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MUTATION PAIRS AND TRIANGULATED QUOTIENTS

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Abstract. We introduce the notion of mutation pairs in pseudo-triangulated categories. Given such a mutation pair, we prove that the corresponding quotient category carries a natural triangulated structure under certain conditions. This result unifies many previous constructions of quotient triangulated categories.

1. Introduction

The notion of triangulated categories was introduced by Grothendieck and Verdier in the sixties of last century. It is important in both geometry and algebra. One way to construct triangulated categories is through quotient categories.

Let $(\mathcal{B}, \mathcal{S})$ be an exact category satisfying the Frobenius condition; that is, $(\mathcal{B}, \mathcal{S})$ has enough $\mathcal{S}$-injectives and enough $\mathcal{S}$-projectives, and the $\mathcal{S}$-injectives and the $\mathcal{S}$-projectives determine the same full subcategory $\mathcal{I}$. Then, as shown by Happel [5], the quotient category $\mathcal{B}/\mathcal{I}$ carries a triangulated structure. Beligiannis obtained a similar result [2, Theorem 7.2] by replacing $\mathcal{B}$ with a triangulated category $\mathcal{C}$ and replacing $\mathcal{S}$ with a proper class of triangles $\mathcal{E}$. Let $\mathcal{C}$ be a triangulated category with AR triangles and $\mathcal{X}$ be a functorially finite subcategory with $\tau\mathcal{X} = \mathcal{X}$, where τ is the AR translation, then Jørgensen [7, Theorem 2.3] showed that the quotient $\mathcal{C}/\mathcal{X}$ is a triangulated category.

The notion of mutation of subcategories in a triangulated category is a generalization of a notion of mutation of cluster tilting objects in a cluster category. Let $(\mathcal{Z}, \mathcal{Z})$ be a $\mathcal{D}$-mutation pair in a triangulated category $\mathcal{C}$, and $\mathcal{Z}$ be an extension-closed subcategory of $\mathcal{C}$, by a result of Iyama-Yoshino [6, Theorem 4.2], the quotient $\mathcal{Z}/\mathcal{D}$ is a triangulated category. Recently Liu-Zhu introduced a notion of $\mathcal{D}$-mutation pairs in right triangulated categories, and then obtained a similar result [8, Theorem 3.11], which unifies the constructions of Iyama-Yoshino and Jørgensen.

Beligiannis and Reiten [4] defined a pretriangulated category $(\mathcal{C}, \Omega, \Sigma, \lhd, \rhd)$ to be a category $\mathcal{C}$ equipped with a left triangulated structure $(\mathcal{C}, \Omega, \lhd)$, and a right triangulated structure $(\mathcal{C}, \Sigma, \rhd)$, for which (Σ, Ω) is an adjoint pair, and certain gluing conditions hold. For example, an abelian category is a pretriangulated category with $\Omega = \Sigma = 0$, and a

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triangulated category is a pretriangulated category with $\Omega = \Sigma^{-1}$. But the converses are not true. Thus the notion of pretriangulated categories is not a good choice to generalize simultaneously some analogous results on abelian categories and triangulated categories. Noticing the imperfection, Nakaoka [9] introduced the notion of pseudo-triangulated categories (see Definition 2.2.1 for detail) which is a natural generalization of abelian categories and triangulated categories. The right triangles and left triangles on pseudo-triangulated categories behave much better than those on pretriangulated categories. To unify the constructions of quotient triangulated structures occurring in exact categories [5] and triangulated categories [6], Nakaoka defined a Frobenius condition on a pseudo-triangulated category, which is similar to that on an exact category, and constructed a quotient triangulated category [9, Theorem 6.17]. As Nakaoka pointed out, his construction cannot cover Beligiannis's result [2, Theorem 7.2].

The main aim of this article is to give a way to unify the existing different constructions of quotient triangulated categories. We define mutation pairs in pseudo-triangulated categories, and show that the corresponding quotient categories carry triangulated structures under certain reasonable conditions. As applications, our result unifies the quotient triangulated category construction considered by Iyama-Yoshino [6], Happel [5], Beligiannis [2], Jørgensen [7], and Nakaoka [9], but not that of [8].

The paper is organized as follows. In Section 2, we list some necessary preliminaries. We first review the definitions and some facts on right triangulated categories and pseudo-triangulated categories, and then define $\mathcal{D}$-mutation pairs in pseudo-triangulated categories and set some conventions throughout this paper. In Section 3, we state and prove our main result Theorem 3.3.1. We show that under certain conditions, the quotient category associated to a given mutation pair has a structure of a triangulated category. At last, we give some examples to illustrate our main result.

2. Preliminaries

Let $\mathcal{C}$ be an additive category and $\mathcal{D}$ a subcategory of $\mathcal{C}$. When we say $\mathcal{D}$ is a subcategory of $\mathcal{C}$, we always mean that $\mathcal{D}$ is an additive full subcategory which is closed under isomorphisms and direct summands. A *pseudokernel* of a morphism $g : B \to C$ is a morphism $f : A \to B$ such that $gf = 0$ and if $h : D \to B$ is a morphism such that $gh = 0$, there exists a morphism $i : D \to A$ such that $h = fi$. We can define the notion of a pseudocokernel dually. A morphism $f : A \to B$ in $\mathcal{C}$ is called $\mathcal{D}$-*epic*, if for any object $D \in \mathcal{D}$, the sequence $\mathcal{C}(D, A) \xrightarrow{\mathcal{C}(D,f)} \mathcal{C}(D, B) \to 0$ is exact. A right $\mathcal{D}$-*approximation* of X in $\mathcal{C}$ is a $\mathcal{D}$-epic map $f : D \to X$ with $D \in \mathcal{D}$. If for any object $X \in \mathcal{C}$, there exists a right $\mathcal{D}$-approximation $f : D \to X$, then $\mathcal{D}$ is called a *contravariantly finite subcategory*. Dually we have the notions of a $\mathcal{D}$-monic map, a left $\mathcal{D}$-approximation and a covariantly finite subcategory. The subcategory $\mathcal{D}$ is called *functorially finite* if $\mathcal{D}$ is both contravariantly finite and covariantly finite.

2.1. Right triangulated categories. Let $\mathcal{C}$ be an additive category and $\Sigma : \mathcal{C} \to \mathcal{C}$ an additive functor. A sextuple (A, B, C, f, g, h) in $\mathcal{C}$ is of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$. A morphism of sextuples from (A, B, C, f, g, h) to (A', B', C', f', g', h') is a triple (a, b, c) of morphisms such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A'.
\end{tikzcd}$$

If in addition $a, b,$ c are isomorphisms in $\mathcal{C}$, then (a, b, c) is called an isomorphism of sextuples.

2.1.1. Definition. ([3], [9]) Let $\mathcal{C}$ be an additive category, $\Sigma : \mathcal{C} \to \mathcal{C}$ an additive functor, and $\rhd$ a class of sextuples. The triple $(\mathcal{C}, \Sigma, \rhd)$ is called a *right triangulated category*, Σ its *suspension functor*, and the elements of $\rhd$ *right triangles*, if the following axioms are satisfied:

(RTR0) $\rhd$ is closed under isomorphisms.

(RTR1) For any object $A \in \mathcal{C}$, the sextuple $0 \to A \xrightarrow{1_A} A \to 0$ is a right triangle; and for any morphism $f : A \to B$ in $\mathcal{C}$, there exists a right triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$.

(RTR2) If $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ is a right triangle, then so is $B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$.

(RTR3) For any two right triangles $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$, and any two morphisms $a : A \to A'$ and $b : B \to B'$ such that $bf = f'a$, there exists a morphism $c : C \to C'$ such that (a, b, c) is a morphism of right triangles.

