PROFINITE TOPOLOGICAL SPACES

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ABSTRACT. It is well known [Hoc69, Joy71] that profinite T_0-spaces are exactly the spectral spaces. We generalize this result to the category of all topological spaces by showing that the following conditions are equivalent:

(1) (X, τ) is a profinite topological space.

(2) The T_0-reflection of (X, τ) is a profinite T_0-space.

(3) (X, τ) is a quasi spectral space (in the sense of [BMM08]).

(4) (X, τ) admits a stronger Stone topology π such that (X, τ, π) is a bitopological quasi spectral space (see Definition 6.1).

1. Introduction

A topological space is *profinite* if it is (homeomorphic to) the inverse limit of an inverse system of finite topological spaces. It is well known [Hoc69, Joy71] that profinite T_0-spaces are exactly the spectral spaces. This can be seen as follows. A direct calculation shows that the inverse limit of an inverse system of finite T_0-spaces is spectral. Conversely, by [Cor75], the category **Spec** of spectral spaces and spectral maps is isomorphic to the category **Pries** of Priestley spaces and continuous order preserving maps. This isomorphism is a restriction of a more general isomorphism between the category **StKSp** of stably compact spaces and proper maps and the category **Nach** of Nachbin spaces and continuous order preserving maps [GHKLMS03]. Priestley spaces are exactly the profinite objects in **Nach**, and the proof of this fact is a straightforward generalization of the proof that Stone spaces are profinite objects in the category of compact Hausdorff spaces and continuous maps. Consequently, spectral spaces are profinite objects in **StKSp**. From this it follows that spectral spaces are profinite objects in the category of T_0-spaces and continuous maps.

We aim to generalize these results to characterize all profinite topological spaces. For this we first note that Nachbin spaces are partially ordered compact spaces such that the order is closed. It is natural, as Nachbin himself did in [Nac65], to work more generally with preordered Nachbin spaces. As it follows from [Dia07, Sec. 2] (see also [JS08, Sec. 1],

and more generally, [Hof02, Sec. 4]), the fact that Priestley spaces are exactly the profinite objects in **Nach** generalizes to the setting of preordered spaces. Consequently, preordered Priestley spaces are exactly the profinite objects in the category **PNach** of preordered Nachbin spaces. Associating with each preordered Nachbin space the topology of open upsets defines a functor K from **PNach** to the category **Top** of all topological spaces and continuous maps. It follows from [HST14, Sec. III.5.3] that $K : \mathbf{PNach} \to \mathbf{Top}$ has a left adjoint $M : \mathbf{Top} \to \mathbf{PNach}$. From this we derive that a topological space is profinite iff it is the K-image of a preordered Priestley space.

To characterize the K-image of **PNach** in **Top**, note that the isomorphism between **StKSp** and **Nach** generalizes as follows. Dropping the T_0 separation axiom from the definition of a stably compact space results in the concept of a quasi stably compact space, which generalizes the concept of a quasi spectral space of [BMM08], and corresponds to that of a representable space of [HST14, Sec. III.5.7]. However, unlike the case with stably compact spaces, the patch topology of a quasi stably compact space may not be compact Hausdorff. In order to obtain a category isomorphic to **PNach**, we have to introduce bitopological analogues of quasi stably compact spaces, which together with the quasi stably compact topology also carry a stronger compact Hausdorff topology. The same way **PNach** is isomorphic to the category of Eilenberg-Moore algebras for the ultrafilter monad on the category of preordered sets [Tho09], the category **BQStKSp** of bitopological quasi stably compact spaces is isomorphic to the category of Eilenberg-Moore algebras for the ultrafilter monad on **Top**. Consequently, applying [HST14, Cor. III.5.4.2] yields that **PNach** is isomorphic to **BQStKSp**, and this isomorphism restricts to an isomorphism of the category **PPries** of preordered Priestley spaces and the category **BQSpec** of bitopological quasi spectral spaces. This characterizes the spaces in the K-image of **PNach** as the bitopological quasi stably compact spaces, and the spaces in the K-image of **PPries** as the bitopological quasi spectral spaces. Thus, profinite topological spaces are exactly the bitopological quasi spectral spaces.

Our main result establishes that each quasi stably compact space (X, τ) can be made into a bitopological quasi stably compact space (X, τ, π), which will become a bitopological quasi spectral space if (X, τ) is quasi spectral. Since a space is quasi spectral iff its T_0-reflection is spectral, we obtain that a topological space is profinite iff so is its T_0-reflection. This yields that for a topological space (X, τ), the following conditions are equivalent:

(1) (X, τ) is a profinite topological space.

(2) The T_0-reflection of (X, τ) is a profinite T_0-space.

(3) (X, τ) is a quasi spectral space.

(4) (X, τ) admits a stronger Stone topology π such that (X, τ, π) is a bitopological quasi spectral space.

2. Preliminaries

In this section we recall the basic topological notions that are used in the paper. We start by the following well-known definition.

