both tensored and cotensored.

For each $\mathcal{Q}$-category $\mathbb{A}$ and each object X in $\mathcal{Q}$, write $\mathbb{A}_X$ for the preordered set consisting of objects of type X in $\mathbb{A}$ together with the underlying order. It is known that if $\mathbb{A}$ is a complete $\mathcal{Q}$-category then $\mathbb{A}_X$ is a complete preordered set for each X in $\mathcal{Q}$. The following proposition was observed in [LZ09] for quantale-enriched categories and in [Shen2014] for the general setting.

3.1. Proposition. *Let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{Q}$-categories, and $F : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ be a type-preserving map. If both $\mathbb{A}$ and $\mathbb{B}$ are tensored, then $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a $\mathcal{Q}$-functor if and only if*

(1) *For any object x in $\mathbb{A}$ and arrow $f : tx \longrightarrow Y$, $f \otimes Fx \leq F(f \otimes x)$;*

(2) *For any object X in $\mathcal{Q}$, $F : \mathbb{A}_X \longrightarrow \mathbb{B}_X$ is order-preserving.*

Dually, if both $\mathbb{A}$ and $\mathbb{B}$ are cotensored, then $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a $\mathcal{Q}$-functor if and only if

(1') *For any object x in $\mathbb{A}$ and arrow $g : Y \longrightarrow tx$, $F(g \rightarrowtail x) \leq g \rightarrowtail Fx$;*

(2') *For any object X in $\mathcal{Q}$, $F : \mathbb{A}_X \longrightarrow \mathbb{B}_X$ is order-preserving.*

3.2. Proposition. *[Stu2006] Let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{Q}$-categories, $F : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ a type-preserving map. If $\mathbb{A}$ is tensored, then $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a left adjoint $\mathcal{Q}$-functor if and only if*

(1) *F preserves tensors in the sense that $F(f \otimes x) = f \otimes Fx$ for all objects x in $\mathbb{A}$ and all arrows $f : tx \longrightarrow Y$;*

(2) *For all objects X in $\mathcal{Q}$, $F : \mathbb{A}_X \longrightarrow \mathbb{B}_X$ is a left adjoint.*

Dually, if $\mathbb{A}$ is cotensored, then $F : \mathbb{A} \longrightarrow \mathbb{B}$ is a right adjoint $\mathcal{Q}$-functor if and only if

(1') *F preserves cotensors in the sense that $F(g \rightarrowtail x) = g \rightarrowtail Fx$ for all objects x in $\mathbb{A}$ and all arrows $g : Y \longrightarrow tx$;*

(2') *For all objects X in $\mathcal{Q}$, $F : \mathbb{A}_X \longrightarrow \mathbb{B}_X$ is a right adjoint.*

3.3. Definition. [SZ2013a] A closure operator on a $\mathcal{Q}$-category $\mathbb{A}$ is a $\mathcal{Q}$-functor $c : \mathbb{A} \longrightarrow \mathbb{A}$ such that $1_{\mathbb{A}} \leq c$ and $c^2 \leq c$.

3.4. Lemma. *If $c : \mathbb{A} \longrightarrow \mathbb{A}$ is a closure operator on a skeletal $\mathcal{Q}$-category $\mathbb{A}$, then $c(\mathbb{A}) = \{x \in \mathbb{A} \mid c(x) = x\}$ and $c : \mathbb{A} \longrightarrow c(\mathbb{A})$ is left adjoint to the inclusion $i : c(\mathbb{A}) \hookrightarrow \mathbb{A}$.*

Proof. Since $\mathbb{A}$ is skeletal and $c^2(x)$ is isomorphic to $c(x)$ for each x in $\mathbb{A}$, it follows immediately that $c(\mathbb{A}) = \{x \in \mathbb{A} \mid c(x) = x\}$.

Since $c \circ i(y) = c(y) = y$ for any y in $c(\mathbb{A})$ and $i \circ c(x) = c(x) \geq x$ for any x in $\mathbb{A}$, it follows that $c \circ i = 1_{c(\mathbb{A})}$ and $i \circ c \geq 1_{\mathbb{A}}$. Hence, c is left adjoint to i. ∎

3.5. Definition. A congruence on a complete skeletal $\mathcal{Q}$-category $\mathbb{A}$ is an equivalence relation R on the underlying set $\mathbb{A}_0$ subject to the following conditions:

(i) $(x,y)\in R$ implies $tx = ty$, that is, equivalent elements have the same type.

(ii) For each object X in $\mathcal{Q}$, the subset $R\cap(\mathbb{A}_X\times\mathbb{A}_X)$ is closed w.r.t. joins in $\mathbb{A}_X\times\mathbb{A}_X$.

(iii) If $(x,y)\in R$, then $(f\otimes x, f\otimes y)\in R$ for all $f: tx\longrightarrow Y$.

A congruence R is complete if it satisfies moreover:

(iv) For each object X in $\mathcal{Q}$, the subset $R\cap(\mathbb{A}_X\times\mathbb{A}_X)$ is closed w.r.t. meets in $\mathbb{A}_X\times\mathbb{A}_X$.

(v) If $(x,y)\in R$, then $(g\rightarrowtail x, g\rightarrowtail y)\in R$ for all $g: Y\longrightarrow tx$ in $\mathcal{Q}$.

For a congruence R on a complete skeletal $\mathcal{Q}$-category $\mathbb{A}$, define a map $c:\mathbb{A}_0\longrightarrow\mathbb{A}_0$ by putting $c(x)$ to be the greatest element in the equivalence class of x (which is a subset of the complete lattice $\mathbb{A}_{tx}$). Then c is clearly type-preserving.

3.6. Lemma. *If R is a congruence on a complete skeletal $\mathcal{Q}$-category $\mathbb{A}$, then $c:\mathbb{A}\longrightarrow\mathbb{A}$ is a closure operator. Furthermore, if R is complete then $c:\mathbb{A}\longrightarrow\mathbb{A}$ is also a right adjoint.*

Proof. It is easy to check that c has the following properties:

(a) $c:\mathbb{A}_X\longrightarrow\mathbb{A}_X$ preserves order for each object X in $\mathcal{Q}$.

(b) For each x in $\mathbb{A}$, $x\leq c(x) = c^2(x)$.

(c) For any object x in $\mathbb{A}$ and any $f: tx\longrightarrow Y$ in $\mathcal{Q}$, $f\otimes c(x)\leq c(f\otimes x)$.

Properties (a) and (c) ensure that $c:\mathbb{A}\longrightarrow\mathbb{A}$ is a $\mathcal{Q}$-functor by virtue of Proposition 3.1, hence a closure operator by (b).