(RTR4) Let $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, $A \xrightarrow{l} M \xrightarrow{m} B' \xrightarrow{n} \Sigma A$ and $A' \xrightarrow{l'} M \xrightarrow{m'} B \xrightarrow{n'} \Sigma A'$ be three right triangles with $m'l = f$. Then there exist two morphisms $g' : B' \to C$ and $h' : C \to \Sigma A'$ such that the following diagram is commutative and the third column is a right triangle.

$$\begin{tikzcd}
 & A' \arrow[r, equal] \arrow[d, "l'"] & A' \arrow[d, "f'"] & \\
A \arrow[r, "l"] \arrow[d, equal] & M \arrow[r, "m"] \arrow[d, "m'"] & B' \arrow[r, "n"] \arrow[d, dashed, "g'"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "n'"] & C \arrow[r, "h"] \arrow[d, dashed, "h'"] & \Sigma A \arrow[d, "\Sigma l"] \\
 & \Sigma A' \arrow[r, equal] & \Sigma A' \arrow[r, "-\Sigma l'"] & \Sigma M
\end{tikzcd}$$

In particular, if the suspension functor Σ is an equivalence, then $\mathcal{C}$ is a triangulated category. A *left triangulated category* $(\mathcal{C}, \Omega, \lhd)$ can be defined dually, with $\Omega : \mathcal{C} \to \mathcal{C}$ being called the *loop functor*, and $\lhd$ the class of *left triangles*.

2.1.2. Remark. Condition (RTR4) is slightly different from that in [3]. But the following two lemmas are still true.

2.1.3. Lemma. *([1, Lemma 1.3]) Let $\mathcal{C}$ be a right triangulated category, and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ a right triangle. Then the following are true.*

(1) The morphism g is a pseudocokernel of f, and h is a pseudocokernel of g.

(2) If Σ is fully-faithful, then f is a pseudokernel of g, and g is a pseudokernel of h.

2.1.4. Lemma. *([8, Proposition 2.13]) Let $\mathcal{C}$ be a right triangulated category, and (a, b, c) a morphism of right triangles:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A'. \end{tikzcd}$$

If a and b are isomorphisms, then so is c.

2.1.5. Definition. ([9, Definition 3.1]) Let $\mathcal{C}$ be a right triangulated category with the suspension functor Σ, and $f : A \to B$ a morphism in $\mathcal{C}$.

(1) The morphism f is called Σ-*null* if it factors through some object in $\Sigma\mathcal{C}$.

(2) The morphism f is called Σ-*epic* if for any morphism $b : B \to B'$, the composition $bf = 0$ implies b is Σ-null.

For a left triangulated category $\mathcal{C}$ with the loop functor Ω, we can define Ω-null morphisms and Ω-monic morphisms dually.

2.2. Pseudo-triangulated categories. We recall some basics on pseudo-triangulated categories from [9].

2.2.1. Definition. ([9, Definition 3.3]) The sextuple $(\mathcal{C}, \Sigma, \Omega, \rhd, \lhd, \psi)$ is called a *pseudo-triangulated category* if $(\mathcal{C}, \Sigma, \rhd)$ is a right triangulated category, $(\mathcal{C}, \Omega, \lhd)$ is a left triangulated category, and (Ω, Σ) is an adjoint pair with the adjugant $\psi : \mathcal{C}(\Omega C, A) \xrightarrow{\sim} \mathcal{C}(C, \Sigma A)$, moreover, the right triangles and left triangles satisfy the following gluing conditions (G1) and (G2):

(G1) If a morphism $g : B \to C$ is Σ-epic, and $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \in \lhd$, then $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{-\psi(e)} \Sigma A \in \rhd$.

(G2) If a morphism $f : A \to B$ is Ω-monic, and $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \rhd$, then $\Omega C \xrightarrow{-\psi^{-1}(h)} A \xrightarrow{f} B \xrightarrow{g} C \in \lhd$.

2.2.2. Remark. Gluing conditions (G1) and (G2) are slightly different from that in [9]. But it is easy to prove that they are actually the same.

2.2.3. Example. ([9, Example 3.4]) Let $\mathcal{C}$ be an additive category.

(1) The category $\mathcal{C}$ is an abelian category if and only if there exists a pseudo-triangulated structure $(\mathcal{C}, \Sigma, \Omega, \rhd, \lhd, \psi)$ such that $\Sigma = \Omega = 0$.

(2) The category $\mathcal{C}$ is a triangulated category if and only if there exists a pseudo-triangulated structure $(\mathcal{C}, \Sigma, \Omega, \rhd, \lhd, \psi)$ such that Σ is the quasi-inverse of Ω.

2.2.4. Definition. ([9, Definition 4.1]) Let $(\mathcal{C}, \Sigma, \Omega, \rhd, \lhd, \psi)$ be a pseudo-triangulated category. A sequence $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ in $\mathcal{C}$ is called an *extension* if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A \in \rhd$, $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \in \lhd$, and $h = -\psi(e)$.

A *morphism of extensions* from $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ to $\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$ is a triple (a, b, c) such that the following diagram is commutative

$$\begin{tikzcd}
\Omega C \arrow[r, "e"] \arrow[d, "\Omega c"] & A \arrow[r, "f"] \arrow[d, "a"] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
\Omega C' \arrow[r, "e'"] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A'.
\end{tikzcd}$$

Note that $ae = e' \cdot \Omega c$ is equivalent to $\Sigma a \cdot h = h'c$. Thus a morphism of extensions is essentially the same as a morphism in $\rhd$ or in $\lhd$.

2.2.5. Example. (cf. [9, Proposition 4.6]) Let $\mathcal{C}$ be a pseudo-triangulated category.

(1) For any objects $A, B \in \mathcal{C}$, the sequence

$$\Omega B \xrightarrow{0} A \xrightarrow{\binom{1_A}{0}} A \oplus B \xrightarrow{(0, 1_B)} B \xrightarrow{0} \Sigma A$$

is an extension.

(2) If $\mathcal{C}$ is abelian, then an extension is nothing other than a short exact sequence.

(3) If $\mathcal{C}$ is a triangulated category, then an extension is nothing other than a distinguished triangle.

The following lemma will be frequently used in the next section.

2.2.6. Lemma. *([9, Proposition 4.7]) Let* $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, $\Omega B' \xrightarrow{k} A \xrightarrow{l} M \xrightarrow{m} B' \xrightarrow{n} \Sigma A$ *and* $\Omega B \xrightarrow{k'} A' \xrightarrow{l'} M \xrightarrow{m'} B \xrightarrow{n'} \Sigma A'$ *be three extensions with* $m'l = f$. *Then there exist two morphisms* $g' : B' \to C$ *and* $h' : C \to \Sigma A'$ *such that the following diagram is commutative and the fourth column is an extension.*

$$\begin{tikzcd}
 & & \Omega B \arrow[r, "\Omega g"] \arrow[d, "k'"] & \Omega C \arrow[d, "-\psi^{-1}(h')"] & \\
 & & A' \arrow[r, equal] \arrow[d, "l'"] & A' \arrow[d, "f'"] & \\
\Omega B' \arrow[r, "k"] \arrow[d] & A \arrow[r, "l"] \arrow[d, equal] & M \arrow[r, "m"] \arrow[d, "m'"] & B' \arrow[r, "n"] \arrow[d, dashed, "g'"] & \Sigma A \arrow[d, equal] \\
\Omega C \arrow[r, "e"] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "n'"] & C \arrow[r, "h"] \arrow[d, dashed, "h'"] & \Sigma A \arrow[d, "\Sigma l"] \\
 & & \Sigma A' \arrow[r, equal] & \Sigma A' \arrow[r, "-\Sigma l'"] & \Sigma M
\end{tikzcd} \tag{2.1}$$

2.2.7. Remark. In Diagram (2.1), if l' and f are $\mathcal{D}$-monic, then f' is also $\mathcal{D}$-monic.

Proof. For any morphism $a : A' \to D$, where $D \in \mathcal{D}$, we need to show that a factors through f'. Since l' is $\mathcal{D}$-monic, there exists a morphism $b : M \to D$ such that $a = bl'$. Since f is $\mathcal{D}$-monic, there exists a morphism $c : B \to D$ such that $bl = cf$. Thus $(b - cm')l = bl - cf = 0$. There exists a morphism $d : B' \to D$ such that $b - cm' = dm$. Hence $a = bl' = bl' - cm'l' = dml' = df'$. ∎

Now we will give some properties of Σ-epic morphisms in a pseudo-triangulated category $\mathcal{C}$. The properties of Ω-monic morphisms are dual.