2.1. DEFINITION. *Let X be a topological space.*

(1) *A subset A of X is* saturated *if it is an intersection of open subsets of X, and it is* irreducible *if from $A = F \cup G$, with F and G closed, it follows that $A = F$ or $A = G$.*

(2) *The space X is* sober *if each irreducible closed set of X is the closure of a point of X.*

(3) *The space X is* stable *if compact saturated subsets of X are closed under finite intersections, and* coherent *if compact open subsets of X are closed under finite intersections and form a basis for the topology.*

(4) *The space X is* locally compact *if for each $x \in X$ and each open neighborhood U of x, there exist an open set V and a compact set K such that $x \in V \subseteq K \subseteq U$.*

(5) *The space X is* stably compact *if X is a compact, locally compact, stable, sober T_0-space.*

(6) *The space X is* spectral *if X is a compact, coherent, sober T_0-space.*

2.2. REMARK. It is customary to call a space X sober if each irreducible closed set is the closure of a unique point. This stronger definition of soberness automatically implies that the space X is T_0. Since we will work with non-T_0-spaces, we prefer the weaker definition of soberness given above, and will explicitly add the T_0 separation axiom when needed. In [HST14, Sec. III.5.6] this weaker condition is referred to as *weakly sober*.

It is well known that each spectral space is stably compact. In fact, stably compact spaces are exactly the retracts of spectral spaces; see [Joh82, Thm. VII.4.6] or [Sim82, Lem. 3.13].

2.3. DEFINITION.

(1) *A continuous map $f : X \to X'$ between stably compact spaces is* proper *if the f-inverse image of a compact saturated subset of X' is compact in X.*

(2) *A continuous map $f : X \to X'$ between spectral spaces is* spectral *if the f-inverse image of a compact open subset of X' is compact in X.*

Let **StKSp** be the category of stably compact spaces and proper maps, and let **Spec** be the category of spectral spaces and spectral maps. It is well known that a continuous map $f : X \to X'$ between spectral spaces is spectral iff it is proper. Thus, **Spec** is a full subcategory of **StKSp**. In fact, **Spec** can be characterized as the profinite objects in **StKSp** or as the profinite objects in the category of T_0-spaces (see [Hoc69, Prop. 10], [Joy71]).

2.4. DEFINITION.

(1) *A preordered space is a triple $(X, \pi, \leq)$, where π is a topology and $\leq$ is a preorder on X (reflexive and transitive relation on X). If $\leq$ is a partial order, then $(X, \pi, \leq)$ is an ordered space.*

(2) *The preordered space $(X, \pi, \leq)$ is a preordered Nachbin space if (X, π) is compact Hausdorff and $\leq$ is closed in the product $X \times X$. If $\leq$ is a partial order, then $(X, \pi, \leq)$ is a Nachbin space.*

(3) *A subset U of X is an upset provided $x \in U$ and $x \leq y$ imply $y \in U$. The concept of a downset is defined dually.*

(4) *The preordered space $(X, \pi, \leq)$ satisfies the Priestley separation axiom if from $x \nleq y$ it follows that there is a clopen upset U containing x and missing y.*

(5) *The preordered space $(X, \pi, \leq)$ is a preordered Priestley space if (X, π) is a Stone space (zero-dimensional compact Hausdorff space) and $(X, \pi, \leq)$ satisfies the Priestley separation axiom. If $\leq$ is a partial order, then $(X, \pi, \leq)$ is a Priestley space.*

Let (X, π) be a compact Hausdorff space. It is well known (see, e.g., [Nac65, Sec. I.1 and I.3] or [BMM02, Prop. 2.3]) that a preorder $\leq$ on X is closed iff for all $x, y \in X$, from $x \nleq y$ it follows that there exist an open upset U and an open downset V such that $x \in U$, $y \in V$, and $U \cap V = \varnothing$. From this it is clear that each preordered Priestley space is a preordered Nachbin space.

Let **PNach** be the category of preordered Nachbin spaces and continuous preorder preserving maps ($x \leq y$ implies $f(x) \leq f(y)$), and let **Nach** be its full subcategory consisting of Nachbin spaces. Let also **PPries** be the full subcategory of **PNach** consisting of preordered Priestley spaces, and **Pries** be the full subcategory of **PPries** consisting of Priestley spaces. It is well known (see, e.g., [Spe72]) that the objects of **Pries** are exactly the profinite objects of **Nach**. This result generalizes to the setting of preordered spaces. Indeed, as follows from [Dia07, Sec. 2] (see also [JS08, Sec. 1], and more generally, [Hof02, Sec. 4]), the objects of **PPries** are exactly the profinite objects of **PNach**.

2.5. REMARK. That Priestley spaces are exactly the profinite objects of **Nach** can be proved directly or by using Priestley duality [Pri70], according to which **Pries** is dually equivalent to the category **Dist** of bounded distributive lattices and bounded lattice homomorphisms. This dual equivalence restricts to the dual equivalence of finite distributive lattices and finite posets. Since **Dist** is locally finite (each finitely generated object in **Dist** is finite), each bounded distributive lattice is the direct limit of a direct system of finite distributive lattices. This, by Priestley duality, yields that each Priestley space is the inverse limit of an inverse system of finite posets. Consequently, Priestley spaces are exactly the profinite objects of **Nach**.
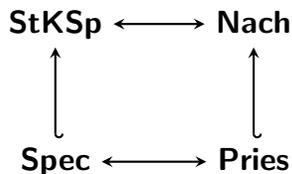
That preordered Priestley spaces are exactly the profinite objects of **PNach** also has a similar alternate proof. As follows from [Bez13, Thm. 5.2], **PPries** is dually equivalent

to the category **BDA** of pairs (B, D), where B is a Boolean algebra and D is a bounded sublattice of B. This dual equivalence restricts to the dual equivalence of finite preorders and finite objects of **BDA**. Since each Boolean algebra and each distributive lattice is the direct limit of a direct system of their finite subobjects, it follows that each $(B, D) \in$ **BDA** is also the direct limit of its finite subobjects in **BDA**. Therefore, by the duality of [Bez13], each preordered Priestley space is the inverse limit of an inverse system of finite preorders. Thus, preordered Priestley spaces are exactly the profinite objects of **PNach**.

