

# BUILDING A MODEL CATEGORY OUT OF COFIBRATIONS AND FIBRATIONS: THE TWO OUT OF THREE PROPERTY FOR WEAK EQUIVALENCES

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**ABSTRACT.** The purpose of this note is to understand the two out of three property of the model category in terms of the weak factorization systems. We will show that if a category with classes of trivial cofibrations, cofibrations, trivial fibrations, and fibrations is given a simplicial structure similar to that of the simplicial model category, then the full subcategory of cofibrant and fibrant objects has the two out of three property, and we will give a list of necessary and sufficient conditions in terms of the simplicial structure for the associated canonical "weak equivalence class" to have the two out of three property.

## 1. Introduction

In [Qui67], Quillen introduced the model category as a general setting in which one can do a homotopy theory. The following is the reformulation of Quillen's closed model category by Joyal and Tierney. It is equivalent to Quillen's definition. See, for example, [MP12].

Let  $\mathcal{M}$  be a category closed under finite limits and finite colimits. A closed model structure on  $\mathcal{M}$  consists of three classes  $\mathcal{W}$ ,  $\mathcal{C}$ , and  $\mathcal{F}$  of morphisms in  $\mathcal{M}$  such that the following two properties hold:

- $\mathcal{W}$  satisfies the two out of three property.
- $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are weak factorization systems.

We will not assume that weak factorization systems are functorial. This view on closed model structures also appears in [PT02] where the orthogonal factorization systems are used instead of the weak factorization systems. See Proposition 3.3 and Definition 3.9 in [PT02].

The purpose of this note is to understand the two out of three property in terms of the weak factorization systems. For this, we will define a Quillen structure on  $\mathcal{M}$  as a pair  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$  of weak factorization systems satisfying  $\mathcal{C}_t \subseteq \mathcal{C}$  and  $\mathcal{F}_t \subseteq \mathcal{F}$  instead of three classes  $(\mathcal{W}, \mathcal{C}, \mathcal{F})$  of morphisms. And we define the class  $\mathcal{W}_{\mathcal{M}}$  of morphisms in  $\mathcal{M}$

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as the class of the compositions of elements in  $\mathcal{C}_t$  followed by elements of  $\mathcal{F}_t$ . Then by Lemma 2.4,  $\mathcal{C} \cap \mathcal{W}_{\mathcal{M}} = \mathcal{C}_t$  and  $\mathcal{F} \cap \mathcal{W}_{\mathcal{M}} = \mathcal{F}_t$  hold. Thus, if  $\mathcal{W}_{\mathcal{M}}$  satisfies the two out of three property, then  $(\mathcal{W}_{\mathcal{M}}, \mathcal{C}, \mathcal{F})$  is indeed a closed model structure on  $\mathcal{M}$  with  $\mathcal{C}_t$  the class of trivial cofibrations and  $\mathcal{F}_t$  the class of trivial fibrations.

In general, it is easier to produce a Quillen structure than a full closed model structure. One can use the Quillen's small object argument to obtain weak factorization systems as is done in cofibrantly generated model categories. But to see that  $\mathcal{W}_{\mathcal{M}}$  has the two out of three property could be difficult. For example, on chapter II.3 in [Qui67], the author, when constructing the closed model structure on the category of simplicial sets, devotes most of it to the proof of the two out of three property. Therefore a good understanding of when  $\mathcal{W}_{\mathcal{M}}$  has the two out of three property could be valuable for practical applications.

In this note, we will show that if  $\mathcal{M}$  is given a simplicial structure similar to that of the simplicial model category  $\mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_{cf})$  has the two out of three property, and we will give a list of necessary and sufficient conditions in terms of the simplicial structure for  $\mathcal{W}_{\mathcal{M}}$  to have the two out of three property. If every object of  $\mathcal{M}$  is cofibrant, we will show that a simple inclusion is equivalent to  $\mathcal{W}_{\mathcal{M}}$  having the two out of three property.

1.1. DEFINITION. *Let  $\mathcal{M}$  be a category closed under finite colimits and finite limits. A Quillen structure on  $\mathcal{M}$  is a pair of weak factorization systems*

$$(\mathcal{C}, \mathcal{F}_t) \quad \text{and} \quad (\mathcal{C}_t, \mathcal{F})$$

*such that*

$$\mathcal{C}_t \subseteq \mathcal{C} \quad \text{and} \quad \mathcal{F}_t \subseteq \mathcal{F}$$

*hold. A category is called a Quillen category if it has a Quillen structure.*

Every model category is a Quillen category. If  $\mathcal{M}$  is a Quillen category, then its opposite is also a Quillen category with the obvious opposite Quillen structure.

1.2. REMARK. Since one of our aims is to show that Quillen categories are the model categories under some conditions, it is natural and inevitable to adopt the notations and the definitions used in the model category theory. Instead of defining them as they are needed, we will use the notations and the definitions in [Qui67] in the context of Quillen categories systematically. (See the remark at the end of this section for some exceptions). For example, if  $\mathcal{M}$  is a Quillen category, we call an element of  $\mathcal{C}_t$  a trivial cofibration and  $\mathcal{M}_c$  will denote the full subcategory of "cofibrant" objects of  $\mathcal{M}$ . We hope that it does not cause any confusion.

Once we have a Quillen category  $\mathcal{M}$ , it is natural to define the class  $\mathcal{W}_{\mathcal{M}}$  of weak equivalences in  $\mathcal{M}$  as follows.

1.3. DEFINITION. *Let  $\mathcal{M}$  be a Quillen category with its Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . We define*

$$\mathcal{W}_{\mathcal{M}} = \{p \cdot i \mid i \in \mathcal{C}_t, p \in \mathcal{F}_t\}.$$

*We call the elements of  $\mathcal{W}_{\mathcal{M}}$  the weak equivalences of  $\mathcal{M}$ .*

We want to know when  $\mathcal{W}_{\mathcal{M}}$  has the two out of three property.

1.4. DEFINITION. Let  $\mathcal{M}$  be a category. Let  $\mathcal{W}$  be a nonempty class of morphisms in  $\mathcal{M}$ . We say that  $\mathcal{W}$  satisfies two out of three property if the following three properties hold: For every  $g, h \in \text{Mor}(\mathcal{M})$  with  $\text{dom}(h) = \text{cod}(g)$ ,

(M)  $g \in \mathcal{W}$  and  $h \in \mathcal{W}$  imply  $hg \in \mathcal{W}$ .

(L)  $hg \in \mathcal{W}$  and  $h \in \mathcal{W}$  imply  $g \in \mathcal{W}$ .

(R)  $hg \in \mathcal{W}$  and  $g \in \mathcal{W}$  imply  $h \in \mathcal{W}$ .

Following the definition of the closed simplicial model category of Quillen, we will define the simplicial Quillen category below. First, we recall that the category  $\text{sSet}$  of simplicial sets is a closed symmetric monoidal category. So it has two functors  $- \otimes - : \text{sSet} \times \text{sSet} \rightarrow \text{sSet}$  and  $\underline{\text{sSet}}(-, -) : \text{sSet}^{\text{op}} \times \text{sSet} \rightarrow \text{sSet}$ , and for every  $x, y, z \in \text{sSet}$ , we have an isomorphism

$$\pi : \text{sSet}(x \otimes y, z) \rightarrow \text{sSet}(x, \underline{\text{sSet}}(y, z))$$

natural in  $x, y, z$ . For a simplicial category  $\mathcal{S}$ , there is a functor  $\underline{\mathcal{S}}(-, -) : \mathcal{S}^{\text{op}} \times \mathcal{S} \rightarrow \text{sSet}$  providing the hom-space of  $\mathcal{S}$  in  $\text{sSet}$  and, for every  $a, b, c \in \mathcal{S}$ , there is a morphism  $\bullet : \underline{\mathcal{S}}(b, c) \times \underline{\mathcal{S}}(a, b) \rightarrow \underline{\mathcal{S}}(a, c)$  in  $\text{sSet}$  providing the simplicial composition.  $\text{sSet}$  is an example of simplicial category with  $\underline{\text{sSet}}(-, -)$  the hom-space.