2.2.8. Lemma. *Let $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C$ be a left triangle. Then the following statements are equivalent.*

(1) The morphism g is Σ-epic.

(2) The sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{-\psi(e)} \Sigma A$ is a right triangle.

(3) The sequence $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{-\psi(e)} \Sigma A$ is an extension.

Proof. (1)$\Rightarrow$ (2) follows from gluing condition (G1). By the definition of an extension we get (2)$\Leftrightarrow$(3). It remains to show (2)$\Rightarrow$ (1). Since $B \xrightarrow{g} C \xrightarrow{-\psi(e)} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$ is a right triangle, we get that g is Σ-epic by Lemma 2.1.3(1). ∎

2.2.9. Lemma. *Let $f : A \to B$ be a morphism in $\mathcal{C}$. Then The following statements are equivalent.*

(1) The morphism f is Σ-epic.

(2) For any right triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$, there exists an object $C' \in \mathcal{C}$ such that $C \cong \Sigma C'$.

(3) There exists a right triangle $A \xrightarrow{f} B \xrightarrow{g'} \Sigma C' \xrightarrow{h'} \Sigma A$.

Proof. (1)$\Rightarrow$ (2). Let $\Omega B \xrightarrow{d} C' \xrightarrow{e} A \xrightarrow{f} B$ be a left triangle. Since f is Σ-epic, $C' \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{-\psi(d)} \Sigma C'$ is a right triangle by Lemma 2.2.8. Now $A \xrightarrow{f} B \xrightarrow{-\psi(d)} \Sigma C' \xrightarrow{-\Sigma e} \Sigma A$ is a right triangle by (RTR2). So $C \cong \Sigma C'$ by Lemma 2.1.4. (2)$\Rightarrow$ (3) and (3)$\Rightarrow$ (1) are trivial. ∎

2.2.10. Lemma. *Let $f : A \to B$, $g : B \to C$ and $h : A \to C$ be morphisms in $\mathcal{C}$ such that $h = gf$.*

(1) If h is Σ-epic, then so is g;

(2) If f and g are Σ-epic, then so is h.

Proof. For (1), see [9, Lemma 4.4]. Now we prove (2). Since $g : B \to C$ is Σ-epic, there exists a right triangle $L \xrightarrow{f'} B \xrightarrow{g} C \xrightarrow{h'} \Sigma L$ by Lemma 2.2.8. By (RTR1) and (RTR4), we get the following commutative diagram

$$\begin{tikzcd}
& L \arrow[r, equal] \arrow[d, "f'"] & L \arrow[d, "a"] & \\
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g'"] \arrow[d, "g"] & M \arrow[r, "h''"] \arrow[d, "b"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "h"] & C \arrow[r, "b'"] \arrow[d, "h'"] & N \arrow[r, "c'"] \arrow[d, "c"] & \Sigma A \\
& \Sigma L \arrow[r, equal] & \Sigma L &
\end{tikzcd}$$

where the third column and the middle two rows are right triangles. Since $f : A \to B$ is Σ-epic, there exists an isomorphism $m : M \xrightarrow{\sim} \Sigma M'$ with $M' \in \mathcal{C}$ by Lemma 2.2.9. Note that $ma : L \to \Sigma M'$ is Σ-epic by definition. There exists a right triangle $L \xrightarrow{ma} \Sigma M' \xrightarrow{m'} \Sigma N' \xrightarrow{n'} \Sigma L$ by Lemma 2.2.9. By (RTR3) and Lemma 2.1.4 there exists an isomorphism $n : N \xrightarrow{\sim} \Sigma N'$ such that the following diagram is commutative.

$$\begin{tikzcd}
L \arrow[r, "a"] \arrow[d, equal] & M \arrow[r, "b"] \arrow[d, "m", "\simeq"'] & N \arrow[r, "c"] \arrow[d, dashed, "n", "\simeq"'] & \Sigma L \arrow[d, equal] \\
L \arrow[r, "ma"] & \Sigma M' \arrow[r, "m'"] & \Sigma N' \arrow[r, "n'"] & \Sigma L
\end{tikzcd}$$

By Lemma 2.2.9 again, h is Σ-epic. ■

2.2.11. Definition. Let $\mathcal{C}$ be a pseudo-triangulated category, and $\mathcal{D} \subseteq \mathcal{Z}$ be two subcategories of $\mathcal{C}$. The pair $(\mathcal{Z}, \mathcal{Z})$ is called a $\mathcal{D}$-*mutation pair* if it satisfies:

(1) For any object $X \in \mathcal{Z}$, there exists an extension $\Omega Y \xrightarrow{e} X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} \Sigma X$ such that $Y \in \mathcal{Z}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation.

(2) For any object $Y \in \mathcal{Z}$, there exists an extension $\Omega Y \xrightarrow{e} X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} \Sigma X$ such that $X \in \mathcal{Z}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation.

2.2.12. Definition. Let $\mathcal{C}$ be a pseudo-triangulated category. A subcategory $\mathcal{Z}$ of $\mathcal{C}$ is said to be *extension-closed* if for any extension in $\mathcal{C}$

$$\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X \tag{2.2}$$

$X, Z \in \mathcal{Z}$ implies $Y \in \mathcal{Z}$.

For an extension (2.2), if $X, Y, Z \in \mathcal{Z}$, we simply say the extension is in $\mathcal{Z}$.

Let $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ be an extension in $\mathcal{C}$, then h is a pseudocokernel of g and e is a pseudokernel of f by Lemma 2.1.3 and its dual. But g may be not a pseudokernel of h and f may be not a pseudocokernel of e. Thus we define a critical assumption below. In the rest of this article, we will work with pseudo-triangulated categories satisfying this particular assumption.

2.2.13. Assumption. For $\Omega C \xrightarrow{e} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ and $\Omega C' \xrightarrow{e'} A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} \Sigma A'$ any two extensions in the pseudo-triangulated category $\mathcal{C}$, the following two conditions hold.

(A1) If a morphism $c : C \to C'$ satisfies $h'c = 0$ and $cg = 0$, then there exists a morphism $c' : C \to B'$ such that $g'c' = c$.

(A2) If a morphism $a : A \to A'$ satisfies $f'a = 0$ and $ae = 0$, then there exists a morphism $a' : B \to A'$ such that $a'f = a$.

2.2.14. Remark. Assumption 2.2.13 is trivially true for both triangulated category and abelian category. In fact, if $\mathcal{C}$ is an abelian category, then g is epic so that $cg = 0$ implies that $c = 0$, thus we can take $c' = 0$ in (A1). Similarly we obtain that $a = 0$ and we can take $a' = 0$ in (A2).

3. Main results

Throughout this section we assume that $\mathcal{C}$ is a pseudo-triangulated category satisfying Assumption 2.2.13 and $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair.

3.1. Quotient categories of pseudo-triangulated categories. Consider the quotient category $\mathcal{Z}/\mathcal{D}$, whose objects are objects of $\mathcal{Z}$ and given two objects X and Y, the set of morphisms $(\mathcal{Z}/\mathcal{D})(X, Y)$ is defined as the quotient group $\mathcal{Z}(X, Y)/[\mathcal{D}](X, Y)$, where $[\mathcal{D}](X, Y)$ is the subgroup of morphisms from X to Y factoring through some object in $\mathcal{D}$. For any morphism $f : X \to Y$ in $\mathcal{Z}$, we denote by $\underline{f}$ the image of f under the quotient functor $\mathcal{Z} \to \mathcal{Z}/\mathcal{D}$.