The category of stably compact spaces is isomorphic to the category of Nachbin spaces [GHKLMS03], and this isomorphism restricts to an isomorphism between the categories of spectral spaces and Priestley spaces [Cor75]. It is obtained as follows. Let (X, τ) be a stably compact space. The *co-compact topology* τ^k is the topology having compact saturated sets as closed sets, and the *patch topology* π is the smallest topology containing τ and τ^k; that is, $\pi = \tau \vee \tau^k$. The *specialization order* $\leq$ of τ is given by $x \leq y$ iff $x \in \mathsf{cl}_\tau(y)$. This is a partial order because (X, τ) is a T_0-space. Then $(X, \pi, \leq)$ turns out to be a Nachbin space. Moreover, a map $f : X \to X'$ between stably compact spaces (X, τ) and (X', τ') is proper iff it is continuous and order preserving between the corresponding Nachbin spaces $(X, \pi, \leq)$ and $(X', \pi', \leq')$. This defines a functor $F :$ **StKSp** $\to$ **Nach**.

Conversely, if $(X, \pi, \leq)$ is a Nachbin space, then let π^u be the topology of open upsets (and π^d be the topology of open downsets). Then (X, π^u) is a stably compact space (and so is (X, π^d)). Moreover, a map $f : X \to X'$ between Nachbin spaces $(X, \pi, \leq)$ and $(X', \pi', \leq')$ is continuous order preserving iff it is a proper map between the corresponding stably compact spaces (X, π^u) and $(X', (\pi')^u)$. This defines a functor $G :$ **Nach** $\to$ **StKSp**. The functors F, G establish the desired isomorphism between **StKSp** and **Nach**.

This isomorphism restricts to an isomorphism between **Spec** and **Pries**. In fact, if (X, τ) is a spectral space, then the patch topology can alternatively be defined as the topology generated by compact opens of (X, τ) and their complements (which are compact opens in the spectral space (X, τ^k)). As a result, we arrive at the following commutative diagram, where the horizontal arrows indicate an isomorphism of categories, while the vertical ones indicate that one category is a full subcategory of another.

$$
\begin{array}{ccc}
\textbf{StKSp} & \longleftrightarrow & \textbf{Nach} \\
\uparrow & & \uparrow \\
\textbf{Spec} & \longleftrightarrow & \textbf{Pries}
\end{array}
$$

We will see later on how to generalize this commutative diagram to involve **PNach** and **PPries**.

3. Profinite topological spaces

Let $(X, \pi, \leq) \in$ **PNach** and let $\tau := \pi^u$ be the topology of open upsets. Then $(X, \tau) \in$ **Top**. Moreover, if $f : (X, \pi, \leq) \to (X', \pi', \leq')$ is continuous and preorder preserving, then

$f : (X, \tau) \to (X', \tau')$ is continuous. Thus, we have a functor $K : \mathbf{PNach} \to \mathbf{Top}$. It follows from general considerations in [HST14, Cor. III.5.3.4] that $K : \mathbf{PNach} \to \mathbf{Top}$ has a left adjoint $M : \mathbf{Top} \to \mathbf{PNach}$. The description of M can be derived from [HST14, Thm. III.5.3.5 and Ex. III.5.3.7(1)] and is closely related to Salbany's construction [Sal00] (see also [BMM08]). More precisely, let $\beta : \mathbf{Set} \to \mathbf{Set}$ be the *ultrafilter functor* that assigns to a set X the set βX of ultrafilters on X, and to a map $f : X \to Y$ the map $\beta f : \beta X \to \beta Y$ given by

$$(\beta f)(\chi) := \{T \subseteq Y \mid f^{-1}(T) \in \chi\}.$$

For $(X, \tau) \in \mathbf{Top}$, let $M(X, \tau) := (\beta X, \Pi, \le)$. Here Π is the Stone topology on βX given by the basis $\{\varphi(S) \mid S \subseteq X\}$, where $\varphi : \wp X \to \wp \beta X$ is the Stone map

$$\varphi(S) := \{\chi \in \beta X \mid S \in \chi\}.$$

Moreover, for $\chi, \xi \in \beta X$, we set $\chi \le \xi$ iff $\chi \cap \tau \subseteq \xi$. It is a consequence of [BMM08, Thm. 3.8] that $(\beta X, \Pi, \le) \in \mathbf{PPries} \subseteq \mathbf{PNach}$.

For a map $f : X \to X'$, we have $(\beta f)^{-1} \varphi(S) = \varphi(f^{-1}S)$, and so $\beta f : (\beta X, \Pi) \to (\beta X', \Pi')$ is continuous. Furthermore, if $f : (X, \tau) \to (X', \tau')$ is continuous, then $\chi \le \xi$ implies $\beta f(\chi) \le' \beta f(\xi)$. Therefore, $M : \mathbf{Top} \to \mathbf{PNach}$ is a functor.