1.5. DEFINITION. Let  $\mathcal{M}$  be a category closed under finite colimits and finite limits. We say that  $\mathcal{M}$  has a simplicial Quillen structure if the following three conditions hold.

1.  $\mathcal{M}$  is a simplicial category (See Definition 1 on chapter II.1 in [Qui67]).

2.  $\mathcal{M}$  satisfies SM0 (cf. Definition 2 on chapter II.2 in [Qui67]):

(a) For every  $x \in \mathcal{M}$  and  $k \in \text{sSet}$ , there exist an object  $k \otimes x \in \mathcal{M}$ , a morphism  $\alpha_{k,x} : k \rightarrow \underline{\mathcal{M}}(x, k \otimes x)$  in  $\text{sSet}$ , and, for every  $y$ , an isomorphism

$$\phi_{k,x,y} : \underline{\mathcal{M}}(k \otimes x, y) \rightarrow \underline{\text{sSet}}(k, \underline{\mathcal{M}}(x, y))$$

such that  $\pi^{-1}(\phi_{k,x,y})$  is the map  $k \times \underline{\mathcal{M}}(k \otimes x, y) \xrightarrow{\alpha_{k,x} \times 1} \underline{\mathcal{M}}(x, k \otimes x) \times \underline{\mathcal{M}}(k \otimes x, y) \xrightarrow{\bullet} \underline{\mathcal{M}}(x, y)$ .

(b) For every  $x \in \mathcal{M}$  and  $k \in \text{sSet}$ , there exist an object  $x^k \in \mathcal{M}$ , a morphism  $\beta_{k,x} : k \rightarrow \underline{\mathcal{M}}(x^k, x)$  in  $\text{sSet}$ , and, for every  $y$ , an isomorphism

$$\psi_{k,y,x} : \underline{\mathcal{M}}(y, x^k) \rightarrow \underline{\text{sSet}}(k, \underline{\mathcal{M}}(y, x))$$

such that  $\pi^{-1}(\psi_{k,y,x})$  is the map  $k \times \underline{\mathcal{M}}(y, x^k) \xrightarrow{(\text{pr}_2, \text{pr}_1 \cdot \beta_{k,x})} \underline{\mathcal{M}}(y, x^k) \times \underline{\mathcal{M}}(x^k, x) \xrightarrow{\bullet} \underline{\mathcal{M}}(y, x)$ .

3.  $\mathcal{M}$  has a Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$  satisfying SM7 (cf. Definition 2 on chapter II.2 in [Qui67]): For every  $(i : a \rightarrow b) \in \mathcal{C}$  and  $(p : x \rightarrow y) \in \mathcal{F}$

$$(i^*, p_*) : \underline{\mathcal{M}}(b, x) \rightarrow \underline{\mathcal{M}}(b, y) \times_{\underline{\mathcal{M}}(a, y)} \underline{\mathcal{M}}(a, x)$$

is a fibration of  $\mathbf{sSet}$  and is a trivial fibration of  $\mathbf{sSet}$  if  $i \in \mathcal{C}_t$  or  $p \in \mathcal{F}_t$ .

We call  $\mathcal{M}$  a simplicial Quillen category if  $\mathcal{M}$  has a simplicial Quillen structure. By an abuse of notation we will call the pair  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$  a simplicial Quillen structure of  $\mathcal{M}$ .

The category  $\mathbf{sSet}$  of simplicial sets is fundamental in model category theory. For example, Hovey showed in [Hov99] that the homotopy category of a closed functorial model category  $\mathcal{M}$  is a module over the homotopy category of  $\mathbf{sSet}$  using the function complex in [DK80] and this module structure coincides with the canonical module structure which exists if  $\mathcal{M}$  is a closed simplicial model category. From the results of Dugger [Dug01] and Rezk, Schwede, and Shipley [RSS01], we know that a large class of closed model categories are Quillen equivalent to closed simplicial model categories. These results seem to tell us that the homotopy categories of closed model categories behave as if they are the homotopy categories of closed simplicial model categories. So adding such a simplicial structure on  $\mathcal{M}$  may not be a serious restriction.

Our first result is that if  $\mathcal{M}$  is a simplicial Quillen category, then  $\mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_{cf})$  has the two out of three property. To make the statement more precise, we need to introduce the following notions, which played a role in the proof of the two out of three property of  $\mathcal{W}_{\mathbf{sSet}}$  in [Qui67].

1.6. DEFINITION. Let  $\mathcal{M}$  be a simplicial Quillen category with its Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . We define

1.  $\mathcal{SC} = \{g \in \text{Mor}(\mathcal{M}) \mid \pi_0 \underline{\mathcal{M}}(g, z) \text{ is bijective for all } z \in \text{ob} \mathcal{M}_f\}$
2.  $\mathcal{SF} = \{g \in \text{Mor}(\mathcal{M}) \mid \pi_0 \underline{\mathcal{M}}(a, g) \text{ is bijective for all } a \in \text{ob} \mathcal{M}_c\}$

1.7. THEOREM. Let  $\mathcal{M}$  be a simplicial Quillen category.

1.  $\mathcal{SC} \cap \text{Mor}(\mathcal{M}_c)_f = \mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_c)_f$  holds where  $\text{Mor}(\mathcal{M}_c)_f$  is the class of morphisms in  $\mathcal{M}_c$  whose codomains are fibrant.
2.  $\mathcal{SF} \cap \text{Mor}(\mathcal{M}_f)^c = \mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_f)^c$  holds where  $\text{Mor}(\mathcal{M}_f)^c$  is the class of morphisms in  $\mathcal{M}_f$  whose domains are cofibrant.

In particular,  $\mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_{cf})$  satisfies two out of three property.

If  $\mathcal{M}$  is indeed a model category, then we know by Theorem 1 in [Qui67] that its homotopy category is equivalent  $\pi \mathcal{M}_{cf}$ .

Our second result is about a necessary and sufficient condition for two out of three property. We need a definition. Please see Definition 3.7 for the definition of the (co)fibrant replacements.

1.8. DEFINITION. Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . We define

1.  $\overline{\mathcal{SC}} = \{g \in \text{Mor}(\mathcal{M}) \mid Qg \in \mathcal{SC} \text{ for some cofibrant replacement } Qg \text{ of } g\}$
2.  $\overline{\mathcal{SF}} = \{g \in \text{Mor}(\mathcal{M}) \mid Rg \in \mathcal{SF} \text{ for some fibrant replacement } Rg \text{ of } g\}$

1.9. THEOREM. Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Then the following are equivalent.

1.  $(M)$ ,  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$ , and  $\mathcal{F} \cap \mathcal{SF} \subseteq \mathcal{F}_t$  hold.
2.  $\mathcal{W}_{\mathcal{M}} = \overline{\mathcal{SC}}$  holds.
3.  $\mathcal{C} \cap \overline{\mathcal{SC}} = \mathcal{C}_t$  holds.
4.  $\mathcal{W}_{\mathcal{M}} = \overline{\mathcal{SF}}$  holds.
5.  $\mathcal{F} \cap \overline{\mathcal{SF}} = \mathcal{F}_t$  holds.
6.  $\mathcal{W}_{\mathcal{M}}$  has two out of three property.