3.1.1. Lemma. *Let*

$$\begin{array}{ccccccccc}
\Omega Z & \xrightarrow{e} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
\downarrow{\scriptstyle \Omega z_i} & & \downarrow{\scriptstyle x_i} & & \downarrow{\scriptstyle y_i} & & \downarrow{\scriptstyle z_i} & & \downarrow{\scriptstyle \Sigma x_i} \\
\Omega Z' & \xrightarrow{e'} & X' & \xrightarrow{f'} & D' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'
\end{array} \qquad (i=1,2)$$

be morphisms of extensions in $\mathcal{Z}$, where $D' \in \mathcal{D}$ and f is $\mathcal{D}$-monic, $i = 1, 2$. Then $\underline{x_1} = \underline{x_2}$ implies that $\underline{z_1} = \underline{z_2}$.

Proof. Since $\underline{x_1} = \underline{x_2}$, there exist morphisms $a_1 : X \to D$ and $a_2 : D \to X'$ such that $x_1 - x_2 = a_2 a_1$, where $D \in \mathcal{D}$. Since f is $\mathcal{D}$-monic, there exists a morphism $a_3 : Y \to D$ such that $a_1 = a_3 f$. Thus $(x_1 - x_2)e = a_2 a_1 e = a_2 a_3 f e = 0$. Then $\Sigma(x_1 - x_2) \cdot h = -\Sigma(x_1 - x_2)(\psi(e)) = -\psi((x_1 - x_2)e) = 0$. Note that $(y_1 - y_2 - f' a_2 a_3) f = (y_1 - y_2) f - f'(x_1 - x_2) = 0$, there exists a morphism $d : Z \to D'$ such that $y_1 - y_2 - f' a_2 a_3 = dg$. Now $(z_1 - z_2 - g'd)g = g'(y_1 - y_2) - g'(y_1 - y_2 - f' a_2 a_3) = 0$, and $h'(z_1 - z_2 - g'd) = h'(z_1 - z_2) = \Sigma(x_1 - x_2) \cdot h = 0$. By (A1), there exists a morphism $d' : Z \to D'$ such that $z_1 - z_2 - g'd = g'd'$. So $z_1 - z_2 = g'(d + d')$ and $\underline{z_1} = \underline{z_2}$. ∎

3.1.2. Lemma. *Let* $\Omega Y \xrightarrow{e} X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} \Sigma X$ *and* $\Omega Y' \xrightarrow{e'} X \xrightarrow{f'} D' \xrightarrow{g'} Y' \xrightarrow{h'} \Sigma X$ *be two extensions in* $\mathcal{Z}$*, where* f *and* f' *are left* $\mathcal{D}$*-approximations. Then* Y *and* Y' *are isomorphic in* $\mathcal{Z}/\mathcal{D}$*.*

Proof. Since f and f' are left $\mathcal{D}$-approximations, we obtain the following commutative diagram

$$\begin{tikzcd}
\Omega Y \arrow[r, "e"] \arrow[d, "\Omega y"] & X \arrow[r, "f"] \arrow[d, equal] & D \arrow[r, "g"] \arrow[d, "d"] & Y \arrow[r, "h"] \arrow[d, "y"] & \Sigma X \arrow[d, equal] \\
\Omega Y' \arrow[r, "e'"] \arrow[d, "\Omega y'"] & X \arrow[r, "f'"] \arrow[d, equal] & D' \arrow[r, "g'"] \arrow[d, "d'"] & Y' \arrow[r, "h'"] \arrow[d, "y'"] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r, "e"] & X \arrow[r, "f"] & D \arrow[r, "g"] & Y \arrow[r, "h"] & \Sigma X.
\end{tikzcd}$$

By Lemma 3.1.1, we get $\underline{y'y} = \underline{1_Y}$. Similarly, we can show that $\underline{yy'} = \underline{1_{Y'}}$. Hence Y and Y' are isomorphic in $\mathcal{Z}/\mathcal{D}$. ∎

For any object $X \in \mathcal{Z}$, by the definition of a $\mathcal{D}$-mutation pair, there exists an extension

$$\Omega TX \xrightarrow{e} X \xrightarrow{f} D \xrightarrow{g} TX \xrightarrow{h} \Sigma X \tag{$*$}$$

where $TX \in \mathcal{Z}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation. By Lemma 3.1.2, TX is unique up to isomorphism in the quotient category $\mathcal{Z}/\mathcal{D}$. So for any object $X \in \mathcal{Z}$, we fix an extension as in $(*)$. For any morphism $x : X \to X'$ in $\mathcal{Z}$, since f is a left $\mathcal{D}$-approximation, we can complete the following commutative diagram:

$$\begin{tikzcd}
\Omega TX \arrow[r, "e"] \arrow[d, dashed, "\Omega y"] & X \arrow[r, "f"] \arrow[d, "x"] & D \arrow[r, "g"] \arrow[d, dashed, "d"] & TX \arrow[r, "h"] \arrow[d, dashed, "y"] & \Sigma X \arrow[d, "\Sigma x"] \\
\Omega TX' \arrow[r, "e'"] & X' \arrow[r, "f'"] & D' \arrow[r, "g'"] & TX' \arrow[r, "h'"] & \Sigma X'.
\end{tikzcd}$$

We define a functor $T : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ by setting $T(X) = TX$ on the objects X of $\mathcal{Z}/\mathcal{D}$ and $T(\underline{x}) = \underline{y}$ on the morphisms $\underline{x} : X \to X'$ of $\mathcal{Z}/\mathcal{D}$. By Lemma 3.1.1, $T(\underline{x})$ is well defined and T is an additive functor.

3.1.3. Lemma. *The functor* $T : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ *is an equivalence.*

Proof. For any object $Y \in \mathcal{Z}$, we fix an extension $\Omega Y \xrightarrow{e} T'Y \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} \Sigma T'Y$, where $T'Y \in \mathcal{Z}$, f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation. We can similarly define an additive functor $T' : \mathcal{Z}/\mathcal{D} \to \mathcal{Z}/\mathcal{D}$ by $T'(Y) = T'Y$. It is easy to check that $T'T \cong id$ and $TT' \cong id$. Thus T is an equivalence. ∎

3.2. Triangles on the quotient categories. Let $\Omega Z \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{x} \Sigma X$ be an extension in $\mathcal{Z}$, where v is $\mathcal{D}$-monic. Then we can obtain the following commutative diagram.

$$\begin{tikzcd}
\Omega Z \arrow[r, "u"] \arrow[d, "\Omega z"] & X \arrow[r, "v"] \arrow[d, equal] & Y \arrow[r, "w"] \arrow[d, "y"] & Z \arrow[r, "x"] \arrow[d, "z"] & \Sigma X \arrow[d, equal] \\
\Omega TX \arrow[r, "e"] & X \arrow[r, "f"] & D \arrow[r, "g"] & TX \arrow[r, "h"] & \Sigma X
\end{tikzcd}$$

The sequence $X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{z} TX$ is called a *standard triangle* in $\mathcal{Z}/\mathcal{D}$. We define $\triangle$ to be the class of distinguished triangles which are isomorphic to standard triangles.

3.2.1. Lemma. *Let $v : X \to Y$ be a morphism in $\mathcal{Z}$. If $\mathcal{Z}$ is extension-closed, then there exists an extension*

$$\Omega Z \xrightarrow{u} X \xrightarrow{\binom{v}{f}} Y \oplus D \xrightarrow{(w,d)} Z \xrightarrow{x} \Sigma X$$

in $\mathcal{Z}$, which induces a distinguished triangle $X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{z} TX$ in $\mathcal{Z}/\mathcal{D}$.