3.1. PROPOSITION. [HST14, Sec. III.5] *The functor M is left adjoint to K.*

We are ready to prove that profinite topological spaces are exactly the K-images of $\mathbf{PPries}$.

3.2. THEOREM. *A topological space (X, τ) is profinite iff (X, τ) is homeomorphic to an object of $K(\mathbf{PPries})$.*
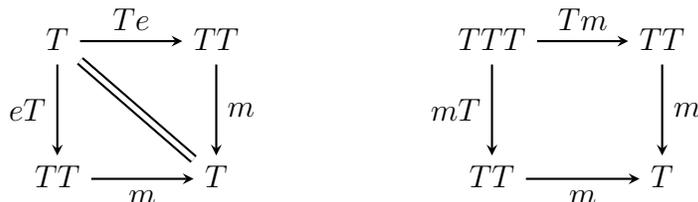
PROOF. First suppose $(X, \pi, \le) \in \mathbf{PPries}$. By [Dia07, Sec. 2], $(X, \pi, \le)$ is an inverse limit of a diagram of finite objects of $\mathbf{PNach}$. Since K has a left adjoint, K preserves limits. Therefore, K preserves the limit of this diagram. As the values of K on finite objects in $\mathbf{PNach}$ are finite topological spaces, we conclude that $K(X, \pi, \le)$ is a profinite topological space.

Conversely, suppose (X, τ) is a profinite topological space. Then (X, τ) is the inverse limit of a diagram $\mathcal{D}$ of finite topological spaces in $\mathbf{Top}$. Applying M to $\mathcal{D}$ produces a diagram $\tilde{\mathcal{D}}$ of finite objects in $\mathbf{PNach}$. Obviously $K(\tilde{\mathcal{D}}) = \mathcal{D}$. Let $(\tilde{X}, \tilde{\pi}, \tilde{\le})$ be the inverse limit of $\tilde{\mathcal{D}}$ in $\mathbf{PNach}$. By [Dia07, Sec. 2], $(\tilde{X}, \tilde{\pi}, \tilde{\le}) \in \mathbf{PPries}$. Since K preserves the limit of $\tilde{\mathcal{D}}$, we have $K(\tilde{X}, \tilde{\pi}, \tilde{\le}) = K(\varprojlim \tilde{\mathcal{D}}) \simeq \varprojlim K(\tilde{\mathcal{D}}) = \varprojlim \mathcal{D} = (X, \tau)$. Thus, (X, τ) is homeomorphic to an object of $K(\mathbf{PPries})$. ∎
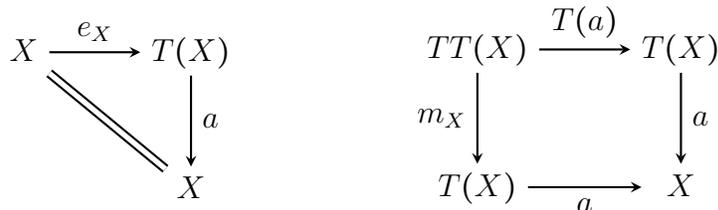
In Section 6 we characterize $K(\mathbf{PPries})$ within $\mathbf{Top}$. This requires preparation, which is the subject of the next section.
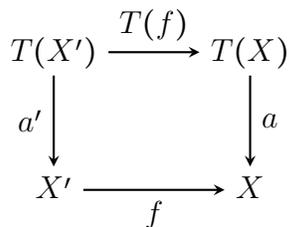
4. Algebras for the ultrafilter monad

We recall that a *monad* $\mathbf{T} = (T, e, m)$ on a category $\mathbf{C}$ consists of an endofunctor $T : \mathbf{C} \to \mathbf{C}$ together with natural transformations $e : \mathrm{Id}_{\mathbf{C}} \to T$ (*unit*) and $m : TT \to T$ (*multiplication*) satisfying $m \circ Te = m \circ eT = \mathrm{Id}_T$ and $m \circ Tm = m \circ mT$.

$$
\begin{array}{ccc}
T & \xrightarrow{Te} & TT \\
{\scriptstyle eT}\downarrow & & \downarrow{\scriptstyle m} \\
TT & \xrightarrow{m} & T
\end{array}
\qquad\qquad
\begin{array}{ccc}
TTT & \xrightarrow{Tm} & TT \\
{\scriptstyle mT}\downarrow & & \downarrow{\scriptstyle m} \\
TT & \xrightarrow{m} & T
\end{array}
$$

A $\mathbf{T}$-*algebra* or an *Eilenberg-Moore algebra* is a pair (X, a), where X is a $\mathbf{C}$-object and $a : T(X) \to X$ is a $\mathbf{C}$-morphism satisfying $a \circ e_X = \mathrm{Id}_X$ and $a \circ T(a) = a \circ m_X$.

$$
\begin{array}{ccc}
X & \xrightarrow{e_X} & T(X) \\
 & & \downarrow{\scriptstyle a} \\
 & & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
TT(X) & \xrightarrow{T(a)} & T(X) \\
{\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle a} \\
T(X) & \xrightarrow{a} & X
\end{array}
$$

A $\mathbf{T}$-*homomorphism* $f : (X', a') \to (X, a)$ is a $\mathbf{C}$-morphism $f : X' \to X$ satisfying $a \circ T(f) = f \circ a'$.

$$
\begin{array}{ccc}
T(X') & \xrightarrow{T(f)} & T(X) \\
{\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\
X' & \xrightarrow{f} & X
\end{array}
$$

It is easy to see that $\mathbf{T}$-algebras form a category, which we denote by $\mathbf{C}^{\mathbf{T}}$.