It remains to show that $c:\mathbb{A}\longrightarrow\mathbb{A}$ is a right adjoint if R is a complete congruence. We apply Proposition 3.2 to accomplish this.

Since $c:\mathbb{A}\longrightarrow\mathbb{A}$ is a $\mathcal{Q}$-functor, one has that $g\rightarrowtail c(x)\geq c(g\rightarrowtail x)$ for all x and $g: Y\longrightarrow tx$ by Proposition 3.1. Meanwhile, condition (v) ensures that $g\rightarrowtail c(x)\leq c(g\rightarrowtail x)$. Therefore, $g\rightarrowtail c(x) = c(g\rightarrowtail x)$. This proves that c preserves cotensors.

Let $\{x_i\}$ be a family of elements in $\mathbb{A}_X$. On one hand, since $(x_i, c(x_i))\in R$ for any x_i and R is closed w.r.t. meets, it follows that $(\bigwedge x_i, \bigwedge c(x_i))\in R$. Thus, $c(\bigwedge x_i)\geq\bigwedge c(x_i)$. On the other hand, since $c:\mathbb{A}_X\longrightarrow\mathbb{A}_X$ preserves order, it is clear that $c(\bigwedge x_i)\leq\bigwedge c(x_i)$. Therefore, $c:\mathbb{A}_X\longrightarrow\mathbb{A}_X$ is meet-preserving, hence a right adjoint since A_X is a complete lattice. ■

3.7. Lemma. *Let $\mathbb{A}$ be a skeletal complete $\mathcal{Q}$-category, $c:\mathbb{A}\longrightarrow\mathbb{A}$ a closure operator. Then $c(\mathbb{A})$, as a subcategory of $\mathbb{A}$, is complete.*

Proof. Let i be the embedding $c(\mathbb{A})\hookrightarrow\mathbb{A}$. It is easy to check that $\mathcal{P}i\circ\mathsf{Y}_{\mathbb{A}}\circ i = \mathsf{Y}_{c(\mathbb{A})}$. Since $c\dashv i$ (Lemma 3.4), $\sup_{\mathbb{A}}\dashv\mathsf{Y}_{\mathbb{A}}$ ($\mathbb{A}$ is cocomplete) and $\mathcal{P}c\dashv\mathcal{P}i$ (Proposition 2.5), then

$$c\circ\sup_{\mathbb{A}}\circ\mathcal{P}c\dashv\mathcal{P}i\circ\mathsf{Y}_{\mathbb{A}}\circ i = \mathsf{Y}_{c(\mathbb{A})},$$

showing that the Yoneda embedding $\mathsf{Y}_{c(\mathbb{A})}$ has a left adjoint, hence $c(\mathbb{A})$ is cocomplete, hence complete. ■

3.8. Theorem. *The forgetful functor* $|\text{-}| : \mathcal{Q}\text{-}\mathbf{Sup} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ *is strictly monadic.*

Proof. Since both of the forgetful functors $\mathcal{Q}\text{-}\mathbf{Sup} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ and $\mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ are right adjoints, it follows that the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{Sup} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$, being a composite of right adjoints, is itself a right adjoint. Thus, by virtue of Beck's theorem (Theorem 1 on page 151 in [Mac1998]), it suffices to show that $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers.

Given a pair of cocontinuous $\mathcal{Q}$-functors $F, G : \mathbb{A} \longrightarrow \mathbb{B}$ between complete skeletal $\mathcal{Q}$-categories, a split coequalizer of $F, G : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ in $\mathbf{Set} \downarrow \mathcal{Q}_0$ is, by definition, a type-preserving map $H : |\mathbb{B}| \longrightarrow C$ with type-preserving maps $C \xrightarrow{K} |\mathbb{B}| \xrightarrow{L} |\mathbb{A}|$ such that

$$H \circ F = H \circ G,\ F \circ L = 1,\ H \circ K = 1,\ G \circ L = K \circ H.$$

Define a relation R on $\mathbb{B}_0$ by

$$R = \{(y_1, y_2) \in \mathbb{B}_0 \times \mathbb{B}_0 \mid H(y_1) = H(y_2)\}.$$

Claim 1: For any $y_1, y_2 \in \mathbb{B}_0$, $(y_1, y_2) \in R$ if and only if there is a pair $(x_1, x_2) \in \mathbb{A}_0 \times \mathbb{A}_0$ such that $G(x_1) = G(x_2)$ and $y_1 = F(x_1)$, $y_2 = F(x_2)$.

Sufficiency is easy. For necessity, let $x_1 = L(y_1)$ and $x_2 = L(y_2)$. Then

$$G(x_1) = G \circ L(y_1) = K \circ H(y_1) = K \circ H(y_2) = G \circ L(y_2) = G(x_2),$$

and

$$F(x_1) = F \circ L(y_1) = y_1, \quad F(x_2) = F \circ L(y_2) = y_2.$$

Claim 2: The relation R is a congruence on $\mathbb{B}$.

This follows from Claim 1 and the fact that both F and G preserve tensors and joins (with respect to the underlying orders).

Thus, R determines a closure operator $c : \mathbb{B} \longrightarrow \mathbb{B}$ by Lemma 3.6. It follows from Lemma 3.7 that $c(\mathbb{B})$ is complete. Since the underlying $\mathcal{Q}$-typed set of $c(\mathbb{B})$ is essentially the $\mathcal{Q}$-typed set C, hence C can be made into a complete $\mathcal{Q}$-category $\mathbb{C}$ (which is isomorphic to $c(\mathbb{B})$) such that $H : \mathbb{B} \longrightarrow \mathbb{C}$ is a cocontinuous $\mathcal{Q}$-functor. This proves that the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers. ■

Since $(\mathcal{Q}\text{-}\mathbf{Inf})^{\mathrm{co}}$ is isomorphic to $\mathcal{Q}^{\mathrm{op}}\text{-}\mathbf{Sup}$ as 2-categories, applying the above theorem to $\mathcal{Q}^{\mathrm{op}}$ yields:

3.9. Theorem. *The forgetful functor* $\mathcal{Q}\text{-}\mathbf{Inf} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ *is strictly monadic.*

Our next task is to show that the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ is strictly monadic. We show that it is a right adjoint first. Given a continuous $\mathcal{Q}$-functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ between complete $\mathcal{Q}$-categories, it follows from Proposition 2.4 and 2.5 that $\mathcal{P}_{\exists} F : \mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$ is bicontinuous. Therefore, by restricting the domain and the codomain of the functor $\mathcal{P}_{\exists} : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Sup}$ one obtains a functor $\mathcal{P}_{\exists}^{\mathrm{inf}} : \mathcal{Q}\text{-}\mathbf{Inf} \longrightarrow \mathcal{Q}\text{-}\mathbf{CD}$ that is left adjoint to the forgetful functor $\mathcal{E}^{\mathrm{inf}} : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathcal{Q}\text{-}\mathbf{Inf}$. Then the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$, as a composite of right adjoints, is a right adjoint.