It would be interesting to know if (M) can be removed from (1) by, if necessary, adding another sufficiently general structure to  $\mathcal{M}$  or its simplicial Quillen structure. Perhaps the first choice would be the cofibrant generation, but it is not symmetric. It would be desirable to find a symmetric structure.

Finally, some interesting and important model categories satisfy  $\mathcal{M} = \mathcal{M}_c$  or  $\mathcal{M} = \mathcal{M}_f$ . In these cases, the following proposition and its dual pinpoint why  $\mathcal{W}_{\mathcal{M}}$  has two out of three property.

1.10. PROPOSITION. Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}_t, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F}_t)$ . We assume that  $\mathcal{M} = \mathcal{M}_c$ . Then, the following are equivalent.

1.  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$  holds.
2.  $\mathcal{W}_{\mathcal{M}} = \mathcal{SC}$  holds.
3.  $\mathcal{W}_{\mathcal{M}}$  has two out of three property.

NOTATIONS. As we remarked above, we adopt the notations and the definitions in [Qui67]. The following are exceptions.

1. We drop the adjective *closed* from the closed model category and the closed simplicial model category.
2. We denote the category of simplicial sets by  $\text{sSet}$ .
3. We denote the boundary of  $\Delta[n]$  with  $\partial\Delta[n]$  instead of  $\overset{\circ}{\Delta}[n]$ .

Finally, the morphisms in  $\mathcal{C}_t$ ,  $\mathcal{C}$ ,  $\mathcal{F}_t$ , and  $\mathcal{F}$  in a Quillen category will be represented by the arrows  $\bullet \xrightarrow{\sim} \bullet$ ,  $\bullet \xrightarrow{\triangleright} \bullet$ ,  $\bullet \xrightarrow{\sim} \bullet$ , and  $\bullet \xrightarrow{\triangleright} \bullet$  respectively.

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## 2. Quillen Category

In this section, we investigate the two out of three property of a Quillen category  $\mathcal{M}$  without extra structure. What we will do is to split (L) into two special cases (L2) and (L3). And in the next section, we will show that, if  $\mathcal{M}$  is a simplicial Quillen category, (L2) is not necessary and (L3) has a simplicial interpretation. We will have the dual result for (R).

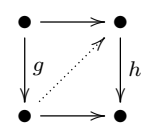
One can produce stronger results with orthogonal factorization systems. For example in Theorem 3.10 in [PT02], the authors show that Theorem 1.9 holds for all Quillen categories if the pair of weak factorization systems is in fact orthogonal.

2.1. WEAK FACTORIZATION SYSTEMS. Here we recall the definition of the weak factorization system and prove a simple lemma saying that the trivial (co)fibrations in the Quillen categories are the trivial (co)fibrations in the model category sense.

2.2. DEFINITION. *Let  $\mathcal{M}$  be a category closed under finite colimits and finite limits. A pair  $(\mathcal{L}, \mathcal{R})$  of classes of morphisms in  $\mathcal{M}$  is called a weak factorization system if the following properties hold.*

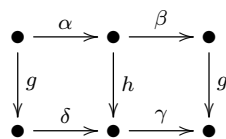
**(Factorization)** *Every morphism  $g$  in  $\mathcal{M}$  can be factored as  $g = p \cdot i$  with  $i \in \mathcal{L}$  and  $p \in \mathcal{R}$ .*

**(Lifting)** *For every  $g \in \mathcal{L}$  and  $h \in \mathcal{R}$ ,  $g \boxtimes h$  holds, i.e., every diagram*



*arrows has a lifting of the dotted arrow.*

**(Retract)** *For every commutative diagram*



*with  $\beta\alpha$  and  $\gamma\delta$  being identities, if  $h$  is in  $\mathcal{L}$  (resp.  $\mathcal{R}$ ) then  $g$  is in  $\mathcal{L}$  (resp.  $\mathcal{R}$ ).*

2.3. REMARK. If  $(\mathcal{L}, \mathcal{R})$  is a weak factorization system, then  $\mathcal{L} = \boxtimes \mathcal{R}$  and  $\mathcal{R} = \mathcal{L} \boxtimes$  hold where  $\boxtimes \mathcal{R} = \{g \mid g \boxtimes h, \forall h \in \mathcal{R}\}$  and  $\mathcal{L} \boxtimes = \{h \mid g \boxtimes h, \forall g \in \mathcal{L}\}$ . Thus  $\mathcal{L}$  is stable under the pushouts,  $\mathcal{R}$  is stable under the pullbacks, and  $\mathcal{L}$  and  $\mathcal{R}$  are stable under the compositions.

2.4. LEMMA. Let  $\mathcal{M}$  be a Quillen category with its Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ .

1.  $\mathcal{C} \cap \mathcal{W}_{\mathcal{M}} = \mathcal{C}_t$ .
2.  $\mathcal{F} \cap \mathcal{W}_{\mathcal{M}} = \mathcal{F}_t$ .

PROOF. We will prove (1). The proof of (2) is dual.

Let  $p \in \mathcal{F}_t$  and  $i \in \mathcal{C}_t$ . Assume that  $pi \in \mathcal{C}$ . Then diagram  $\begin{array}{ccc} \bullet & \xrightarrow{i} & \bullet \\ \downarrow pi & & \downarrow p \\ \bullet & \xrightarrow{=} & \bullet \end{array}$  has a lifting by

the lifting axiom of  $(\mathcal{C}, \mathcal{F}_t)$ . Then  $pi$  is a retract of  $i$ . Hence, by the retract axiom of  $\mathcal{C}_t$ ,  $pi \in \mathcal{C}_t$ . The other inclusion is clear by the definition. ■

2.5. ON (M), (L), AND (R). Let  $\mathcal{M}$  be a Quillen category with its Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . We want to give more explicit descriptions of (M), (L), and (R) using  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ .

First, we consider (M). By Lemma 2.4  $\mathcal{C} \cap \mathcal{W}_{\mathcal{M}} \cap \mathcal{F}$  is the class of the isomorphisms in  $\mathcal{M}$ . So  $\mathcal{W}_{\mathcal{M}}$  becomes a subcategory of  $\mathcal{M}$  if (M) holds. We consider the following variants of the condition (M).

- (M0) For every  $i \in \mathcal{C}_t$  and  $p \in \mathcal{F}_t$  there exist  $j \in \mathcal{C}_t$  and  $q \in \mathcal{F}_t$  such that  $qj = ip$  holds.
- (ML) For every  $i \in \mathcal{C}_t$ ,  $p \in \mathcal{F}_t$ ,  $j \in \mathcal{C}$  and  $q \in \mathcal{F}_t$  satisfying  $qj = ip$ ,  $j \in \mathcal{C}_t$  holds.
- (MR) For every  $i \in \mathcal{C}_t$ ,  $p \in \mathcal{F}_t$ ,  $j \in \mathcal{C}_t$  and  $q \in \mathcal{F}$  satisfying  $qj = ip$ ,  $q \in \mathcal{F}_t$  holds.

Since  $\mathcal{C}_t$  and  $\mathcal{F}_t$  are closed under the compositions, the following lemma holds.

2.6. LEMMA. (M) holds for  $\mathcal{W}_{\mathcal{M}}$  iff (M0) holds.

Next, we turn to (L) and (R). We will show that under (M), (L) is equivalent to a pair of special cases of which the following will be important for us.

$$\text{For every } g \in \mathcal{C} \text{ and } h \in \mathcal{C}_t, \text{ if } hg \in \mathcal{C}_t \text{ then } g \in \mathcal{C}_t. \tag{1}$$

Also under (M), (R) is equivalent to two special cases, and one of them is

$$\text{For every } g \in \mathcal{F}_t \text{ and } h \in \mathcal{F}, \text{ if } hg \in \mathcal{F}_t \text{ then } g \in \mathcal{F}_t. \tag{2}$$

Please see Lemma 2.8 and Lemma 2.9 for the precise statements. Note that (1) and (2) are dual to each other.