We are mainly interested in the *ultrafilter monad* $\boldsymbol{\beta} = (\beta, e, m)$. The unit e is given by the embeddings $e_X : X \to \beta X$ assigning to $x \in X$ the principal ultrafilter $e_X(x) \coloneqq \chi_x$; and the multiplication m consists of the maps $m_X : \beta\beta X \to \beta X$ given by

$$
m_X(\Xi) = \{ S \subseteq X \mid \varphi(S) \in \Xi \},
$$

where $\Xi \in \beta\beta X$ and φ is the Stone map.

By Manes' theorem [Man69] (see also [HST14, Thm. III.2.3.3]), the category of $\boldsymbol{\beta}$-algebras is isomorphic to the category of compact Hausdorff spaces and continuous maps. Under this isomorphism, a compact Hausdorff space (X, π) corresponds to the $\boldsymbol{\beta}$-algebra $(X, \lim_\pi)$, where the map $\lim_\pi : \beta X \to X$ assigns to an ultrafilter its unique limit point.

Let **Pre** be the category of preordered sets and preorder preserving maps. The extensions of the monad **β** to **Pre** and **Top** are well studied (see, e.g., [HST14]).

We first consider the extension of **β** to **Pre**. If $\leq$ is a preorder on a set X, then we extend it to βX as follows:

$$\chi \,\overline{\leq}\, \xi \text{ iff } (\forall S \in \chi)(\forall T \in \xi)(\uparrow S \cap T \neq \varnothing).$$

Here $\uparrow S := \{x \in X \mid \exists s \in S \text{ with } s \leq x\}$ and $\downarrow S$ is defined dually. The extension $\overline{\leq}$ can equivalently be defined as follows:

$$\chi \,\overline{\leq}\, \xi \text{ iff } (\forall S \in \chi)(\forall T \in \xi)(S \cap \downarrow T \neq \varnothing).$$

This in turn is equivalent to:

$$\chi \,\overline{\leq}\, \xi \text{ iff } (\forall S)(S \in \xi \Rightarrow \downarrow S \in \chi).$$

The last condition shows that the extension of $\leq$ to βX is the well-known ultrafilter extension in modal logic (see, e.g., [BRV01, Sec. 2.5]). It is straightforward to see that $\overline{\leq}$ is a preorder on βX. Moreover, if $f : X \to X'$ is preorder preserving, then so is $\beta f : \beta X \to \beta X'$. We thus obtain an extension $\bar{\beta} : \textbf{Pre} \to \textbf{Pre}$ of the functor β to the category of preorders.

To extend further the monad structure, we must show that for a preorder X, the maps $e_X : X \to \beta X$ and $m_X : \beta \beta X \to \beta X$ preserve the corresponding preorders. This however holds more generally for functors satisfying the Beck-Chevalley condition (see [HST14, Thm. III.1.12.1]); that the latter condition is satisfied by β is proved in [HST14, Ex. III.1.12.3(3)].

We thus obtain a monad $\bar{\boldsymbol{\beta}}$ on **Pre**. The structure of a $\bar{\boldsymbol{\beta}}$-algebra on a preorder X amounts to a preorder preserving map $\lim_\pi : \beta X \to X$, which by Manes' theorem gives a compact Hausdorff topology π on X. As is discussed in [HST14, Ex. III.5.2.1(3)], the map $\lim_\pi$ is preorder preserving precisely if the preorder is π-closed in the product $X \times X$. Furthermore, morphisms of such algebras are simply continuous preorder preserving maps. This yields the following theorem:

4.1. THEOREM. (see, e.g., [HST14, Ex. III.5.2.1(3)]) *The category* **Pre**$^{\bar{\boldsymbol{\beta}}}$ *of* $\bar{\boldsymbol{\beta}}$*-algebras on the category of preorders is concretely isomorphic to* **PNach**.

We next turn to the extension of the monad **β** to **Top**.

4.2. DEFINITION. ([Sal00]) *For a topology τ on a set X let $\bar{\tau}$ be the topology on βX given by the basis $\{\varphi(U) \mid U \in \tau\}$, where $\varphi : \wp X \to \wp \beta X$ is the Stone map. For a topological space (X, τ), let $\bar{\beta}(X, \tau) := (\beta X, \bar{\tau})$.*

It is easy to check that if $f : (X, \tau) \to (X', \tau')$ is continuous, then so is $\beta f : (\beta X, \bar{\tau}) \to (\beta X', \bar{\tau}')$. Therefore, we obtain an endofunctor $\bar{\beta} : \textbf{Top} \to \textbf{Top}$. As explained in [HST14, Sec. III.5.6], the maps $e_X : (X, \tau) \to (\beta X, \bar{\tau})$ and $m_X : (\beta \beta X, \bar{\bar{\tau}}) \to (\beta X, \bar{\tau})$ are continuous, and so the endofunctor $\bar{\beta}$ extends to a monad $\bar{\boldsymbol{\beta}}$ on **Top**.