3.10. Lemma. *Let $\mathbb{A}$ be a skeletal completely distributive $\mathcal{Q}$-category; $c : \mathbb{A} \longrightarrow \mathbb{A}$ be a right adjoint and a closure operator. Then $c(\mathbb{A})$ is completely distributive.*

Proof. This follows from Proposition 2.12(1) and the fact that $c : \mathbb{A} \longrightarrow c(\mathbb{A})$ is both a left and a right adjoint. ■

3.11. Theorem. *The forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ is strictly monadic.*

Proof. It suffices to check that the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers. We only include here a sketch of the proof since it is similar to that of Theorem 3.8.

Suppose $F, G : \mathbb{A} \longrightarrow \mathbb{B}$ are bicontinuous $\mathcal{Q}$-functors between completely distributive skeletal $\mathcal{Q}$-categories and $H : |\mathbb{B}| \longrightarrow C$ is a split coequalizer of $F, G : |\mathbb{A}| \longrightarrow |\mathbb{B}|$ in $\mathbf{Set} \downarrow \mathcal{Q}_0$. By definition there exist type-preserving maps $C \xrightarrow{K} |\mathbb{B}| \xrightarrow{L} |\mathbb{A}|$ such that

$$H \circ F = H \circ G,\ F \circ L = 1,\ H \circ K = 1,\ G \circ L = K \circ H.$$

Define a relation R on $\mathbb{B}_0$ by $R = \{(y_1, y_2) \in \mathbb{B}_0 \times \mathbb{B}_0 \mid H(y_1) = H(y_2)\}$. Then R is a complete congruence on $\mathbb{B}$. This follows easily from Claim 1 in Theorem 3.8 and the fact that both F and G preserve tensors, cotensors, joins and meets (with respect to the underlying orders). By virtue of Lemma 3.6, the relation R determines a $\mathcal{Q}$-functor $c : \mathbb{B} \longrightarrow \mathbb{B}$ which is both a closure operator and a right adjoint. Then $c(\mathbb{B})$ is completely distributive by Lemma 3.10. Since the underlying $\mathcal{Q}$-typed set of $c(\mathbb{B})$ is isomorphic to C, it follows that C can be made into a completely distributive $\mathcal{Q}$-category $\mathbb{C}$ (isomorphic to $c(\mathbb{B})$) such that $H : \mathbb{B} \longrightarrow \mathbb{C}$ is a bicontinuous $\mathcal{Q}$-functor. This proves that the forgetful functor $|\text{-}| : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ creates split coequalizers. ■

In the remainder of this section, we show that the forgetful functor $\mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathcal{Q}\text{-}\mathbf{Inf}$ is monadic. But, we do not know whether so is the forgetful functor $\mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathcal{Q}\text{-}\mathbf{Sup}$.

Consider the adjunction $\mathcal{P}_\exists \dashv \mathcal{E} : \mathcal{Q}\text{-}\mathbf{Cat} \rightharpoonup \mathcal{Q}\text{-}\mathbf{Sup}$. The corresponding monad is given by

$$\mathbf{P}_\exists = \{\mathcal{P}_\exists : \mathcal{Q}\text{-}\mathbf{Cat} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}, \mathsf{Y} : 1 \Rightarrow \mathcal{P}_\exists, \sup : \mathcal{P}_\exists^2 \Rightarrow \mathcal{P}_\exists\}.$$

The monad $\mathbf{P}_\exists$ is an example of monads that are of Kock-Zöberlein type. The following proposition, extracted from [Kock1995, Zob1976], is taken from [Hof2013].

3.12. Proposition. *Let $\mathbf{T} = (T, e, m)$ be a monad on a locally ordered category $\mathbf{C}$ with T a 2-functor. Then the following are equivalent:*

(1) $Te_X \le e_{TX}$ *for all objects X.*

(2) $Te_X \dashv m_X$ *for all objects X.*

(3) $m_X \dashv e_{TX}$ *for all objects X.*

(4) *For any object X and morphism $h : TX \longrightarrow X$, the pair (X, h) is a $\mathbf{T}$-algebra if and only if $h \circ e_X = 1_X$. In this case, $h \dashv e_X$.*

A monad on a locally-ordered category is said to be of Kock-Zöberlein type, if it satisfies one (hence all) of the equivalent conditions in Proposition 3.12. This kind of monads are examples of lax-idempotent 2-monads on 2-categories introduced by G.M. Kelly and S. Lack [KL1997], so, we'll call them lax-idempotent in this paper.

A 2-functor $T : \mathbf{C} \longrightarrow \mathbf{D}$ between locally-ordered categories is lax-idempotent monadic if it is monadic and the corresponding monad is lax-idempotent.

3.13. Proposition. *[Stu2013] The monad* $\mathbf{P}_\exists = (\mathcal{P}_\exists, \mathsf{Y}, \sup)$ *is lax-idempotent.*

Proof. The conclusion was proved in [Stu2013]. Here we repeat the proof for later use. For any $\mathcal{Q}$-category $\mathbb{A}$, since $\sup_{\mathcal{P}\mathbb{A}} = \mathcal{P}\mathsf{Y}_\mathbb{A}$ (Example 2.9) and $\mathcal{P}_\exists\mathsf{Y}_\mathbb{A} \dashv \mathcal{P}\mathsf{Y}_\mathbb{A}$ (Proposition 2.4), it follows that $\mathcal{P}_\exists\mathsf{Y}_\mathbb{A} \dashv \sup_{\mathcal{P}\mathbb{A}}$. Hence $\mathcal{P}_\exists\mathsf{Y}_\mathbb{A} = \mathcal{P}_\exists\mathsf{Y}_\mathbb{A} \circ \sup_{\mathcal{P}\mathbb{A}} \circ \mathsf{Y}_{\mathcal{P}\mathbb{A}} \leq \mathsf{Y}_{\mathcal{P}\mathbb{A}}$, completing the proof. ∎

3.14. Corollary. *[Stu2013] For a* $\mathcal{Q}$*-category* $\mathbb{A}$*, the following are equivalent:*

(1) $\mathbb{A}$ *is complete.*

(2) *The Yoneda embedding* $\mathsf{Y}_\mathbb{A} : \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ *has a left inverse* $\mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$*.*

3.15. Corollary. *Given a* $\mathcal{Q}$*-category* $\mathbb{A}$ *and a* $\mathcal{Q}$*-functor* $F : \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$*,* $(\mathbb{A}, F)$ *is a* $\mathbf{P}_\exists$*-algebra if and only if* $\mathbb{A}$ *is a skeletal complete* $\mathcal{Q}$*-category and* $F = \sup_\mathbb{A}$*.*

It follows from Corollary 3.15 that the category of $\mathbf{P}_\exists$-algebras is equivalent to the category of skeletal complete $\mathcal{Q}$-categories and cocontinuous $\mathcal{Q}$-functors.