Consider the following variants of (L). We note that (L3) is (1).

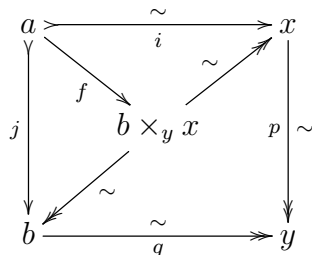
- (L0) For every  $g, i \in \mathcal{C}_t$ ,  $j \in \mathcal{C}$ , and  $h, p, q \in \mathcal{F}_t$  satisfying  $piqj = hg$ ,  $j \in \mathcal{C}_t$  holds.
- (L1) For every  $i \in \mathcal{C}_t$ ,  $j \in \mathcal{C}$ , and  $p, q \in \mathcal{F}_t$  satisfying  $qj = pi$ ,  $j \in \mathcal{C}_t$  holds.
- (L2) For every  $i \in \mathcal{C}_t$ ,  $j \in \mathcal{C}$ , and  $q \in \mathcal{F}_t$  satisfying  $qj = i$ ,  $j \in \mathcal{C}_t$  holds.
- (L3) For every  $i, k \in \mathcal{C}_t$  and  $j \in \mathcal{C}$  satisfying  $kj = i$ ,  $j \in \mathcal{C}_t$  holds.
- (L4) For every  $i, k \in \mathcal{C}_t$ ,  $j \in \mathcal{C}$ , and  $q \in \mathcal{F}_t$  satisfying  $qkj = i$ ,  $j \in \mathcal{C}_t$  holds.

2.7. LEMMA.

1. (L) implies (L1), (L2), (L3), and (L4).
2. (L1) holds iff (L2) holds.
3. (L2) and (L3) hold iff (L4) holds.
4. (L) holds iff (L0) holds.

PROOF. (1) Every  $j$  in (L1)-(L4) belongs to  $\mathcal{W}_M$  by (L). So  $j \in \mathcal{C}_t$  by Lemma 2.4.(1).

(2) (L2) is a special case of (L1). Conversely, let us decompose the diagram in (L1) into



to the upper triangle implies  $k \in \mathcal{C}_t$ . Then, applying Lemma 2.4 to the left triangle, we get  $j \in \mathcal{C}_t$ .

(3) (L2) and (L3) are special cases of (L4). Conversely,  $k \cdot j \in \mathcal{C}_t$  by (L2). Then  $j \in \mathcal{C}_t$  by (L3).

(4) Suppose (L) holds. By (L),  $q \cdot j \in \mathcal{W}_M$ . Then (1) and (L1) imply  $j \in \mathcal{C}_t$ . The converse is clear. ■

2.8. LEMMA.

1. Assume that (M) holds. Then, (L2) and (L3) hold iff (L) holds.
2. (M) and (L2) hold iff (ML) holds.

PROOF. (1) Assume that (L2) and (L3) hold. We will show (L0) holds. Using (M0), we can find  $k \in \mathcal{C}_t$  and  $r \in \mathcal{F}_t$  such that  $rk = iq$ . Lemma 2.7.(2) and (L2) imply that  $k \cdot j \in \mathcal{C}_t$ . Then,  $j \in \mathcal{C}_t$  by (L3). Hence, (L) holds. The other direction follows from Lemma 2.7.(1)

(2) Suppose (M0) and (L2) hold. By (M0), the diagram in (ML) can be turn into a commutative diagram in (L1). Hence Lemma 2.7.(2) and (L2) imply (ML) holds. Conversely, suppose that (ML) holds. (L2) is one instance of (ML). (M0) follows from (ML) using the  $(\mathcal{C}, \mathcal{F}_t)$ -factorization. ■

There are the dual of (L0) through (L4). We will index them (R0) through (R4) so that (L0) is dual to (R0), and so on. We note that (R3) is (2).

The following lemma is the dual of Lemma 2.8.



2.9. LEMMA.

1. Assume that (M) holds. Then, (R2) and (R3) hold iff (R) holds.
2. (M) and (R2) hold iff (MR) holds.

3. Simplicial Quillen categories

In this section, we study the two out of three property of simplicial Quillen categories. After considering some consequences of the axiom SM0 in section 3.1 we collect some properties of simplicial Quillen categories in section 3.2. Finally, in section 3.3, we will prove our statements in the introduction. The proofs rely on sSet being a simplicial model category.

3.1. SM0 AND SIMPLICIAL HOMOTOPY. Here we will make some remarks on consequences of the axiom SM0 and use them to show that two simplicially homotopic maps in simplicial Quillen categories induce the same morphism between the simplicial homotopy classes of maps.

On chapter II.1 [Qui67], Quillen defined an abstract notion of cylinder object and its dual path object in the simplicial context. The axiom SM0 in Definition 1.5 guarantees that they exist. What is not explained in [Qui67] but follows from the definition is that

$$x \otimes k \text{ is a bifunctor} \tag{3}$$

and

$$\phi_{k,x,y} \text{ is a natural in each variable.} \tag{4}$$

We also have the dual of (3) and (4).

Two simplicially homotopic maps in simplicial model categories induce the same morphism between the simplicial homotopy classes of maps. The same result holds without the two out of three property. In fact, it is true for every simplicial category satisfying SM0.

Like simplicial model categories, the simplicial homotopy between two maps in simplicial Quillen categories is the simplicial homotopy in the underlying simplicial category.

3.2. DEFINITION. [cf. Definition 4 on Chapter II.1 in [Qui67]] *Let  $\mathcal{M}$  be a simplicial Quillen category. Let  $g, h \in \mathcal{M}(x, y)$ . We say that  $g$  and  $h$  are simplicially homotopic,  $g \stackrel{s}{\sim} h$ , if there is a commutative diagram*

$$\begin{array}{ccc}
 \partial \mathbf{J} & \xrightarrow{g+h} & \mathcal{M}(x, y) \\
 \downarrow & \nearrow H & \\
 \mathbf{J} & & 
 \end{array}$$

where  $\mathbf{J}$  is a generalized interval and  $\partial \mathbf{J}$  is the boundary of  $\mathbf{J}$ .

3.3. LEMMA. [cf. Proposition 9.6.7 in [Hir03]] *Let  $\mathcal{M}$  be a simplicial category satisfying SM0. Let  $g, h \in \text{Mor}(\mathcal{M})$ . We assume that  $g$  and  $h$  are simplicially homotopic. Then the following two properties hold.*

1.  $\pi_0 \underline{\mathcal{M}}(g, z) = \pi_0 \underline{\mathcal{M}}(h, z)$  for every  $z \in \mathcal{M}$ .
2.  $\pi_0 \underline{\mathcal{M}}(a, g) = \pi_0 \underline{\mathcal{M}}(a, h)$  for every  $a \in \mathcal{M}$ .

PROOF. We will prove (2). The proof of (1) is dual.

The point is that if  $g$  and  $h$  are simplicially homotopic,  $g_* = \underline{\mathcal{M}}(a, g)$  and  $h_* = \underline{\mathcal{M}}(a, h)$  are simplicially homotopic in  $\text{sSet}$  for every  $a \in \mathcal{M}$ . This is a formal consequence of SM0 of  $\mathcal{M}$  and  $\text{sSet}$  together with (3), (4), and their duals.