In order to obtain an analogue of Theorem 4.1 for **Top**$^{\bar{\boldsymbol{\beta}}}$, we require some general definitions and facts from [HST14], together with formulations of the particular cases in the topological setting.

4.3. DEFINITION. *A preorder-enriched category is a pair* $(\mathbf{C}, \le)$*, where* $\mathbf{C}$ *is a category and for each pair of objects* X, Y *in* $\mathbf{C}$*, the hom-set* $\mathbf{C}(X, Y)$ *is equipped with a preorder* $\le$ *so that* $g \le g'$ *implies* $g \circ f \le g' \circ f$ *and* $h \circ g \le h \circ g'$ *for all* $f : X' \to X$*,* $g, g' : X \to Y$*, and* $h : Y \to Y'$.

4.4. NOTATION. For brevity, we will abuse notation and call preorder-enriched categories simply *preordered categories*.

We will be concerned with two examples of preordered categories. First, we view **Pre** as a preordered category by equipping the sets $\mathbf{Pre}((X, \le), (X', \le'))$ with pointwise preorders; that is, for preorder preserving maps $f, g : X \to X'$, we define $f \le g$ provided $f(x) \le' g(x)$ for all $x \in X$. Secondly, **Top** can be viewed as a preordered category as follows. For $f, g \in \mathbf{Top}((X, \tau), (X', \tau'))$, define $f \le g$ provided $f(x) \le_{\tau'} g(x)$ for all $x \in X$, where $\le_{\tau'}$ is the specialization preorder of (X', τ').

4.5. DEFINITION. *For preordered categories* $(\mathbf{C}, \le)$*,* $(\mathbf{C}', \le')$*, a functor* $F : \mathbf{C} \to \mathbf{C}'$ *is preorder preserving provided for all objects* X, Y *of* $\mathbf{C}$*, the map* $F_{X,Y} : \mathbf{C}(X, Y) \to \mathbf{C}'(FX, FY)$ *is preorder preserving. A monad on a preordered category is* preorder preserving *provided its underlying endofunctor is preorder preserving.*

As follows from [HST14, Sec. III.5.1, III.5.4, and III.5.6], the monad $\bar{\boldsymbol{\beta}}$ is preorder preserving on both **Pre** and **Top**.

4.6. DEFINITION. *A preorder preserving monad* $\mathbf{T} = (T, e, m)$ *on a preordered category* $(\mathbf{C}, \le)$ *is of Kock-Zöberlein type (KZ-monad) if for every object* X *the morphism* $m_X : TTX \to TX$ *is left adjoint to* $e_{TX} : TX \to TTX$. *Since* $m_X \circ e_{TX}$ *is the identity morphism on* TX*, this amounts to requiring the inequality* $\mathrm{Id}_{TTX} \le e_{TX} \circ m_X$ *in each* $\mathbf{C}(TTX, TTX)$*. Dually, a* co-KZ-monad *is the one for which* m_X *is* right *adjoint to* e_{TX}*, i. e. the opposite inequality* $e_{TX} \circ m_X \le \mathrm{Id}_{TTX}$ *holds for all objects* X.

This is a particular case of [Koc95], where (among many other things) it is proved that for a KZ-monad (resp., co-KZ-monad) $\mathbf{T}$ on a preordered category $\mathbf{C}$, if a morphism $a : TX \to X$ defines a $\mathbf{T}$-algebra structure on X, then it is a left (resp., right) adjoint retraction for $e_X : X \to TX$. In particular, any two $\mathbf{T}$-algebra structures $a_1, a_2 : TX \to X$ on an object X are $\le$-equivalent (meaning that $a_1 \le a_2 \le a_1$ in $\mathbf{C}(TX, X)$).

It follows from [HST14, Thm. III.5.4.1] that the monad $\bar{\boldsymbol{\beta}}$ on **Top** is a co-KZ-monad (the preorder used in [HST14] is opposite to the preorder we use). In particular, if a topological space (X, τ) admits a $\bar{\boldsymbol{\beta}}$-algebra structure, then $e_X : X \to \bar{\beta}X$ has a right adjoint (with respect to the specialization preorders on X and βX). Spaces with this property are called *representable* in [HST14]. It follows from [HST14, Sec. III.5.7] that representable spaces have the same features as stably compact spaces, with the only exception that in general they are not T_0-spaces.

4.7. DEFINITION. *We call a topological space* X quasi stably compact *if it is compact, locally compact, stable, and sober. Let* **QStKSp** *be the category of quasi stably compact spaces and proper maps.*

4.8. REMARK. In presence of local compactness, the conditions of being compact, stable, and sober can be replaced by a single condition of being supersober, where we recall (see [GHKLMS03, Def. VI-6.12] or [Sal00]) that a topological space (X, τ) is *supersober* or *strongly sober* provided for each ultrafilter ∇ on X, the set $\bigcap \{\mathsf{cl}_\tau(A) \mid A \in \nabla\}$ of limit points of ∇ is the closure of a point of X.[1]

4.9. REMARK. Clearly a quasi stably compact space is stably compact iff it is a T_0-space. In fact, a topological space X is quasi stably compact iff its T_0-reflection X_0 is stably compact, where we recall that the T_0-reflection is the quotient space of X by the equivalence relation $\sim$ given by $x \sim y$ iff $\mathsf{cl}(x) = \mathsf{cl}(y)$. This can be seen by observing that X has property $\mathcal{P}$ iff X_0 has property $\mathcal{P}$, where $\mathcal{P}$ is being compact, locally compact, stable, or sober.