3.16. Theorem. *[Stu2013] The forgetful functor* $\mathcal{Q}$-$\mathbf{Sup} \longrightarrow \mathcal{Q}$-$\mathbf{Cat}$ *is lax-idempotent monadic.*

A 2-functor $T : \mathbf{C} \longrightarrow \mathbf{D}$ between locally-ordered categories is colax-idempotent monadic if $T^{\rm co} : \mathbf{C}^{\rm co} \longrightarrow \mathbf{D}^{\rm co}$ is lax-idempotent monadic. Since the 2-category $(\mathcal{Q}\text{-}\mathbf{Cat})^{\rm co}$ is isomorphic to $\mathcal{Q}^{\rm op}$-$\mathbf{Cat}$, and $(\mathcal{Q}\text{-}\mathbf{Inf})^{\rm co}$ to $\mathcal{Q}^{\rm op}$-$\mathbf{Sup}$, applying the above theorem to $\mathcal{Q}^{\rm op}$ we obtain:

3.17. Corollary. *The forgetful functor* $\mathcal{Q}$-$\mathbf{Inf} \longrightarrow \mathcal{Q}$-$\mathbf{Cat}$ *is colax-idempotent monadic.*

Now we come to the last conclusion in this section.

3.18. Proposition. *The forgetful functor* $\mathcal{Q}$-$\mathbf{CD} \longrightarrow \mathcal{Q}$-$\mathbf{Inf}$ *is lax-idempotent monadic.*

Proof. Consider the monad $\mathbf{P}_\exists^{\rm inf}$ generated by the adjunction $\mathcal{P}_\exists^{\rm inf} \dashv \mathcal{E}^{\rm inf}$ (see the paragraph following Theorem 3.9). By the same argument for $\mathbf{P}_\exists$ one deduces that the monad $\mathbf{P}_\exists^{\rm inf}$ is lax-idempotent. So, it remains to check that the forgetful functor $\mathcal{E}^{\rm inf} : \mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathcal{Q}\text{-}\mathbf{Inf}$ is monadic.

Let $\mathbb{A}$ be a complete skeletal $\mathcal{Q}$-category and $F : \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$ a continuous $\mathcal{Q}$-functor. If $(\mathbb{A}, F)$ is a $\mathbf{P}_\exists^{\rm inf}$-algebra, then F is a left inverse of the Yoneda embedding $\mathsf{Y}_\mathbb{A}$ by Proposition 3.12(4), hence $\mathbb{A}$ is complete and $F = \sup_\mathbb{A}$ by corollaries 3.14 and 3.15. Thus, $\sup_\mathbb{A}$ is a right adjoint, showing that $\mathbb{A}$ is completely distributive. Therefore, the correspondence $(\mathbb{A}, F) \mapsto \mathbb{A}$ defines a functor $\mathcal{Q}\text{-}\mathbf{Inf}^{\mathbf{P}_\exists^{\rm inf}} \longrightarrow \mathcal{Q}\text{-}\mathbf{CD}$ that is inverse to the comparison functor

$$\mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathcal{Q}\text{-}\mathbf{Inf}^{\mathbf{P}_{\exists}^{\inf}}, \quad \mathbb{A} \mapsto (\mathbb{A}, \sup_{\mathbb{A}}).$$

The conclusion thus follows. ∎

The monadicity of the forgetful functor $\mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ does not follow from that of the forgetful functors $\mathcal{Q}\text{-}\mathbf{Inf} \longrightarrow \mathcal{Q}\text{-}\mathbf{Cat}$ and $\mathcal{Q}\text{-}\mathbf{CD} \longrightarrow \mathcal{Q}\text{-}\mathbf{Inf}$, since the composite of monadic functors need not be monadic, see [Bor1994], page 214.

4. $\mathcal{Q}$-powersets as algebras

It is well-known (e.g. [Joh1982, MM1992]) that the contravariant powerset functor $\mathcal{P} : \mathbf{Set}^{\rm op} \longrightarrow \mathbf{Set}$ is monadic with a left adjoint given by $\mathcal{P}^{\rm op} : \mathbf{Set} \longrightarrow \mathbf{Set}^{\rm op}$; and that the algebras corresponding to the monad generated by the adjunction $\mathcal{P}^{\rm op} \dashv \mathcal{P}$ are powersets (or equivalently, complete atomic Boolean algebras). In this section, we establish a $\mathcal{Q}$-version of this conclusion. That is, if we denote by $|\mathcal{P}^{\dagger}| \dashv |\mathcal{P}|$ the adjunction obtained by composing the following

$$\mathbf{Set} \downarrow \mathcal{Q}_0 \underset{\mathcal{I}}{\overset{|\text{-}|}{\leftrightarrows}} \mathcal{Q}\text{-}\mathbf{Cat} \underset{\mathcal{P}^{\dagger}}{\overset{\mathcal{P}}{\leftrightarrows}} \mathcal{Q}\text{-}\mathbf{Cat}^{\rm op} \underset{|\text{-}|^{\rm op}}{\overset{\mathcal{I}^{\rm op}}{\leftrightarrows}} (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\rm op},$$

then the functor $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\rm op} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ is monadic and the corresponding Eilenberg-Moore algebras are exactly the $\mathcal{Q}$-powersets of $\mathcal{Q}$-typed sets.

The monadicity of the powerset functor $\mathcal{P} : \mathbf{Set}^{\rm op} \longrightarrow \mathbf{Set}$ is a special case of a general result in topos theory [MM1992] that states that for each topos $\mathbf{E}$, the opposite category $\mathbf{E}^{\rm op}$ is monadic over $\mathbf{E}$. In particular, for each set X, $(\mathbf{Set} \downarrow X)^{\rm op}$ is monadic over $\mathbf{Set} \downarrow X$ with the (internal) powerset functor being a monadic one. We'd like to remark that the conclusion presented here shows that, for each non-empty set X, there exist many monadic functors from $(\mathbf{Set} \downarrow X)^{\rm op}$ to $\mathbf{Set} \downarrow X$.

Before proceeding, we spell out some facts of the adjunction $|\mathcal{P}^{\dagger}| \dashv |\mathcal{P}|$.