Let  $H : \mathbf{J} \rightarrow \underline{\mathcal{M}}(x, y)$  be a simplicial homotopy from  $g$  to  $h$ . There is a morphism  $x \rightarrow y^{\mathbf{J}}$  that corresponds to  $H$  by SM0. We also denote this map with  $H$  and let  $H_* : \underline{\mathcal{M}}(a, x) \rightarrow \underline{\mathcal{M}}(a, y^{\mathbf{J}})$  be the induced morphism in  $\text{sSet}$ . By SM0 of  $\mathcal{M}$ , we have  $\underline{\mathcal{M}}(a, y^{\mathbf{J}}) \cong \text{sSet}(\mathbf{J}, \underline{\mathcal{M}}(a, y))$ . Then by SM0 of  $\text{sSet}$ , we have the following diagram of solid arrows.

$$\begin{array}{ccccc}
 \Delta[0] \times \partial \mathbf{J} & \xrightarrow{k} & \underline{\mathcal{M}}(a, x) \times \partial \mathbf{J} & \xrightarrow{g_* + h_*} & \underline{\mathcal{M}}(a, y) \\
 \downarrow & & \downarrow & \nearrow H_* & \\
 \Delta[0] \times \mathbf{J} & \xrightarrow{k} & \underline{\mathcal{M}}(a, x) \times \mathbf{J} & & 
 \end{array}$$

Thus for any  $k : \Delta[0] \rightarrow \underline{\mathcal{M}}(a, x)$  in  $\underline{\mathcal{M}}(a, x)_0$ , we have  $g_*(k) \stackrel{s}{\sim} h_*(k)$  in  $\underline{\mathcal{M}}(a, y)$ . ■

3.4. CHARACTERIZATIONS OF  $\mathcal{C} \cap \mathcal{SC}$  AND  $\mathcal{F} \cap \mathcal{SF}$ . Here, we collect some properties of simplicial Quillen categories including the two out of three property of  $\overline{\mathcal{SC}}$  and  $\overline{\mathcal{SF}}$ . The main result is Proposition 3.14 that connects the properties (1) and (2) with the simplicial structure of  $\mathcal{M}$ .

We begin by recording that  $\mathcal{SC}$  and  $\mathcal{SF}$  satisfy the two out of three property.

3.5. LEMMA. *Let  $\mathcal{M}$  be a simplicial Quillen category. Then  $\mathcal{SC}$  and  $\mathcal{SF}$  satisfy two out of three property.*

PROOF. It follows from the definitions of  $\mathcal{SC}$  and  $\mathcal{SF}$ . ■

Next, we recall the (co)fibrant replacement of morphisms.

3.6. DEFINITION. *Let  $\mathcal{M}$  be a Quillen category with its Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Let  $x \in \text{ob}\mathcal{M}$ .*

1. *A cofibrant replacement of  $x$  is a morphism  $Qx \rightarrow x$  in  $\mathcal{F}_t$  such that  $Qx$  is cofibrant.*
2. *A fibrant replacement of  $x$  is a morphism  $x \rightarrow Rx$  in  $\mathcal{C}_t$  such that  $Rx$  is fibrant.*

3.7. DEFINITION. Let  $\mathcal{M}$  be a Quillen category with its Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Let  $g \in \mathcal{M}(x, y)$ .

1. A cofibrant replacement of  $g$  is a commutative diagram  $Qx \xrightarrow{\alpha} x$  such that  $\alpha$  and  $\beta$  are cofibrant replacements of  $x$  and  $y$  respectively.
- $$\begin{array}{ccc} Qx & \xrightarrow{\alpha} & x \\ \downarrow Qg & & \downarrow g \\ Qy & \xrightarrow{\beta} & y \end{array}$$

2. A fibrant replacement of  $g$  is a commutative diagram  $x \xrightarrow{\alpha} Rx$  such that  $\alpha$  and  $\beta$  are fibrant replacements of  $x$  and  $y$  respectively.
- $$\begin{array}{ccc} x & \xrightarrow{\alpha} & Rx \\ \downarrow g & & \downarrow Rg \\ y & \xrightarrow{\beta} & Ry \end{array}$$

We note that in Definition 1.8, we did not ask every cofibrant replacement to be in  $\mathcal{SC}$ . But using a similar trick as in the proof of Lemma 3.9, one can show that if one cofibrant replacement of  $g$  is in  $\mathcal{SC}$ , then every cofibrant replacement of  $g$  is in  $\mathcal{SC}$ .

$\overline{\mathcal{SC}}$  and  $\overline{\mathcal{SF}}$  also have two out of three property. But first we need to prepare a lemma.

3.8. LEMMA. Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Then the following hold.

1.  $\mathcal{C}_t \subseteq \mathcal{SC}$  and  $\mathcal{F}_t \subseteq \mathcal{SF}$ .
2.  $\mathcal{C}_t \cap \text{Mor}(\mathcal{M}_f) \subseteq \mathcal{SF}$  and  $\mathcal{F}_t \cap \text{Mor}(\mathcal{M}_c) \subseteq \mathcal{SC}$ .
3. If (M) holds, then  $\mathcal{C}_t \subseteq \overline{\mathcal{SC}}$  and  $\mathcal{F}_t \subseteq \overline{\mathcal{SF}}$ .
4.  $\mathcal{C}_t \subseteq \overline{\mathcal{SF}}$  and  $\mathcal{F}_t \subseteq \overline{\mathcal{SC}}$ .

PROOF. (1) We prove the first inclusion. The proof of the second is dual.

Let  $i \in \mathcal{C}_t$  and  $z \in \text{ob}\mathcal{M}_f$ . Let  $i^* := \pi_0 \underline{\mathcal{M}}(i, z)$ . First, we show that  $i^*$  is injective. Let  $g, h \in \mathcal{M}(b, z)$ . Suppose that  $g \cdot i \stackrel{s}{\sim} h \cdot i$ . Then there is a commutative diagram of solid arrows  $\partial \mathbf{J} \xrightarrow{g+h} \underline{\mathcal{M}}(b, z)$ . SM7 implies  $i^* : \underline{\mathcal{M}}(b, z) \rightarrow \underline{\mathcal{M}}(a, z)$  is a trivial fibration of  $\text{sSet}$ .

$$\begin{array}{ccc} \partial \mathbf{J} & \xrightarrow{g+h} & \underline{\mathcal{M}}(b, z) \\ \downarrow & \nearrow & \downarrow i^* \\ \mathbf{J} & \longrightarrow & \underline{\mathcal{M}}(a, z) \end{array}$$

In  $\text{sSet}$ ,  $\partial \mathbf{J} \rightarrow \mathbf{J}$  is a cofibration. Hence, the above diagram has a lifting of the dotted arrow. Therefore  $i^*$  is injective.

Next, we show that  $i^*$  is surjective. Let  $f \in \mathcal{M}(a, z)$ . Since  $z$  is fibrant, the diagram

$a \xrightarrow{f} z$  of solid arrows has a lifting where  $*$  is the terminal object. Therefore  $i^*$  is surjective.

$$\begin{array}{ccc} a & \xrightarrow{f} & z \\ \downarrow i & \nearrow & \downarrow \\ b & \longrightarrow & * \end{array}$$

(2) We prove  $\mathcal{F}_t \cap \text{Mor}(\mathcal{M}_c) \subseteq \mathcal{SC}$ . The proof of  $\mathcal{C}_t \cap \text{Mor}(\mathcal{M}_f) \subseteq \mathcal{SF}$  is dual.