Proof. Let $\Omega TX \xrightarrow{e} X \xrightarrow{f} D \xrightarrow{g} TX \xrightarrow{h} \Sigma X$ be the extension given by the mutation pair, where f is a left $\mathcal{D}$-approximation and g is a right $\mathcal{D}$-approximation. The dual of Lemma 2.2.8 implies that f is Ω-monic. Since $(0, 1_D)\binom{v}{f} = f$, we get that $\binom{v}{f}$ is also Ω-monic by the dual of Lemma 2.2.10(1). Thus we obtain an extension $\Omega Z \xrightarrow{u} X \xrightarrow{\binom{v}{f}} Y \oplus D \xrightarrow{(w,d)} Z \xrightarrow{x} \Sigma X$. By Lemma 2.2.6, there exist two morphisms $z : Z \to TX$ and $a : TX \to \Sigma Y$ such that the following diagram is commutative and the fourth column is an extension.

$$\begin{tikzcd}
 & & \Omega D \arrow[r, "\Omega g"] \arrow[d, "0"] & \Omega TX \arrow[d, "-\psi^{-1}(a)"] & \\
 & & Y \arrow[r, equal] \arrow[d, "\binom{1_Y}{0}"] & Y \arrow[d, "w"] & \\
\Omega Z \arrow[r, "u"] \arrow[d] & X \arrow[r, "\binom{v}{f}"] \arrow[d, equal] & Y \oplus D \arrow[r, "(w,d)"] \arrow[d, "(0,1_D)"] & Z \arrow[r, "x"] \arrow[d, dashed, "z"] & \Sigma X \arrow[d, equal] \\
\Omega TX \arrow[r, "e"] & X \arrow[r, "f"] & D \arrow[r, "g"] \arrow[d, "0"] & TX \arrow[r, "h"] \arrow[d, dashed, "a"] & \Sigma X \arrow[d, "\Sigma\binom{v}{f}"] \\
 & & \Sigma Y \arrow[r, equal] & \Sigma Y \arrow[r, "-\Sigma\binom{1_Y}{0}"] & \Sigma(Y \oplus D)
\end{tikzcd} \tag{3.1}$$

Since $Y, TX \in \mathcal{Z}$ and $\mathcal{Z}$ is extension-closed, we get $Z \in \mathcal{Z}$. The morphism f is $\mathcal{D}$-monic implies that $\binom{v}{f}$ is also $\mathcal{D}$-monic. Thus $X \xrightarrow{\binom{v}{f}} Y \oplus D \xrightarrow{(w,d)} Z \xrightarrow{z} TX$ is a standard triangle. So $X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{z} TX$ is a distinguished triangle. ∎

3.2.2. LEMMA. *Let*

$$\begin{tikzcd}
\Omega Z \arrow[r, "u"] \arrow[d, "\Omega c"] & X \arrow[r, "v"] \arrow[d, "a"] & Y \arrow[r, "w"] \arrow[d, "b"] & Z \arrow[r, "x"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
\Omega Z' \arrow[r, "u'"] & X' \arrow[r, "v'"] & Y' \arrow[r, "w'"] & Z' \arrow[r, "x'"] & \Sigma X'
\end{tikzcd}$$

be a morphism of extensions in $\mathcal{Z}$, where v and v' are $\mathcal{D}$-monic. Then we have a morphism of standard triangles in $\mathcal{Z}/\mathcal{D}$:

$$\begin{tikzcd}
X \arrow[r, "\underline{v}"] \arrow[d, "\underline{a}"] & Y \arrow[r, "\underline{w}"] \arrow[d, "\underline{b}"] & Z \arrow[r, "\underline{z}"] \arrow[d, "\underline{c}"] & TX \arrow[d, "T\underline{a}"] \\
X' \arrow[r, "\underline{v'}"] & Y' \arrow[r, "\underline{w'}"] & Z' \arrow[r, "\underline{z'}"] & TX'.
\end{tikzcd}$$

PROOF. Let $T\underline{a} = \underline{p}$. By the definition of standard triangles and the definition of the functor T we have the following two commutative diagrams

$$\begin{tikzcd}
\Omega Z \arrow[r, "u"] \arrow[d, "\Omega z"] & X \arrow[r, "v"] \arrow[d, equal] & Y \arrow[r, "w"] \arrow[d, "y"] & Z \arrow[r, "x"] \arrow[d, "z"] & \Sigma X \arrow[d, equal] \\
\Omega TX \arrow[r, "e"] \arrow[d, "\Omega Ta"] & X \arrow[r, "f"] \arrow[d, "a"] & D \arrow[r, "g"] \arrow[d, "d"] & TX \arrow[r, "h"] \arrow[d, "p"] & \Sigma X \arrow[d, "\Sigma a"] \\
\Omega TX' \arrow[r, "e'"] & X' \arrow[r, "f'"] & D' \arrow[r, "g'"] & TX' \arrow[r, "h'"] & \Sigma X',
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega Z \arrow[r, "u"] \arrow[d, "\Omega c"] & X \arrow[r, "v"] \arrow[d, "a"] & Y \arrow[r, "w"] \arrow[d, "b"] & Z \arrow[r, "x"] \arrow[d, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
\Omega Z' \arrow[r, "u'"] \arrow[d, "\Omega z'"] & X' \arrow[r, "v'"] \arrow[d, equal] & Y' \arrow[r, "w'"] \arrow[d, "y'"] & Z' \arrow[r, "x'"] \arrow[d, "z'"] & \Sigma X' \arrow[d, equal] \\
\Omega TX' \arrow[r, "e'"] & X' \arrow[r, "f'"] & D' \arrow[r, "g'"] & TX' \arrow[r, "h'"] & \Sigma X'.
\end{tikzcd}$$

By Lemma 3.1.1, we get $T\underline{a} \cdot \underline{z} = \underline{p} \cdot \underline{z} = \underline{z'} \cdot \underline{c}$. ■

3.2.3. LEMMA. *Let $\Omega Z \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{x} \Sigma X$ and $\Omega Z' \xrightarrow{u'} X \xrightarrow{\binom{v}{f}} Y \oplus D \xrightarrow{(w',g)} Z' \xrightarrow{x'} \Sigma X$ be two extensions in $\mathcal{Z}$, where v is $\mathcal{D}$-monic and f is a left $\mathcal{D}$-approximation. Then we have an isomorphism of distinguished triangles in $\mathcal{Z}/\mathcal{D}$:*

$$\begin{tikzcd}
X \arrow[r, "\underline{v}"] \arrow[d, equal] & Y \arrow[r, "\underline{w'}"] \arrow[d, equal] & Z' \arrow[r, "\underline{z'}"] \arrow[d, "\underline{c'}"] & TX \arrow[d, equal] \\
X \arrow[r, "\underline{v}"] & Y \arrow[r, "\underline{w}"] & Z \arrow[r, "\underline{z}"] & TX.
\end{tikzcd}$$

PROOF. By Lemma 2.2.6, we have the following commutative diagram

$$\begin{tikzcd}
 & & \Omega Y \arrow[r, "\Omega w"] \arrow[d, "0"] & \Omega Z \arrow[d, "-\psi^{-1}(h)"] & \\
 & & D \arrow[r, equal] \arrow[d, "\binom{0}{1_D}"] & D \arrow[d, "g"] & \\
\Omega Z' \arrow[r, "u'"] \arrow[d] & X \arrow[r, "\binom{v}{f}"] \arrow[d, equal] & Y \oplus D \arrow[r, "(w',g)"] \arrow[d, "(1_Y,0)"] & Z' \arrow[r, "x'"] \arrow[d, "c'"] & \Sigma X \arrow[d, equal] \\
\Omega Z \arrow[r, "u"] & X \arrow[r, "v"] & Y \arrow[r, "w"] \arrow[d, "0"] & Z \arrow[r, "x"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma\binom{v}{f}"] \\
 & & \Sigma D \arrow[r, equal] & \Sigma D \arrow[r, "-\Sigma\binom{0}{1_D}"] & \Sigma(Y \oplus D)
\end{tikzcd}$$

where the fourth column is an extension. Since v is $\mathcal{D}$-monic, there exists a morphism $y : Y \to D$ such that $f = yv$. Thus we have the following commutative diagram

$$\begin{tikzcd}
\Omega Z' \arrow[r, "u"] \arrow[d, "\Omega c"] & X \arrow[r, "v"] \arrow[d, equal] & Y \arrow[r, "w"] \arrow[d, "\binom{1_Y}{y}"] & Z \arrow[r, "x"] \arrow[d, "c"] & \Sigma X \arrow[d, equal] \\
\Omega Z \arrow[r, "u'"] \arrow[d, "\Omega c'"] & X \arrow[r, "\binom{v}{f}"] \arrow[d, equal] & Y \oplus D \arrow[r, "(w',g)"] \arrow[d, "(1_Y,0)"] & Z' \arrow[r, "x'"] \arrow[d, "c'"] & \Sigma X \arrow[d, equal] \\
\Omega Z' \arrow[r, "u"] & X \arrow[r, "v"] & Y \arrow[r, "w"] & Z \arrow[r, "x"] & \Sigma X.
\end{tikzcd}$$