First, the functor $|\mathcal{P}|$ sends each $\mathcal{Q}$-typed set A to the underlying $\mathcal{Q}$-typed set $|\mathcal{P}A|$ of $\mathcal{P}A$, where A is regarded as a discrete $\mathcal{Q}$-category. The $\mathcal{Q}$-typed set $|\mathcal{P}A|$ is called the $\mathcal{Q}$-powerset of A.

Second, for each type-preserving map $F : A \longrightarrow B$ between $\mathcal{Q}$-typed sets, $|\mathcal{P}|F$ is the underlying type-preserving map of $\mathcal{P}F : \mathcal{P}B \longrightarrow \mathcal{P}A$. Similarly, $|\mathcal{P}^{\dagger}|F$ is the underlying type-preserving map of $\mathcal{P}^{\dagger}F : \mathcal{P}^{\dagger}B \longrightarrow \mathcal{P}^{\dagger}A$. So, for a type-preserving map F between $\mathcal{Q}$-typed sets, we simply write $\mathcal{P}F$ ($\mathcal{P}^{\dagger}F$, resp.) for $|\mathcal{P}|F$ ($|\mathcal{P}^{\dagger}|F$, resp.) if no confusion would arise.

Third, the unit and counit of the adjunction $|\mathcal{P}^{\dagger}| \dashv |\mathcal{P}|$ are respectively given by

$$\epsilon_A = \mathsf{Y}_{|\mathcal{P}^{\dagger}A|} \circ \mathsf{Y}^{\dagger}_A : A \longrightarrow |\mathcal{P}^{\dagger}A| \longrightarrow |\mathcal{P}|\mathcal{P}^{\dagger}A||$$

and

$$\gamma_A = \mathsf{Y}^{\dagger}_{|\mathcal{P}A|} \circ \mathsf{Y}_A : A \longrightarrow |\mathcal{P}A| \longrightarrow |\mathcal{P}^{\dagger}|\mathcal{P}A||$$

for any $\mathcal{Q}$-typed set A.

The following lemma is a counterpart of the Beck-Chevalley condition in [MM1992], Theorem 2, page 206.

4.1. Lemma. *Let*

$$\begin{array}{ccc} A & \xrightarrow{F} & C \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle K} \\ B & \xrightarrow[G]{} & D \end{array}$$

be a pullback square in $\mathbf{Set}\downarrow\mathcal{Q}_0$. *Then the square of $\mathcal{Q}$-distributors between discrete $\mathcal{Q}$-categories*

$$\begin{array}{ccc} A & \xrightarrow{F_\natural} & C \\ {\scriptstyle H^\natural}\uparrow & & \uparrow{\scriptstyle K^\natural} \\ B & \xrightarrow[G_\natural]{} & D \end{array}$$

commutes; or equivalently, the square of $\mathcal{Q}$-functors

$$\begin{array}{ccc} \mathcal{P}C & \xrightarrow{\mathcal{P}F} & \mathcal{P}A \\ {\scriptstyle \exists_K}\downarrow & & \downarrow{\scriptstyle \exists_H} \\ \mathcal{P}D & \xrightarrow[\mathcal{P}G]{} & \mathcal{P}B \end{array}$$

commutes.

Proof. By hypothesis, we can assume that the underlying set of A is

$$\{(y,z)\in B\times C \mid Gy = Kz\},$$

the type function is given by $t[(y,z)] = ty = tz$ for all $(y,z)\in A$, and that both H and F are projections. For all $b\in B$ and $c\in C$,

$$(F_\natural\circ H^\natural)(b,c) = \bigvee_{(y,z)\in A}\mathrm{id}_C(z,c)\circ\mathrm{id}_B(b,y) = \begin{cases} 1_{tb}, & Gb = Kc; \\ \bot_{tb,tc}, & \text{otherwise.}\end{cases}$$

It follows that

$$K^\natural\circ G_\natural(b,c) = \bigvee_{d\in D}\mathrm{id}_D(d,Kc)\circ\mathrm{id}_D(Gb,d) = \mathrm{id}_D(Gb,Kc) = F_\natural\circ H^\natural(b,c).$$

That is, the second square commutes.

It remains to check that the second square commutes if and only if so does the third one. Since $\exists_H \circ \mathcal{P}F(\phi) = \phi \circ F_\natural \circ H^\natural$ and $\mathcal{P}G \circ \exists_K(\phi) = \phi \circ K^\natural \circ G_\natural$ for all $\phi \in \mathcal{P}C$, the commutativity of the third square follows trivially from that of the second one. Conversely, if the third square commutes, then for all $c \in C$,

$$F_\natural \circ H^\natural(-,c) = \mathrm{id}_C(-,c) \circ F_\natural \circ H^\natural = \exists_H \circ \mathcal{P}F(\mathrm{id}_C(-,c))$$

and

$$K^\natural \circ G_\natural(-,c) = \mathrm{id}_C(-,c) \circ K^\natural \circ G_\natural = \mathcal{P}G \circ \exists_K(\mathrm{id}_C(-,c)),$$

hence, the second one commutes. ∎

4.2. Theorem. *The functor* $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ *is monadic.*

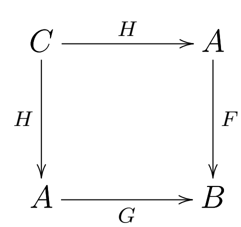
Proof. Since $\mathbf{Set} \downarrow \mathcal{Q}_0$ is a complete category, we apply Corollary 3 on page 180 in [MM1992] to prove the conclusion. That is, we show that $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ reflects isomorphisms and preserves coequalizers of reflexive pairs.

Since $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ is faithful, it reflects both monomorphisms and epimorphisms. Since the slice category $\mathbf{Set} \downarrow \mathcal{Q}_0$ is a topos, an arrow in $\mathbf{Set} \downarrow \mathcal{Q}_0$ is an isomorphism if and only if it is both a monomorphism and an epimorphism. Consequently, $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ reflects isomorphisms.

It remains to check that $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ preserves coequalizers of reflexive pairs. Recall that a pair of arrows $r, s : X \longrightarrow Y$ in a category is reflexive if there exists an arrow $i : Y \longrightarrow X$ such that $r \circ i = 1_Y = s \circ i$. So, a reflexive pair in $(\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}}$ is a pair of arrows $F, G : A \longrightarrow B$ in $\mathbf{Set} \downarrow \mathcal{Q}_0$ together with an arrow $K : B \longrightarrow A$ such that $K \circ F = K \circ G = 1_A$. We must show that if $H : C \longrightarrow A$ is an equalizer of F and G in $\mathbf{Set} \downarrow \mathcal{Q}_0$ then $|\mathcal{P}|H = \mathcal{P}H : |\mathcal{P}C| \longrightarrow |\mathcal{P}A|$ is a coequalizer of $\mathcal{P}F$ and $\mathcal{P}G$. That is, for each $L : |\mathcal{P}A| \longrightarrow D$ in $\mathbf{Set} \downarrow \mathcal{Q}_0$ with $L \circ \mathcal{P}F = L \circ \mathcal{P}G$, there exists a unique $\bar{L} : |\mathcal{P}C| \longrightarrow D$ such that $\bar{L} \circ \mathcal{P}H = L$.