4.10. REMARK. In view of the previous remarks, it would be natural to use the term 'stably compact' in the general case, avoiding the prefix 'quasi.' This is what Salbany does in [Sal00, p. 483]. In the T_0-case we could then use the term 'stably compact T_0-space.' However, since it is already established to assume T_0 in the definition of stably compact, we opted to add the prefix 'quasi' in the non-T_0-case.

By [HST14, Thm. III.5.7.2], a topological space is quasi stably compact iff it is representable. We recall that a continuous map $f : (X, \tau) \to (X', \tau')$ between representable spaces is a *pseudo-homomorphism* if for some adjoints α of $e_X : X \to \beta X$ and α' of $e_{X'} : X' \to \beta X'$, there is a $\leq$-equivalence $f \circ \alpha \simeq \alpha' \circ \beta f$. As pointed out in [HST14, Def. III.5.4.3(2)], this condition does not depend on the particular choice of adjoints α and α' (i.e. replacing "some" by "any" above would give an equivalent definition). By [HST14, Prop. III.5.7.6], a continuous map between representable spaces is a pseudo-homomorphism iff it is a proper map. Consequently, we obtain the following theorem:

4.11. THEOREM. ([HST14, Sec. III.5.7]) *The category of representable spaces and pseudo-homomorphisms is concretely isomorphic to* **QStKSp**.

As we saw, if we have a $\bar{\boldsymbol{\beta}}$-algebra structure on a topological space (X, τ), then the topology is a quasi stably compact topology. But more is true.

4.12. LEMMA. *If a topological space (X, τ) has a $\bar{\boldsymbol{\beta}}$-algebra structure, then (X, τ) admits a stronger compact Hausdorff topology π such that the following conditions are satisfied:*

(1) (X, τ) *is a quasi stably compact space;*

(2) (X, π) *is a compact Hausdorff space;*

(3) $\tau \subseteq \pi$;

(4) *compact saturated sets in (X, τ) are closed in (X, π).*

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[1]We note that in [GHKLMS03, Def. VI-6.12] it is assumed that the point is unique, and hence the space is T_0. We do not assume uniqueness because we do not have the T_0 separation axiom.

PROOF. The structure of a $\bar{\boldsymbol{\beta}}$-algebra on a topological space (X, τ) gives in particular the structure of a $\boldsymbol{\beta}$-algebra on the set X, so applying Manes' theorem yields a compact Hausdorff topology π on X, determined by the map $\lim_\pi : \beta X \to X$. That this is not only a $\boldsymbol{\beta}$-algebra structure but also a $\bar{\boldsymbol{\beta}}$-algebra structure means that $\lim_\pi : (\beta X, \bar\tau) \to (X, \tau)$ is continuous. Since $\bar{\boldsymbol{\beta}}$ is a co-KZ-monad, $\lim_\pi$ is a right adjoint to $e_{(X,\tau)} : (X, \tau) \to (\beta X, \bar\tau)$. As noted above, this implies that (X, τ) is a representable space, hence a quasi stably compact space. Therefore, Conditions (1) and (2) are satisfied.

To verify Condition (3), observe that it is equivalent to showing that $\lim_\pi(\chi)$ is a τ-limit point of χ for any ultrafilter $\chi \in \beta X$. This amounts to showing that any τ-neighborhood of $\lim_\pi(\chi)$ belongs to χ; in other words, that $e_X(\lim_\pi(\chi)) \cap \tau \subseteq \chi$. But this means that $e_X(\lim_\pi(\chi)) \leq_{\bar\tau} \chi$, where $\leq_{\bar\tau}$ is the specialization preorder of $\bar\tau$, which holds for any χ since $\lim_\pi$ is adjoint to e_X.

It remains to verify Condition (4). Observe that $m_X = \lim_\Pi : \bar{\boldsymbol{\beta}}\bar{\boldsymbol{\beta}}(X, \tau) \to \bar{\boldsymbol{\beta}}(X, \tau)$ equips $(\beta X, \bar\tau)$ with a $\bar{\boldsymbol{\beta}}$-algebra structure, and $\lim_\pi$ is a homomorphism of $\bar{\boldsymbol{\beta}}$-algebras. In particular, $\lim_\pi$ is a homomorphism of $\boldsymbol{\beta}$-algebras, and hence $\lim_\pi : (\beta X, \Pi) \to (X, \pi)$ is a continuous map. On the other hand, since $\lim_\pi \circ m_X = \lim_\pi \circ \beta(\lim_\pi)$, m_X is adjoint to $e_{(\beta X, \bar\tau)}$, and $\lim_\pi$ is adjoint to $e_{(X,\tau)}$, we conclude that $\lim_\pi$ is a pseudo-homomorphism, hence $\lim_\pi : (\beta X, \bar\tau) \to (X, \tau)$ is a proper map. Now, if K is a compact saturated set in (X, τ), then $\lim_\pi^{-1}(K)$ is compact saturated in $(\beta X, \bar\tau)$ since $\lim_\pi$ is proper. By [BMM08, Thm. 2.12], $\bar\tau$ is the intersection of Π and the Alexandroff topology of $\leq$-upsets, where we recall from Section 3 that $\chi \leq \xi$ iff $\chi \cap \tau \subseteq \xi$. Therefore, $\lim_\pi^{-1}(K)$ is a Π-closed $\leq$-upset. As $\lim_\pi$ is an onto continuous map between compact Hausdorff spaces $(\beta X, \Pi)$ and (X, π), it follows that $\lim_\pi(\lim_\pi^{-1}(K)) = K$ is π-closed. ∎