Since $(1_Y, 0)\binom{1_Y}{y} = 1_Y$, we get that $c'c$ is an isomorphism by Lemma 2.1.4. On the other hand, since (w', g) is a pseudocokernel of $\binom{v}{f}$ and $(y, -1_D)\binom{v}{f} = 0$, there exists a morphism $d : Z' \to D$ such that $(y, -1_D) = d(w', g)$. Thus $y = dw'$ and $dg = -1_D$. Note that c' is a pseudocokernel of g and $(1_{Z'} + gd)g = g - g = 0$, there exists a morphism $c'' : Z \to Z'$ such that $c''c' = 1_{Z'} + gd$. So $\underline{c''c'} = \underline{1_{Z'}}$. Therefore, $\underline{c'}$ is an isomorphism in $\mathcal{Z}/\mathcal{D}$. The lemma holds by Lemma 3.2.2. ∎

3.3. MAIN THEOREM. Now we can state and prove our main theorem.

3.3.1. THEOREM. *Let $\mathcal{C}$ be a pseudo-triangulated category satisfying Assumption 2.2.13. If $(\mathcal{Z}, \mathcal{Z})$ is a $\mathcal{D}$-mutation pair and $\mathcal{Z}$ is extension-closed, then $(\mathcal{Z}/\mathcal{D}, T, \triangle)$ is a triangulated category.*

PROOF. We will check that the distinguished triangles in $\triangle$ satisfy the axioms of triangulated categories.

(TR1) For any morphism $v : X \to Y$, there is a distinguished triangle $X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{z} TX$ by Lemma 3.2.1. It is easy to see that $\Omega 0 \to X \xrightarrow{1_X} X \to 0 \to \Sigma X$ is an extension and 1_X is $\mathcal{D}$-monic. Thus $X \xrightarrow{1_X} X \to 0 \to TX \in \Delta$.

(TR2) We only need to consider the standard triangles. By Lemma 3.2.3, we assume that $X \xrightarrow{\underline{v}} Y \xrightarrow{\underline{w}} Z \xrightarrow{\underline{z}} TX$ is a distinguished triangle induced by the extension $\Omega Z \xrightarrow{u} X \xrightarrow{\binom{v}{f}} Y \oplus D \xrightarrow{(w,d)} Z \xrightarrow{x} \Sigma X$ in $\mathcal{Z}$, where $f : X \to D$ is a left $\mathcal{D}$-approximation. By Diagram (3.1), we get an extension $\Omega TX \xrightarrow{-\psi^{-1}(a)} Y \xrightarrow{w} Z \xrightarrow{z} TX \xrightarrow{a} \Sigma Y$ in $\mathcal{Z}$. Since $\binom{1_Y}{0}$ and f are $\mathcal{D}$-monic, we obtain that w is $\mathcal{D}$-monic by Remark 2.2.7. Let $\Omega TY \xrightarrow{e_Y} Y \xrightarrow{f_Y} D_Y \xrightarrow{g_Y} TY \xrightarrow{h_Y} \Sigma Y$ be the extension given by the mutation pair, where f_Y is a left $\mathcal{D}$-approximation and g_Y is a right $\mathcal{D}$-approximation. The following commutative diagram

$$\begin{tikzcd}
\Omega TX \arrow[r, "-\psi^{-1}(a)"] \arrow[d, "\Omega z'"] & Y \arrow[r, "w"] \arrow[d, equal] & Z \arrow[r, "z"] \arrow[d, "y'"] & TX \arrow[r, "a"] \arrow[d, "z'"] & \Sigma Y \arrow[d, equal] \\
\Omega TY \arrow[r, "e_Y"] & Y \arrow[r, "f_Y"] & D_Y \arrow[r, "g_Y"] & TY \arrow[r, "h_Y"] & \Sigma Y
\end{tikzcd}$$

shows that $Y \xrightarrow{\underline{w}} Z \xrightarrow{\underline{z}} TX \xrightarrow{\underline{z'}} TY$ is a standard triangle. It remains to show that $\underline{z'} = -T\underline{v}$. The commutative Diagram (3.1) implies the following commutative diagram

$$\begin{tikzcd}
\Omega TX \arrow[r, "e"] \arrow[d, "-1_{\Omega TX}"] & X \arrow[r, "f"] \arrow[d, "v"] & D \arrow[r, "g"] \arrow[d, "-d"] & TX \arrow[r, "h"] \arrow[d, "-1_{TX}"] & \Sigma X \arrow[d, "\Sigma v"] \\
\Omega TX \arrow[r, "-\psi^{-1}(a)"] & Y \arrow[r, "w"] & Z \arrow[r, "z"] & TX \arrow[r, "a"] & \Sigma Y.
\end{tikzcd}$$

Composing the above two commutative diagrams, we obtain the following commutative diagram

$$\begin{tikzcd}
\Omega TX \arrow[r, "e"] \arrow[d, "-\Omega z'"] & X \arrow[r, "f"] \arrow[d, "v"] & D \arrow[r, "g"] \arrow[d, "-y'd"] & TX \arrow[r, "h"] \arrow[d, "-z'"] & \Sigma X \arrow[d, "\Sigma v"] \\
\Omega TY \arrow[r, "e_Y"] & Y \arrow[r, "f_Y"] & D_Y \arrow[r, "g_Y"] & TY \arrow[r, "h_Y"] & \Sigma Y,
\end{tikzcd}$$

which implies that $T\underline{v} = -\underline{z'}$.

(TR3) We only need to consider the case of standard triangles. Suppose there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\underline{v}"] \arrow[d, "\underline{a}"] & Y \arrow[r, "\underline{w}"] \arrow[d, "\underline{b}"] & Z \arrow[r, "\underline{z}"] & TX \arrow[d, "T\underline{a}"] \\
X' \arrow[r, "\underline{v'}"] & Y' \arrow[r, "\underline{w'}"] & Z' \arrow[r, "\underline{z'}"] & TX'
\end{tikzcd}$$

with rows being standard triangles. Since $\underline{b}\underline{v} = \underline{v'}\underline{a}$, there exist two morphisms $a_1 : X \to D$ and $a_2 : D \to Y'$ such that $bv - v'a = a_2a_1$, where $D \in \mathcal{D}$. Since v is $\mathcal{D}$-monic, there exists a morphism $a_3 : Y \to D$ such that $a_1 = a_3v$. Thus $(b - a_2a_3)v = bv - a_2a_1 = v'a$. So by (RTR3) there exists a morphism $c : Z \to Z'$ such that the following diagram is commutative

$$\begin{tikzcd}
\Omega Z \arrow[r, "u"] \arrow[d, "\Omega c"] & X \arrow[r, "v"] \arrow[d, "a"] & Y \arrow[r, "w"] \arrow[d, "b-a_2a_3"] & Z \arrow[r, "x"] \arrow[d, dashed, "c"] & \Sigma X \arrow[d, "\Sigma a"] \\
\Omega Z' \arrow[r, "u'"] & X' \arrow[r, "v'"] & Y' \arrow[r, "w'"] & Z' \arrow[r, "x'"] & \Sigma X'.
\end{tikzcd}$$

Hence (TR3) follows from Lemma 3.2.2.