Let  $p \in \mathcal{M}(x, y)$ . Assume that  $p \in \mathcal{F}_t$  and  $x, y \in \mathcal{M}_c$ . Then there are liftings  $r$  and  $H$  of the following diagrams of solid arrows where  $\sigma$  is the constant homotopy.

$$\begin{array}{ccc}
 \emptyset & \longrightarrow & x \\
 \downarrow & \nearrow r & \downarrow p \\
 y & \xrightarrow{=} & y
 \end{array}
 \qquad
 \begin{array}{ccc}
 x \otimes \partial\Delta[1] & \xrightarrow{r \cdot p + 1_x} & x \\
 \downarrow & \nearrow H & \downarrow p \\
 x \otimes \Delta[1] & \xrightarrow{\sigma} x \xrightarrow{p} & y
 \end{array}$$

Since  $p \in \mathcal{F}_t$  and  $y \in \text{ob}\mathcal{M}_c$ , such  $r$  exists. Since  $x \in \text{ob}\mathcal{M}_c$ ,  $x \otimes \partial\Delta[1] \rightarrow x \otimes \Delta[1]$  belongs to  $\mathcal{C}$  by SM7 and Proposition 3 on Chapter II.2 in [Qui67]. Hence  $H$  also exists. Then  $p$  is a simplicial homotopy equivalence with its inverse  $r$ . So,  $p \in \mathcal{SC}$  by Lemma 3.3.

(3) follows from (1) and (M0).

(4) We prove the first inclusion. The proof of the second is dual.

Let  $g \in \mathcal{C}_t$  and let  $\beta$  be a fibrant replacement of  $\text{cod}(g)$ . Then  $\begin{array}{ccc} \bullet & \xrightarrow{\beta \cdot g} & \bullet \\ \downarrow g & & \downarrow = \\ \bullet & \xrightarrow{\beta} & \bullet \end{array}$  is a fibrant replacement of  $g$ . Hence  $g \in \overline{\mathcal{SF}}$ . ■

Recall that the elements of  $\overline{\mathcal{SC}}$  and  $\overline{\mathcal{SF}}$  are those having cofibrant replacements in  $\mathcal{SC}$  and  $\mathcal{SF}$  respectively.

3.9. LEMMA. *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, F_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Then  $\overline{\mathcal{SC}}$  and  $\overline{\mathcal{SF}}$  satisfy two out of three property.*

PROOF. We will prove it for  $\overline{\mathcal{SC}}$ . The proof of  $\overline{\mathcal{SF}}$  is dual.

First, we prove the property (M). Let  $g : x \rightarrow y$  and  $h : y \rightarrow z$  be morphisms in  $\overline{\mathcal{SC}}$ . Let  $Qg$  and  $Qh$  be cofibrant replacements of  $g$  and  $h$  respectively such that  $Qg, Qh \in \mathcal{SC}$ .

$$\begin{array}{ccc}
 Qx \xrightarrow{\alpha} x & & Qy \xrightarrow{\gamma} y \\
 \downarrow Qg & & \downarrow Qh \\
 Qy \xrightarrow{\beta} y & & Qz \xrightarrow{\delta} z
 \end{array}$$

Consider the commutative diagram  $\begin{array}{ccc} Qw \xrightarrow{p_w} w & \xrightarrow{\beta'} & Qy \\ & \downarrow \gamma' & \downarrow \beta \\ & Qy & \xrightarrow{\gamma} y \end{array}$  obtained by the pullback and

a cofibrant replacement  $p_w$  of  $w$ . Then, since  $\beta' \cdot p_w, \gamma' \cdot p_w \in \mathcal{F}_t \cap \text{Mor}(\mathcal{M}_c)$ ,  $\beta' \cdot p_w, \gamma' \cdot p_w \in \mathcal{SC}$  by Lemma 3.8.(2). Since  $Qx \in \mathcal{M}_c$  and  $\beta' \cdot p_w \in \mathcal{F}_t$ , there is a lifting  $r$  of  $Qg$  against  $\beta' \cdot p_w$  such that  $Qg = \beta' \cdot p_w \cdot r$ . Since  $Qg, \beta' \cdot p_w \in \mathcal{SC}$ , every such a lifting is in  $\mathcal{SC}$  by Lemma 3.5. Then  $Qh \cdot \gamma' \cdot p_w \cdot r$  is in  $\mathcal{SC}$  and it is a cofibrant replacement of  $hg$ .

The proofs of (L) and (R) are similar, so we omit. ■

3.10. LEMMA. *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}_t, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F}_t)$ .*

1.  $\mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_c) \subseteq \mathcal{S}\mathcal{C}$ . *If  $\mathcal{C} \cap \mathcal{S}\mathcal{C} \subseteq \mathcal{C}_t$ , then  $\mathcal{S}\mathcal{C} \cap \text{Mor}(\mathcal{M}_c) \subseteq \mathcal{W}_{\mathcal{M}}$  also holds.*
2.  $\mathcal{W}_{\mathcal{M}} \cap \text{Mor}(\mathcal{M}_f) \subseteq \mathcal{S}\mathcal{F}$ . *If  $\mathcal{F} \cap \mathcal{S}\mathcal{F} \subseteq \mathcal{F}_t$ , then  $\mathcal{S}\mathcal{F} \cap \text{Mor}(\mathcal{M}_f) \subseteq \mathcal{W}_{\mathcal{M}}$  also holds.*

PROOF. We will prove (1). The proof of (2) is dual.

The first inclusion follows from (1) and (2) of Lemma 3.8. Let  $g \in \mathcal{S}\mathcal{C} \cap \text{Mor}(\mathcal{M}_c)$ . Let  $g = p \cdot i$  be a  $(\mathcal{C}, \mathcal{F}_t)$ -factorization of  $g$ .  $p \in \mathcal{S}\mathcal{C}$  by Lemma 3.8.(2).  $i \in \mathcal{S}\mathcal{C}$  by Lemma 3.5. Then by our assumption  $\mathcal{C} \cap \mathcal{S}\mathcal{C} \subseteq \mathcal{C}_t$ ,  $i \in \mathcal{C}_t$ . Thus  $g \in \mathcal{W}_{\mathcal{M}}$ . ■

The following lemma was proved for sSet during the proof of Lemma 7 on chapter II.3 in [Qui67]. The proof is formal so that they also work for the simplicial Quillen category with little modification.

3.11. LEMMA. *Let  $\mathcal{M}$  be a simplicial Quillen category. If  $i \in \mathcal{S}\mathcal{C}$ , then  $\underline{\mathcal{M}}(i, z)$  is a weak equivalence in sSet for all  $z \in \text{ob}\mathcal{M}_f$ .*

PROOF. Let  $i \in \text{Mor}(a, b)$ . Let  $z \in \text{ob}\mathcal{M}_f$ . Let  $k \in \text{obsSet}$ . By SM0 and the dual of (4), we have the following commutative diagram

$$\begin{array}{ccc}
 \pi_0 \text{sSet}(k, \underline{\mathcal{M}}(b, z)) & \xrightarrow{i^*} & \pi_0 \text{sSet}(k, \underline{\mathcal{M}}(a, z)) \\
 \cong \downarrow & & \downarrow \cong \\
 \pi_0 \underline{\mathcal{M}}(b, z^k) & \xrightarrow{i^*} & \pi_0 \underline{\mathcal{M}}(a, z^k)
 \end{array}$$

By SM7,  $z^k$  is fibrant. So  $i \in \mathcal{S}\mathcal{C}$  implies that the bottom  $i^*$  is bijective. Hence the top  $i^*$  is also bijective. Then by Proposition 9.5.16 and Proposition 9.6.9 in [Hir03],  $\underline{\mathcal{M}}(i, z)$  is a weak equivalence in sSet. ■

3.12. REMARK. In the previous lemma, if the domain and the codomain of  $i$  are cofibrant, then  $\underline{\mathcal{M}}(a, z)$  and  $\underline{\mathcal{M}}(b, z)$  are fibrant by SM7( and cofibrant). Then using K. Brown’s lemma and using a similar argument as the proof of Lemma 3.8.(1), one can show that  $\underline{\mathcal{M}}(i, z)$  being a weak equivalence for all  $z \in \text{ob}\mathcal{M}_f$  implies  $i \in \mathcal{S}\mathcal{C}$ .