Uniqueness. It is obvious that, as an equalizer, $H : C \longrightarrow A$ is a monomorphism in $\mathbf{Set} \downarrow \mathcal{Q}_0$. Hence H is a fully faithful $\mathcal{Q}$-functor between discrete $\mathcal{Q}$-categories C and A. Thus, $\bar{L} = \bar{L} \circ \mathcal{P}H \circ \exists_H = L \circ \exists_H$ by Proposition 2.7(1).

Existence. It suffices to verify that $L \circ \exists_H \circ \mathcal{P}H = L$. First, we check that the square



is a pullback in $\mathbf{Set} \downarrow \mathcal{Q}_0$. Given a pair of arrows $F', G' : D \longrightarrow A$ with $F \circ F' = G \circ G'$, since $K \circ F = K \circ G = 1_A$, we have that

$$F' = K \circ F \circ F' = K \circ G \circ G' = G'.$$

Because $H : C \longrightarrow A$ is an equalizer of F and G, there is a unique $U : D \longrightarrow A$ such that $F' = H \circ U = G'$. This proves that the square is a pullback. Then it follows from Lemma 4.1 that $\mathcal{P}G \circ \exists_F = \exists_H \circ \mathcal{P}H$. Finally, since $K \circ F = 1_A$, it follows that $F : A \longrightarrow B$ is a fully faithful $\mathcal{Q}$-functor if we treat A and B as discrete $\mathcal{Q}$-categories. Thus, $\mathcal{P}F \circ \exists_F = 1_{\mathcal{P}A}$ by Proposition 2.7(1). Therefore,

$$L = L \circ \mathcal{P}F \circ \exists_F = L \circ \mathcal{P}G \circ \exists_F = L \circ \exists_H \circ \mathcal{P}H.$$

The proof is thus completed. ∎

4.3. Remark. In general, the monadic functor $|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\rm op} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ is different from the "internal" powerset functor for the topos $\mathbf{Set} \downarrow \mathcal{Q}_0$. It is easily verified that $|\mathcal{P}|$ coincides with the internal powerset functor if $\mathcal{Q}$ is given by

$$\mathcal{Q}(X, Y) = \begin{cases} \mathbf{2} = \{0, 1\}, & \text{if } X = Y; \\ \mathbf{1} = \{0\}, & \text{otherwise.} \end{cases}$$

Furthermore, if there exist different objects X, Y in $\mathcal{Q}$ with $\mathcal{Q}(X, Y)$ containing at least two elements, then the functor $|\mathcal{P}|$ cannot be isomorphic to the internal powerset functor $[-, B] : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\rm op} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$ for any B in $\mathbf{Set} \downarrow \mathcal{Q}_0$. To see this, we first note that for each A in $\mathbf{Set} \downarrow \mathcal{Q}_0$ and Z in $\mathcal{Q}_0$, an element in $[A, B]$ with type Z is exactly a function $A_Z \longrightarrow B_Z$, where A_Z is the set of elements in A with type Z, and likewise for B_Z. Now, let C be a $\mathcal{Q}$-typed set consisting of only one element with type X. Then there is exactly one element in $[C, B]$ that is of type Y (namely, the unique map from the empty set to B_Y), but there are at least two elements in $|\mathcal{P}|C$ that are of type Y. Therefore, $[-, B]$ and $|\mathcal{P}|$ cannot be isomorphic.

In the following we describe the Eilenberg-Moore algebras of the monad generated by the adjunction $|\mathcal{P}^\dagger| \dashv |\mathcal{P}|$. The corresponding monad is given by

$$|\mathbf{B}| = \{|\mathcal{B}| : \mathbf{Set} \downarrow \mathcal{Q}_0 \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0, \epsilon : 1 \Rightarrow |\mathcal{B}|, \delta : |\mathcal{B}|^2 \Rightarrow |\mathcal{B}|\}$$

where

- $|\mathcal{B}|F = |\mathcal{P}|(|\mathcal{P}^\dagger|F) : |\mathcal{P}||\mathcal{P}^\dagger A|| \longrightarrow |\mathcal{P}||\mathcal{P}^\dagger B||$ for any type-preserving map $F : A \longrightarrow B$,
- $\epsilon_A = \mathsf{E}_{\mathcal{P}^\dagger A} \circ \mathsf{Y}^\dagger_A : A \longrightarrow |\mathcal{P}^\dagger A| \longrightarrow |\mathcal{P}||\mathcal{P}^\dagger A||$ for any $\mathcal{Q}$-typed set A,
- $\delta_A = \mathcal{P}\gamma_{|\mathcal{P}^\dagger A|} : |\mathcal{B}|^2 A \longrightarrow |\mathcal{B}|A$ for any $\mathcal{Q}$-typed set A.

For each $\mathcal{Q}$-typed set B, $(|\mathcal{P}B|, \mathcal{P}\gamma_B)$ is a $|\mathbf{B}|$-algebra. The following theorem says that all $|\mathbf{B}|$-algebras are of this form.

4.4. Theorem. *Every $|\mathbf{B}|$-algebra is of the form $(|\mathcal{P}B|, \mathcal{P}\gamma_B)$ for some $\mathcal{Q}$-typed set B.*

Proof. Suppose that (A, F) is a $|\mathbf{B}|$-algebra. That is, A is a $\mathcal{Q}$-typed set, $F : |\mathcal{B}|A \longrightarrow A$ is a type-preserving map such that $F \circ \epsilon_A = 1_A$ and $F \circ \delta_A = F \circ |\mathcal{B}|F$. We show that there is some $\mathcal{Q}$-typed set B such that (A, F) is isomorphic to $(|\mathcal{P}B|, \mathcal{P}\gamma_B)$.

Consider the pullback

$$\begin{array}{ccc} B & \xrightarrow{\ i\ } & |\mathcal{P}^\dagger A| \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \mathcal{P}^\dagger F} \\ |\mathcal{P}^\dagger A| & \xrightarrow[\gamma_{|\mathcal{P}^\dagger A|}]{} & |\mathcal{P}^\dagger|\mathcal{B}A|| \end{array}$$

in $\mathbf{Set} \downarrow \mathcal{Q}_0$. We claim that B satisfies the requirement. The proof is divided into three steps.