(TR4) Let $X \xrightarrow{\underline{v}} Y \xrightarrow{\underline{w}} Z \xrightarrow{\underline{z}} TX$, $X' \xrightarrow{\underline{v'}} Y \xrightarrow{\underline{w'}} Z' \xrightarrow{\underline{z'}} TX'$ and $X \xrightarrow{\underline{w'v}} Z' \xrightarrow{\underline{q}} Y' \xrightarrow{\underline{t}} TX$ be distinguished triangles. Let $f : X \to D$ be a left $\mathcal{D}$-approximation of X. Since $\begin{pmatrix} w' & 0 \\ 0 & 1_D \end{pmatrix} \begin{pmatrix} v \\ f \end{pmatrix} = \begin{pmatrix} w'v \\ f \end{pmatrix}$, for simplicity we may assume that v,v' and $w'v$ are $\mathcal{D}$-monic by Lemma 3.2.3. Now we may assume that the above three distinguished triangles are induced by the following three extensions $\Omega Z \xrightarrow{u} X \xrightarrow{v} Y \xrightarrow{w} Z \xrightarrow{x} \Sigma X$, $\Omega Z' \xrightarrow{u'} X' \xrightarrow{v'} Y' \xrightarrow{w'} Z' \xrightarrow{x'} \Sigma X'$ and $\Omega Y' \xrightarrow{n} X \xrightarrow{w'v} Z' \xrightarrow{q} Y' \xrightarrow{r} \Sigma X$. By Lemma 2.2.6, we get the following commutative diagram

$$\begin{tikzcd}
 & & \Omega Z' \arrow[r, "\Omega q"] \arrow[d, "u'"] & \Omega Y' \arrow[d, "n'"] & \\
 & & X' \arrow[r, equal] \arrow[d, "v'"] & X' \arrow[d, "p'"] & \\
\Omega Z \arrow[r, "u"] \arrow[d, "\Omega q'"] & X \arrow[r, "v"] \arrow[d, equal] & Y \arrow[r, "w"] \arrow[d, "w'"] & Z \arrow[r, "x"] \arrow[d, dashed, "q'"] & \Sigma X \arrow[d, equal] \\
\Omega Y' \arrow[r, "n"] & X \arrow[r, "w'v"] & Z' \arrow[r, "q"] \arrow[d, "x'"] & Y' \arrow[r, "r"] \arrow[d, dashed, "r'"] & \Sigma X \arrow[d, "\Sigma v"] \\
 & & \Sigma X' \arrow[r, equal] & \Sigma X' \arrow[r, "-\Sigma v'"] & \Sigma Y,
\end{tikzcd}$$

where the fourth column is an extension. Since v' and $w'v$ are $\mathcal{D}$-monic, we get that p' is $\mathcal{D}$-monic too by Remark 2.2.7. Thus by Lemma 3.2.2 we get the following commutative diagram:

$$\begin{tikzcd}
 & X' \arrow[r, equal] \arrow[d, "\underline{v'}"] & X' \arrow[d, "\underline{p'}"] & \\
X \arrow[r, "\underline{v}"] \arrow[d, equal] & Y \arrow[r, "\underline{w}"] \arrow[d, "\underline{w'}"] & Z \arrow[r, "\underline{z}"] \arrow[d, "\underline{q'}"] & TX \arrow[d, equal] \\
X \arrow[r, "\underline{w'v}"] & Z' \arrow[r, "\underline{q}"] \arrow[d, "\underline{z'}"] & Y' \arrow[r, "\underline{t}"] \arrow[d, "\underline{t'}"] & TX \\
 & TX' \arrow[r, equal] & TX', &
\end{tikzcd}$$

with rows and columns being standard triangles. It remains to show that the following diagram is commutative:

$$
\begin{tikzcd}
Y' \arrow[r, "\underline{t}"] \arrow[d, "\underline{t}'"'] & TX \arrow[d, "T\underline{v}"] \\
TX' \arrow[r, "-T\underline{v}'"] & TY
\end{tikzcd}
\tag{3.2}
$$

We first claim that there exist morphisms of extensions

$$
\begin{tikzcd}
\Omega Y' \arrow[r, "n"] \arrow[d, "\Omega b"] & X \arrow[r, "w'v"] \arrow[d, "v"] & Z' \arrow[r, "q"] \arrow[d, "a"] & Y' \arrow[r, "r"] \arrow[d, "b"] & \Sigma X \arrow[d, "\Sigma v"] \\
\Omega TY \arrow[r, "e'"] & Y \arrow[r, "f'"] & D' \arrow[r, "g'"] & TY \arrow[r, "h'"] & \Sigma Y
\end{tikzcd}
\tag{3.3}
$$

and

$$
\begin{tikzcd}
\Omega Y' \arrow[r, "n'"] \arrow[d, "\Omega b'"] & X' \arrow[r, "p'"] \arrow[d, "v'"] & Z \arrow[r, "q'"] \arrow[d, "a'"] & Y' \arrow[r, "r'"] \arrow[d, "b'"] & \Sigma X' \arrow[d, "\Sigma v'"] \\
\Omega TY \arrow[r, "e'"] & Y \arrow[r, "f'"] & D' \arrow[r, "g'"] & TY \arrow[r, "h'"] & \Sigma Y,
\end{tikzcd}
\tag{3.4}
$$

such that $f' = aw' + a'w$, $\underline{b} = T\underline{v} \cdot \underline{t}$ and $\underline{b'} = T\underline{v'} \cdot \underline{t'}$.

In fact, since $w'v$ is $\mathcal{D}$-monic, there exists a morphism $a : Z' \to D'$ such that $f'v = aw'v$. Then by (RTR3) there exists a morphism $b : Y' \to TY$ such that Diagram (3.3) is commutative. Because $(f' - aw')v = 0$, there exists a morphism $a' : Z \to D'$ such that $a'w = f' - aw'$. Thus $f' = a'w + aw'$ and $f'v' = a'wv' + aw'v' = a'p'$. Then by (RTR3) there exists a morphism $b' : Y' \to TY$ such that Diagram (3.4) is commutative. By the construction of a standard triangle, we have the following commutative diagram

$$
\begin{tikzcd}
\Omega Y' \arrow[r, "n"] \arrow[d, "\Omega t"] & X \arrow[r, "w'v"] \arrow[d, equal] & Z' \arrow[r, "q"] \arrow[d, "s"] & Y' \arrow[r, "r"] \arrow[d, "t"] & \Sigma X \arrow[d, equal] \\
\Omega TX \arrow[r, "e"] & X \arrow[r, "f"] & D \arrow[r, "g"] & TX \arrow[r, "h"] & \Sigma X.
\end{tikzcd}
$$

On the other hand, letting $T\underline{v} = \underline{l}$, we have the following commutative diagram

$$
\begin{tikzcd}
\Omega TX \arrow[r, "e"] \arrow[d, "\Omega l"] & X \arrow[r, "f"] \arrow[d, "v"] & D \arrow[r, "g"] \arrow[d, "d"] & TX \arrow[r, "h"] \arrow[d, "l"] & \Sigma X \arrow[d, "\Sigma v"] \\
\Omega TY \arrow[r, "e'"] & Y \arrow[r, "f'"] & D' \arrow[r, "g'"] & TY \arrow[r, "h'"] & \Sigma Y.
\end{tikzcd}
$$

Composing the last two diagrams, we immediately obtain the following commutative diagram

$$
\begin{tikzcd}
\Omega Y' \arrow[r, "n"] \arrow[d, "\Omega(lt)"] & X \arrow[r, "w'v"] \arrow[d, "v"] & Z' \arrow[r, "q"] \arrow[d, "ds"] & Y' \arrow[r, "r"] \arrow[d, "lt"] & \Sigma X \arrow[d, "\Sigma v"] \\
\Omega TY \arrow[r, "e'"] & Y \arrow[r, "f'"] & D' \arrow[r, "g'"] & TY \arrow[r, "h'"] & \Sigma Y.
\end{tikzcd}
\tag{3.5}
$$

Comparing Diagram (3.3) and Diagram (3.5), we get $\underline{b} = \underline{l} \cdot \underline{t} = T\underline{v} \cdot \underline{t}$ by Lemma 3.1.1. We can similarly show $\underline{b'} = T\underline{v'} \cdot \underline{t'}$.