The following lemma which plays a key role in our paper was also proved in [Qui67] in the context of sSet. The proof also works with our setting with little changes. But we will give the proof for the sake of completeness. Our proof uses the two out of three property of  $\mathcal{W}_{\text{sSet}}$ . Because of this, the proof is shorter than that in [Qui67].

3.13. LEMMA. [cf. Lemma 7 on Chapter 2.3 in [Qui67]] *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Let  $i \in \mathcal{C} \cap \mathcal{S}\mathcal{C}$ . Then, for every  $q \in \mathcal{F}$  with  $\text{cod}(q) \in \text{ob}\mathcal{M}_f$ ,  $(i_*, q^*)$  is a trivial fibration of sSet. In particular,  $i \sqsupseteq q$  holds.*

PROOF. We may assume that simplicial Quillen category  $\mathcal{M}$  is not the simplicial model category  $\mathbf{sSet}$ .

Let  $q \in \mathcal{M}(x, y) \cap \mathcal{F}$ . We assume that  $y \in \mathcal{M}_f$ . Note that  $x \in \text{ob}\mathcal{M}_f$  too. Let  $a \xrightarrow{\alpha} x$  be a commutative diagram. It induces the following commutative diagram.

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & x \\ \downarrow i & & \downarrow q \\ b & \xrightarrow{\beta} & y \end{array}$$

$$\begin{array}{ccccc} \underline{\mathcal{M}}(b, x) & \xrightarrow{(i^*, p_*)} & \underline{\mathcal{M}}(b, y) \times_{\underline{\mathcal{M}}(a, y)} \underline{\mathcal{M}}(a, x) & \xrightarrow{f} & \underline{\mathcal{M}}(a, x) \\ & & \downarrow & & \downarrow \\ & & \underline{\mathcal{M}}(b, y) & \xrightarrow{\underline{\mathcal{M}}(i, y)} & \underline{\mathcal{M}}(a, y) \end{array}$$

Since  $x, y \in \text{ob}\mathcal{M}_f$ ,  $\underline{\mathcal{M}}(i, y)$  and  $f \cdot (i^*, p_*)$  are trivial fibrations in  $\mathbf{sSet}$  by Lemma 3.11 and SM7 of  $\mathcal{M}$ . Since  $f$  is a pull-back of  $\underline{\mathcal{M}}(i, y)$ ,  $f$  is also a trivial fibration in  $\mathbf{sSet}$ . Then by two out of three property of  $\mathbf{sSet}$ ,  $(i^*, p_*)$  is an weak equivalence. Therefore  $(i^*, p_*)$  is a trivial fibration by SM7.

Now, we have a lifting of the following diagram of solid arrows

$$\begin{array}{ccc} \partial\Delta[0] & \longrightarrow & \underline{\mathcal{M}}(b, x) \\ \downarrow & & \downarrow (i^*, p_*) \\ \Delta[0] & \longrightarrow & \underline{\mathcal{M}}(b, y) \times_{\underline{\mathcal{M}}(a, y)} \underline{\mathcal{M}}(a, x), \end{array}$$

where the bottom map is given by the first diagram. Then the lifting is a lifting of the first diagram. ■

3.14. PROPOSITION. *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ .*

1. *For every  $i \in \mathcal{C}$ , the following are equivalent*
  - (a)  $i \in \mathcal{SC}$ .
  - (b) *For every fibrant replacement  $j$  of  $\text{cod}(i)$ ,  $j \cdot i \in \mathcal{C}_t$ .*
  - (c) *There is a morphism  $j \in \mathcal{C}_t$  such that  $j \cdot i \in \mathcal{C}_t$ .*
  
2. *For every  $p \in \mathcal{F}$ , the following are equivalent*
  - (a)  $p \in \mathcal{SF}$ .
  - (b) *For every cofibrant replacement  $q$  of  $\text{dom}(p)$ ,  $p \cdot q \in \mathcal{F}_t$ .*
  - (c) *There is a morphism  $q \in \mathcal{F}_t$  such that  $p \cdot q \in \mathcal{F}_t$ .*

PROOF. We will prove (1). The proof of (2) is dual.

(a)  $\Rightarrow$  (b) Let  $i \in \mathcal{M}(a, b)$ . Let  $i_b : b \rightarrow Rb$  be a fibrant replacement of  $b$ . We factor  $i_b \cdot i$  as  $i_b \cdot i = p \cdot \alpha$   $a \xrightarrow{\sim \alpha} x$  where  $p \in \mathcal{F}$  and  $\alpha \in \mathcal{C}_t$ . By Lemma 3.13, there is a lifting

$$\begin{array}{ccc} a & \xrightarrow{\sim \alpha} & x \\ i \downarrow & & p \downarrow \\ b & \xrightarrow{\sim i_b} & Rb \end{array}$$

of the above diagram. So,  $p$  has a section. Then  $i_b \cdot i$  is a retract of  $\alpha$ . Hence,  $i_b \cdot i \in \mathcal{C}_t$ .

(b)  $\Rightarrow$  (c) is clear.

(c)  $\Rightarrow$  (a) By Lemma 3.8.(1),  $j$  and  $j \cdot i$  are in  $\mathcal{SC}$ . Then  $i$  is in  $\mathcal{SC}$  too by Lemma 3.5. ■

Since the identities for (co)fibrant objects are (co)fibrant replacements, we have the following corollary of Proposition 3.14.

3.15. COROLLARY. *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Then*

$$\mathcal{SC} \cap \text{Mor}(\mathcal{M})_f \subseteq \mathcal{C}_t, \quad \mathcal{SF} \cap \text{Mor}(\mathcal{M})^c \subseteq \mathcal{F}_t$$

hold where  $\text{Mor}(\mathcal{M})_f$  is the class of morphisms whose codomain are fibrant and  $\text{Mor}(\mathcal{M})^c$  is the class of morphisms whose domain are cofibrant.

3.16. PROOFS. Here we prove the results stated in the introduction. We will reproduce the statements for the convenience of readers.

3.17. THEOREM 1.7. *Let  $\mathcal{M}$  be a simplicial Quillen category.*

1.  $\mathcal{SC} \cap \text{Mor}(\mathcal{M}_c)_f = \mathcal{W}_M \cap \text{Mor}(\mathcal{M}_c)_f$  holds where  $\text{Mor}(\mathcal{M}_c)_f$  is the class of morphisms in  $\mathcal{M}_c$  whose codomains are fibrant.
2.  $\mathcal{SF} \cap \text{Mor}(\mathcal{M}_f)^c = \mathcal{W}_M \cap \text{Mor}(\mathcal{M}_f)^c$  holds where  $\text{Mor}(\mathcal{M}_f)^c$  is the class of morphisms in  $\mathcal{M}_f$  whose domains are cofibrant.

In particular,  $\mathcal{W}_M \cap \text{Mor}(\mathcal{M}_{cf})$  satisfies two out of three property.

PROOF. We will prove (1). The proof of (2) is dual.

By Lemma 3.10,  $\mathcal{W}_M \cap \text{Mor}(\mathcal{M}_c)_f \subseteq \mathcal{SC} \cap \text{Mor}(\mathcal{M}_c)_f$  holds. Let  $g \in \mathcal{SC} \cap \text{Mor}(\mathcal{M}_c)_f$ . We can factor  $g = p \cdot i$  so that  $p \in \mathcal{F}_t$  and  $i \in \mathcal{C}$ . Since  $g \in \text{Mor}(\mathcal{M}_c)$ ,  $p \in \text{Mor}(\mathcal{M}_c)$ . So,  $p \in \mathcal{SC}$  by Lemma 3.8.(2). Hence,  $i \in \mathcal{SC}$ . Since  $g \in \text{Mor}(\mathcal{M})_f$ ,  $\text{cod}(i) \in \text{ob}\mathcal{M}_f$ . Then by Corollary 3.15, we have  $i \in \mathcal{C}_t$ . Therefore,  $g \in \mathcal{W}_M$ . Hence  $\mathcal{W}_M \cap \text{Mor}(\mathcal{M}_c)_f \supseteq \mathcal{SC} \cap \text{Mor}(\mathcal{M}_c)_f$  holds. ■

To prove Theorem 1.9, we record the following corollary of Proposition 3.14.