Step 1. $i = i'$. This follows easily from the triangular identity $\mathcal{P}^\dagger\epsilon_A \circ \gamma_{|\mathcal{P}^\dagger A|} = 1_{|\mathcal{P}^\dagger A|}$ and the equality $\mathcal{P}^\dagger\epsilon_A \circ \mathcal{P}^\dagger F = \mathcal{P}^\dagger(F \circ \epsilon_A) = 1_{|\mathcal{P}^\dagger A|}$. Consequently, i is an equalizer of $\mathcal{P}^\dagger F$ and $\gamma_{|\mathcal{P}^\dagger A|}$.

Step 2. $K_A = \mathcal{P}i \circ \epsilon_A : (A, F) \longrightarrow (|\mathcal{P}B|, \mathcal{P}\gamma_B)$ is a homomorphism between $|\mathbf{B}|$-algebras, i.e., $K_A \circ F = \mathcal{P}\gamma_B \circ |\mathcal{B}|K_A$. To see this, we calculate:

$$\begin{aligned} K_A \circ F &= \mathcal{P}i \circ \epsilon_A \circ F \\ &= \mathcal{P}i \circ |\mathcal{B}|F \circ \epsilon_{|\mathcal{B}|A} && \text{(naturailty of } \epsilon) \\ &= \mathcal{P}(\mathcal{P}^\dagger F \circ i) \circ \epsilon_{|\mathcal{B}|A} && (\mathcal{P}^\dagger F : |\mathcal{P}^\dagger A| \longrightarrow |\mathcal{P}^\dagger(|\mathcal{B}|A)|) \\ &= \mathcal{P}(\gamma_{|\mathcal{P}^\dagger A|} \circ i) \circ \epsilon_{|\mathcal{B}|A} && (i \text{ equalizes } \mathcal{P}^\dagger F \text{ and } \gamma_{|\mathcal{P}^\dagger A|}) \\ &= \mathcal{P}i \circ \mathcal{P}\gamma_{|\mathcal{P}^\dagger A|} \circ \epsilon_{|\mathcal{B}|A} \\ &= \mathcal{P}i \circ \delta_A \circ \epsilon_{|\mathcal{B}|A} \\ &= \mathcal{P}i && (\delta_A \circ \epsilon_{|\mathcal{B}|A} = 1_{|\mathcal{B}|A}) \\ &= \mathcal{P}i \circ |\mathcal{B}|F \circ |\mathcal{B}|\epsilon_A && (F \circ \epsilon_A = 1_A) \\ &= \mathcal{P}(\gamma_{|\mathcal{P}^\dagger A|} \circ i) \circ |\mathcal{B}|\epsilon_A && (i \text{ equalizes } \mathcal{P}^\dagger F \text{ and } \gamma_{|\mathcal{P}^\dagger A|}) \\ &= \mathcal{P}(|\mathcal{P}^\dagger|(\mathcal{P}i) \circ \gamma_B) \circ |\mathcal{B}|\epsilon_A && \text{(naturailty of } \gamma) \\ &= \mathcal{P}\gamma_B \circ |\mathcal{B}|(\mathcal{P}i) \circ |\mathcal{B}|\epsilon_A \\ &= \mathcal{P}\gamma_B \circ |\mathcal{B}|K_A. \end{aligned}$$

Step 3. $K_A : (A, F) \longrightarrow (|\mathcal{P}B|, \mathcal{P}\gamma_B)$ is an isomorphism between $|\mathbf{B}|$-algebras. It suffices to check that $K_A : A \longrightarrow |\mathcal{P}B|$ is an isomorphism in $\mathbf{Set} \downarrow \mathcal{Q}_0$.

Let $L_A = F \circ \exists_i$. On the one hand, it follows from the calculations in Step 2 that

$$K_A \circ L_A = \mathcal{P}i \circ \epsilon_A \circ F \circ \exists_i = \mathcal{P}i \circ \exists_i = 1_{|\mathcal{P}B|},$$

where the last equality holds due to Proposition 2.7(1).

On the other hand, by virtue of Lemma 4.1 and the definition of δ_A one has that

$$\exists_i \circ \mathcal{P}i = \mathcal{P}\gamma_{|\mathcal{P}^\dagger A|} \circ \exists_{\mathcal{P}^\dagger F} = \delta_A \circ \exists_{\mathcal{P}^\dagger F}.$$

Since $\mathcal{P}^\dagger F : |\mathcal{P}^\dagger A| \longrightarrow |\mathcal{P}^\dagger(|\mathcal{B}|A)|$ is fully faithful ($\mathcal{P}^\dagger\epsilon_A \circ \mathcal{P}^\dagger F = 1_{|\mathcal{P}^\dagger A|}$), it holds that

$$|\mathcal{B}|F \circ \exists_{\mathcal{P}^\dagger F} = \mathcal{P}(\mathcal{P}^\dagger F) \circ \exists_{\mathcal{P}^\dagger F} = 1_{|\mathcal{B}|A}$$

by Proposition 2.7(1). Consequently,

$$\begin{aligned} L_A \circ K_A &= F \circ \exists_i \circ \mathcal{P}i \circ \epsilon_A \\ &= F \circ \delta_A \circ \exists_{\mathcal{P}^\dagger F} \circ \epsilon_A \\ &= F \circ |\mathcal{B}|F \circ \exists_{\mathcal{P}^\dagger F} \circ \epsilon_A && ((A,F) \text{ is a } |\mathbf{B}|\text{-algebra}) \\ &= F \circ \epsilon_A && (|\mathcal{B}|F \circ \exists_{\mathcal{P}^\dagger F} = 1_{|\mathcal{B}|A}) \\ &= 1_A. \end{aligned}$$

Therefore, $K_A : A \longrightarrow |\mathcal{P}B|$ is an isomorphism between $\mathcal{Q}$-typed sets. ∎

4.5. Remark. From the point of view of fuzzy sets [Zad1965], a $\mathcal{Q}$-typed set is nothing but a "fuzzy set valued in $\mathcal{Q}_0$". Viewed in this perspective, the functor

$$|\mathcal{P}| : (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}} \longrightarrow \mathbf{Set} \downarrow \mathcal{Q}_0$$

is a fuzzy counterpart of the contravariant powerset functor $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Set}$; the above theorem can be thought of as a fuzzy version of the Stone duality between sets and complete atomic Boolean algebras. Thus, it is not surprising that the functor $|\mathcal{P}|$ has applications in the theory of fuzzy sets. Interested readers are referred to [Hoh2014, SZ2013b, Stu2014] for more discussions on related topics.