Now we can show Diagram (3.2) is commutative. We note that $h'(b+b') = \Sigma v \cdot r + \Sigma v' \cdot r' = 0$ and $(b+b')q'w = bqw' + b'q'w = g'aw' + g'a'w = g'f' = 0$. Since w, q' are Σ-epic, so is $q'w$ by Lemma 2.2.10(2). Thus we obtain an extension $\Omega Y' \xrightarrow{\alpha} Y'' \xrightarrow{\beta} Y \xrightarrow{q'w} Y' \xrightarrow{\gamma} \Sigma Y''$ by Lemma 2.2.8. So $b+b'$ factors through D' by (A1). Then $T\underline{v} \cdot \underline{t} + T\underline{v'} \cdot \underline{t'} = \underline{b} + \underline{b'} = 0$. ∎

3.4. Examples. Since triangulated categories and abelian categories are pseudo-triangulated categories, and Assumption 2.2.13 is trivially true for both cases, we can apply Theorem 3.3.1 to several situations.

3.4.1. Example. ([6, Theorem 4.2]) Let $(\mathcal{Z}, \mathcal{Z})$ be a $\mathcal{D}$-mutation pair in a triangulated category $\mathcal{C}$. If $\mathcal{Z}$ is extension-closed, then the quotient $\mathcal{Z}/\mathcal{D}$ is a triangulated category. We note that our definition of mutation pair is weaker than Iyama-Yoshino's definition ([6, Definition 2.5]), since we do not need to require $\mathcal{D}$ to be a rigid subcategory.

3.4.2. Example. ([5, Theorem 2.6]) Let $\mathcal{C}$ be an abelian category and $(\mathcal{B}, \mathcal{S})$ a Frobenius subcategory. Then the quotient $\mathcal{B}/\mathcal{I}$ is a triangulated category, where $\mathcal{I}$ is the subcategory of $\mathcal{B}$ consisting of all $\mathcal{S}$-injectives.

Proof. Note that $\mathcal{B}$ is an extension-closed subcategory of $\mathcal{C}$, by Theorem 3.3.1 we only need to show that $(\mathcal{B}, \mathcal{B})$ is an $\mathcal{I}$-mutation pair. In fact, for any object $X \in \mathcal{B}$, since $\mathcal{B}$ has enough $\mathcal{S}$-injectives, there exists a short exact sequence $0 \to X \xrightarrow{f} I \xrightarrow{g} Y \to 0$ in $\mathcal{S}$, where $I \in \mathcal{I}$. It is easy to check that f is a left $\mathcal{I}$-approximation. Since the $\mathcal{S}$-projectives coincide with the $\mathcal{S}$-injectives, g is a right $\mathcal{I}$-approximation by definition. The second condition can be showed similarly. ∎

3.4.3. Example. ([2, Theorem 7.2]) Let $\mathcal{C}$ be a triangulated category, and $\mathcal{E}$ a proper class of distinguished triangles on $\mathcal{C}$ (see [2, Definition 2.2]), which is closed under translations and satisfies the analogous formal properties of a proper class of short exact sequences in an exact category. An object $I \in \mathcal{C}$ is called an *$\mathcal{E}$-injective*, if for any distinguished triangle $A \to B \to C \to \Sigma A$ in $\mathcal{E}$, the induced sequence $0 \to \mathcal{C}(C, I) \to \mathcal{C}(B, I) \to \mathcal{C}(A, I) \to 0$ is exact. Denote by $\mathcal{I}$ the full subcategory of $\mathcal{C}$ consisting of $\mathcal{E}$-injective objects. We say that $\mathcal{C}$ *has enough $\mathcal{E}$-injectives* if for any object $A \in \mathcal{C}$ there exists a distinguished triangle $A \to I \to C \to \Sigma A$ in $\mathcal{C}$ with $I \in \mathcal{I}$. If $\mathcal{C}$ has enough $\mathcal{E}$-injectives and enough $\mathcal{E}$-projectives and $\mathcal{I} = \mathcal{P}$, where $\mathcal{P}$ is the subcategory of $\mathcal{E}$-projectives, then it is easy to see that $(\mathcal{C}, \mathcal{C})$ is an $\mathcal{I}$-mutation pair, thus $\mathcal{C}/\mathcal{I}$ is a triangulated category by Theorem 3.3.1. We remark that $\mathcal{C}(\mathcal{I}, \Sigma\mathcal{I})$ is not zero because $\mathcal{I}$ is closed under Σ. So $(\mathcal{C}, \mathcal{C})$ is not an $\mathcal{I}$-mutation pair in the sense of Iyama-Yoshino.

3.4.4. Example. ([7, Theorem 2.3]) Let $\mathcal{C}$ be a triangulated category with a Serre functor S, and $\mathcal{X}$ a functorially finite subcategory with $\tau\mathcal{X} = \mathcal{X}$, where τ is the AR translation determined by S. Let $X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} \Sigma X$ be a distinguished triangle in $\mathcal{C}$, then by [7, Lemma 2.2], f is a left $\mathcal{X}$-approximation if and only if g is a right $\mathcal{X}$-approximation. Thus it is easy to see that $(\mathcal{C}, \mathcal{C})$ is a $\mathcal{X}$-mutation pair. So the quotient $\mathcal{C}/\mathcal{X}$ is a triangulated category by Theorem 3.3.1. We remark that $(\mathcal{C}, \mathcal{C})$ may be not a $\mathcal{X}$-mutation pair in the sense of Iyama-Yoshino. For example, if $\mathcal{C}$ is a cluster category, then $\mathcal{C}(\mathcal{X}, \Sigma\mathcal{X})$ may be non-zero.

3.4.5. Example. ([9, Theorem 6.17]) Let $\mathcal{C}$ be a pseudo-triangulated category and $\mathcal{Z}$ an extension-closed subcategory. A morphism $f : X \to Y$ (resp. $g : Y \to Z$) in $\mathcal{Z}$ is called an *inflation* (resp. *deflation*) if there exists an extension $\Omega Z \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\mathcal{Z}$. Let $\mathcal{D}$ be a subcategory of $\mathcal{Z}$. An object I in $\mathcal{D}$ is *injective* if $\mathcal{Z}(Y, I) \xrightarrow{\mathcal{Z}(f,I)} \mathcal{Z}(X, I) \to 0$ is exact for any inflation $f : X \to Y$. The full subcategory of $\mathcal{D}$ consisting of injectives is denoted by $\mathcal{I}_{\mathcal{D}}$. We say $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ has *enough injectives* if for any object $X \in \mathcal{Z}$, there exists an inflation $f : X \to I$ such that $I \in \mathcal{I}_{\mathcal{D}}$. We say $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is *Frobenius* if it has enough injectives and projectives, and the injectives coincide with the projectives.

If $(\mathcal{C}, \mathcal{Z}, \mathcal{D})$ is Frobenius and $\mathcal{C}$ satisfies Assumption 2.2.13, then it is easy to check that $(\mathcal{Z}, \mathcal{Z})$ is an $\mathcal{I}_{\mathcal{D}}$-mutation pair, thus the quotient $\mathcal{Z}/\mathcal{I}_{\mathcal{D}}$ is a triangulated category by Theorem 3.3.1.

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