3.18. COROLLARY. *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ .*

1. (1) holds iff  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$  holds.
2. (2) holds iff  $\mathcal{F} \cap \mathcal{SF} \subseteq \mathcal{F}_t$  holds.

3.19. REMARK. By Lemma 3.8.(1),  $\mathcal{C} \cap \mathcal{SC} \supseteq \mathcal{C}_t$  and  $\mathcal{F} \cap \mathcal{SF} \supseteq \mathcal{F}_t$  always hold. So Corollary 3.18 tells us that (1) is precisely what is needed to characterize the acyclicity of the elements of  $\mathcal{C}$  in terms of the simplicial structure of  $\mathcal{M}$ .

3.20. THEOREM 1.9. *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}, \mathcal{F}_t)$  and  $(\mathcal{C}_t, \mathcal{F})$ . Then the following are equivalent.*

1.  $(M)$ ,  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$ , and  $\mathcal{F} \cap \mathcal{SF} \subseteq \mathcal{F}_t$  hold.
2.  $\mathcal{W}_{\mathcal{M}} = \overline{\mathcal{SC}}$  holds.
3.  $\mathcal{C} \cap \overline{\mathcal{SC}} = \mathcal{C}_t$  holds.
4.  $\mathcal{W}_{\mathcal{M}} = \overline{\mathcal{SF}}$  holds.
5.  $\mathcal{F} \cap \overline{\mathcal{SF}} = \mathcal{F}_t$  holds.
6.  $\mathcal{W}_{\mathcal{M}}$  has two out of three property.

PROOF. (1) $\Rightarrow$ (2) Let  $g \in \overline{\mathcal{SC}}$ . We factor  $g$  as  $g = p \cdot i$  so that  $p \in \mathcal{F}$  and  $i \in \mathcal{C}_t$ . By Lemma 3.8.(3) and (M),  $i \in \overline{\mathcal{SC}}$ . So by Lemma 3.9,  $p \in \overline{\mathcal{SC}}$ . Let  $p \in \mathcal{M}(x, y)$ . Let

$$\begin{array}{ccc}
 Qx & \xrightarrow{\sim} & x \\
 \downarrow Qp & & \downarrow p \\
 Qy & \xrightarrow{\sim} & y
 \end{array}$$

$\alpha$    $\beta$

be a cofibrant replacement of  $p$  such that  $Qp \in \mathcal{SC}$ . Since  $Qp \in \mathcal{SC}$  and  $Qx, Qy \in \mathcal{M}_c$ ,  $Qp \in \mathcal{W}_{\mathcal{M}}$  by Lemma 3.10 and  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$ . Then  $p \cdot \alpha \in \mathcal{F}_t$  by the Lemma 2.4. So,  $p \in \mathcal{SF}$  by the implication (c) $\Rightarrow$ (a) of Proposition 3.14.(2). Then by our assumption  $\mathcal{F} \cap \mathcal{SF} \subseteq \mathcal{F}_t$ ,  $p \in \mathcal{F}_t$ . Therefore  $\overline{\mathcal{SC}} \subseteq \mathcal{W}_{\mathcal{M}}$ .

Conversely, let  $g \in \mathcal{W}_{\mathcal{M}}$  and factor  $g$  as  $g = p \cdot i$  where  $p \in \mathcal{F}_t$  and  $i \in \mathcal{C}_t$ . By (3) and (4) of Lemma 3.8 and our assumption (M),  $p, i \in \overline{\mathcal{SC}}$ . So by Lemma 3.9,  $g \in \overline{\mathcal{SC}}$ .

(2) $\Rightarrow$ (6) follows from Lemma 3.9.

(6) $\Rightarrow$ (1) follows from Corollary 3.18 and Lemma 2.7.(1).

(2) $\Rightarrow$ (3) follows from Lemma 2.4.

(3) $\Rightarrow$ (2) Let  $f \in \mathcal{W}_{\mathcal{M}}$ . Then  $f = p \cdot i$  for some  $p \in \mathcal{F}_t$  and  $i \in \mathcal{C}_t$ .  $p \in \overline{\mathcal{SC}}$  by Lemma 3.8.(4).  $i \in \mathcal{C}_t \subseteq \mathcal{C} \cap \overline{\mathcal{SC}}$ , hence  $i \in \overline{\mathcal{SC}}$ . So  $f \in \overline{\mathcal{SC}}$  by Lemma 3.9. Conversely, let  $f \in \overline{\mathcal{SC}}$ . Let  $f = p \cdot i$  be a  $(\mathcal{C}, \mathcal{F}_t)$ -factorization of  $f$ . Since  $p \in \overline{\mathcal{SC}}$  by Lemma 3.8.(4),  $i \in \overline{\mathcal{SC}}$  by Lemma 3.9. Then  $i \in \mathcal{C} \cap \overline{\mathcal{SC}} \subseteq \mathcal{C}_t$ , and  $f \in \mathcal{W}_{\mathcal{M}}$ .

By duality, (4) and (5) are equivalent to (6). ■

3.21. REMARK. If  $\mathcal{W}_{\mathcal{M}}$  has two out of three property, then  $\mathcal{M}$  is a simplicial model category and  $\mathcal{W}_{\mathcal{M}} = \overline{\mathcal{SC}}$  holds by a general result. See Theorem 9.7.4.(4) in [Hir03] and Remark 3.12.



**3.22. PROPOSITION 1.10.** *Let  $\mathcal{M}$  be a simplicial Quillen category with its simplicial Quillen structure  $(\mathcal{C}_t, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F}_t)$ . We assume that  $\mathcal{M} = \mathcal{M}_c$ . Then, the following are equivalent.*

1.  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$  holds.
2.  $\mathcal{W}_{\mathcal{M}} = \mathcal{SC}$  holds.
3.  $\mathcal{W}_{\mathcal{M}}$  has two out of three property.

PROOF. (1) $\Rightarrow$ (2) follows from Lemma 3.10 and our assumption  $\mathcal{M}_c = \mathcal{M}$ .

(2) $\Rightarrow$ (3) follows from Lemma 3.5.

(3) $\Rightarrow$ (1) follows from Corollary 3.18 and Lemma 2.7.(1). ■

**3.23. REMARK.** If  $\mathcal{M} = \mathcal{M}_c$  and  $\mathcal{W}_{\mathcal{M}}$  has two out of three property, then  $\mathcal{M}$  is a simplicial model category and  $\mathcal{W}_{\mathcal{M}} = \mathcal{SC}$  holds by a general result. See Corollary 9.7.5 in [Hir03].  $\mathcal{M}_c = \mathcal{M}$  is a rather strong restriction on  $\mathcal{M}$ . It implies  $\mathcal{SC} = \overline{\mathcal{SC}}$ . And  $\mathcal{M}_c = \mathcal{M}$  and  $\mathcal{C} \cap \mathcal{SC} \subseteq \mathcal{C}_t$  imply (M) and  $\mathcal{F} \cap \mathcal{SF} \subseteq \mathcal{F}_t$ . So, Proposition 1.10 also follows from Theorem 1.9. But the proof given above is shorter.

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