Finally, consider the adjunction $|\text{-}|^{\mathrm{op}} \circ \mathcal{P}^\dagger \dashv \mathcal{P} \circ \mathcal{I}^{\mathrm{op}}$ obtained by composing the following adjunctions

$$\mathcal{Q}\text{-}\mathbf{Cat} \underset{\mathcal{P}^\dagger}{\overset{\mathcal{P}}{\rightleftarrows}} \mathcal{Q}\text{-}\mathbf{Cat}^{\mathrm{op}} \underset{|\text{-}|^{\mathrm{op}}}{\overset{\mathcal{I}^{\mathrm{op}}}{\rightleftarrows}} (\mathbf{Set} \downarrow \mathcal{Q}_0)^{\mathrm{op}}.$$

Let τ be the counit of the adjunction $|\text{-}|^{\mathrm{op}} \circ \mathcal{P}^\dagger \dashv \mathcal{P} \circ \mathcal{I}^{\mathrm{op}}$ and let $\mathbf{B}$ be the monad on $\mathcal{Q}$-$\mathbf{Cat}$ corresponding to this adjunction. Then the following theorem says that the Eilenberg-Moore algebras of $\mathbf{B}$ are also the $\mathcal{Q}$-powersets of $\mathcal{Q}$-typed sets.

4.6. Theorem. *If $(\mathbb{A}, F)$ is a $\mathbf{B}$-algebra, then there exists a $\mathcal{Q}$-typed set B such that $(\mathbb{A}, F)$ is isomorphic to $(\mathcal{P}B, \mathcal{P}\tau_B)$.*

Proof. Consider the pullback

$$\begin{array}{ccc} B & \xrightarrow{\quad i \quad} & |\mathcal{P}^\dagger\mathbb{A}| \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle \mathcal{P}^\dagger F} \\ |\mathcal{P}^\dagger\mathbb{A}| & \xrightarrow[\tau_{|\mathcal{P}^\dagger\mathbb{A}|}]{} & |\mathcal{P}^\dagger(\mathcal{B}\mathbb{A})| \end{array}$$

in $\mathbf{Set} \downarrow \mathcal{Q}_0$, where $\mathcal{B} = \mathcal{P} \circ \mathcal{I}^{\mathrm{op}} \circ |\text{-}|^{\mathrm{op}} \circ \mathcal{P}^\dagger$. Then B satisfies the requirement. The proof is similar to that of Theorem 4.4 and is thus omitted here. ∎

References

J. Bénabou. Introduction to bicategories. In: Reports of the Midwest Category Seminar, Volume 47 of Lecture Notes in Mathematics, pages 1-77. Springer, Berlin, 1967.

R. Betti, A. Carboni. Cauchy-completion and the associated sheaf. Cahiers de Topologie et Géométrie Différentielle Catégoriques 23(3): 243-256, 1982.

R. Betti, A. Carboni, R. H. Street, R. F. C. Walters. Variation through enrichment. Journal of Pure and Applied Algebra 29: 109-127, 1983.

F. Borceux. Handbook of Categorical Algebra: Volume 2, Categories and Structures, Volume 51 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1994.

D. Hofmann. Duality for distributive spaces. Theory and Applications of Categories 28(3): 66-122, 2013.

U. Höhle. Categorical foundations of topology with applications to quantaloid enriched topological spaces. Fuzzy Sets and Systems 256: 166-210, 2014.

P. T. Johnstone. Stone Spaces. Cambridge University Press, Cambridge, 1982.

G. M. Kelly, S. Lack. On property-like structures. Theory and Applications of Categories, 3(9): 213-250, 1997.

A. Kock. Monads for which structures are adjoint to units. Journal of Pure and Applied Algebra 104(1): 41-59, 1995.

S. Lack. A 2-categories companion. In: Towards Higher Categories, Volume 152 of The IMA Volumes in Mathematics and its Applications, pages 105-191. Springer, New York, 2010.

H. Lai, D. Zhang. Concept lattices of fuzzy contexts: Formal concept analysis vs. rough set theory. International Journal of Approximate Reasoning 50(5): 695-707, 2009.

S. Mac Lane. Categories for the Working Mathematician, 2nd Edition. Volume 5 of Graduate Texts in Mathematics. Springer, New York, 1998.

S. Mac Lane, I. Moerdijk. Sheaves in Geometry and Logic: A First Introduction to Topos Theory. Springer, New York, 1992.

G. N. Raney. Completely distributive complete lattices. Proceedings of the American Mathematical Society 3(5): 677-680, 1952.

K. I. Rosenthal. The Theory of Quantaloids. Volume 348 of Pitman Research Notes in Mathematics Series. Longman, Harlow, 1996.

L. Shen. Adjunctions in Quantaloid-enriched Categories. PhD thesis, Sichuan University, Chengdu, 2014. arXiv:1408.0321.

L. Shen, D. Zhang. Categories enriched over a quantaloid: Isbell adjunctions and Kan adjunctions. Theory and Applications of Categories 28(20): 577-615, 2013.

L. Shen, D. Zhang. Formal concept analysis on fuzzy sets. In: Proceedings of the IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), 2013 Joint, pages 215-219, IEEE Conference Publications, 2013.

R. H. Street. Cauchy characterization of enriched categories. Rendiconti del Seminario Matemático e Fisico di Milano 51: 217-233, 1981.

R. H. Street. Enriched categories and cohomology. Proceedings of the Symposium on Categorical Algebra and Topology (Cape Town, 1981). Quaestiones Mathematicae 6: 265-283, 1983.

I. Stubbe. Categorical structures enriched in a quantaloid: categories, distributors and functors. Theory and Applications of Categories 14(1): 1-45, 2005.

I. Stubbe. Categorical structures enriched in a quantaloid: tensored and cotensored categories. Theory and Applications of Categories 16(14): 283-306, 2006.

I. Stubbe. Towards "dynamic domains": totally continuous cocomplete $\mathcal{Q}$-categories. Theoretical Computer Science 373: 142-160, 2007.

I. Stubbe. The double power monad is the composite power monad. Preprint, 2013.

I. Stubbe. An introduction to quantaloid-enriched categories. Fuzzy Sets and Systems 256: 95-116, 2014.

R. F. C. Walters. Sheaves and Cauchy-complete categories. Cahiers de Topologie et Géométrie Différentielle Catégoriques 22: 283-286, 1981.

R. F. C. Walters. Sheaves on sites as Cauchy-complete categories. Journal of Pure and Applied Algebra 24: 95-102, 1982.

R. J. Wood. Ordered sets via adjunctions. In: Categorical Foundations, Volume 97 of Encyclopedia of Mathematics and its Applications, pages 5-47. Cambridge University Press, Cambridge, 2004.

L. A. Zadeh. Fuzzy sets. Information and Control 8: 338-353, 1965.

V. Zöberlein. Doctrines on 2-categories. Mathematische Zeitschrift 148(3): 267-279, 1976.

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