Lemma 4.12 motivates the following definition.

4.13. DEFINITION. *Let (X, τ, π) be a bitopological space. We call (X, τ, π) a bitopological quasi stably compact space if*

(1) *(X, τ) is a quasi stably compact space;*

(2) *(X, π) is a compact Hausdorff space;*

(3) *$\tau \subseteq \pi$;*

(4) *compact saturated sets in (X, τ) are closed in (X, π).*

*Let **BQStKSp** be the category of bitopological quasi stably compact spaces and bicontinuous maps.*

4.14. REMARK. If (X, τ, π) and (X', τ', π') are bitopological quasi stably compact spaces and $f : (X, \tau, \pi) \to (X', \tau', \pi')$ is bicontinuous, then $f : (X, \tau) \to (X', \tau')$ is proper. To see this, let K be compact saturated in (X', τ'). Then K is closed in (X', π'). Since $f : (X, \pi) \to (X', \pi')$ is continuous, $f^{-1}(K)$ is closed in (X, π). Therefore, $f^{-1}(K)$ is compact in (X, π), hence compact in (X, τ). Thus, f is proper.

4.15. THEOREM. *The category* **Top**$^{\bar{\beta}}$ *of algebras over the monad* $\bar{\beta}$ *on* **Top** *is concretely isomorphic to* **BQStKSp**.

PROOF. By Lemma 4.12, if a topological space (X, τ) has a $\bar{\beta}$-algebra structure, then (X, τ) admits a stronger compact Hausdorff topology π such that (X, τ, π) is a bitopological quasi stably compact space.

Conversely, suppose (X, τ, π) is a bitopological quasi stably compact space. Since (X, π) is compact Hausdorff, by Manes' theorem, $\lim_\pi : (\beta X, \Pi) \to (X, \pi)$ is continuous. From $\tau \subseteq \pi$ it follows that $e_X(\lim_\pi(\chi)) \leq_{\bar{\tau}} \chi$. To see that $\lim_\pi$ is a right adjoint to $e_{(X,\tau)}$, we must show that $\lim_\pi$ is a preorder preserving map from $(\beta X, \overline{\leq_\tau})$ to $(X, \leq_\tau)$, where $\leq_\tau$ is the specialization preorder of τ. This in turn follows from continuity of $\lim_\pi : (\beta X, \overline{\tau}) \to (X, \tau)$. To see the latter, let $\chi \in \beta X$. Set $x := \lim_\pi(\chi)$. Suppose U is a τ-open neighborhood of x. Since (X, τ) is locally compact, there are an open neighborhood V of x and a compact set K with $V \subseteq K \subseteq U$. Since the saturation of K is compact and is contained in U, without loss of generality we may assume that K is compact saturated. We set $U' := \varphi(V)$. Then $x = \lim_\pi(\chi) \in V$ implies $V \in \chi$, so $\chi \in \varphi(V) = U'$, and hence U' is a $\bar{\tau}$-neighborhood of χ. To see that $\lim_\pi(U') \subseteq U$, let $x' \in \lim_\pi(U')$. Then there is $\chi' \in U' = \varphi(V)$ with $\lim_\pi(\chi') = x'$. Therefore, $V \in \chi'$. But $\lim_\pi(\chi') = x'$ yields that x' belongs to the π-closure of V. Since K is compact saturated in (X, τ), by Condition (4), it is π-closed. Thus, $x' \in K \subseteq U$, and hence $\lim_\pi(U') \subseteq U$. Consequently, $\lim_\pi : (\beta X, \bar{\tau}) \to (X, \tau)$ is continuous. This gives that $\lim_\pi$ is a right adjoint to e_X, and so $\lim_\pi$ equips (X, τ) with a $\bar{\beta}$-algebra structure.

Finally, a $\bar{\beta}$-algebra homomorphism from $(X, \tau, \lim_\pi)$ to $(X', \tau', \lim_{\pi'})$ is a continuous map from (X, τ) to (X', τ') compatible with $\lim_\pi$ and $\lim_{\pi'}$. Therefore, it is also a continuous map from (X, π) to (X', π'). Such homomorphisms are precisely the bicontinuous maps between bitopological quasi stably compact spaces. Thus, we indeed obtain the required isomorphism of categories. ∎

4.16. COROLLARY. *The categories* **BQStKSp** *and* **PNach** *are concretely isomorphic.*

PROOF. By [HST14, Cor. III.5.4.2], **Pre**$^{\bar{\beta}}$ and **Top**$^{\bar{\beta}}$ are concretely isomorphic. By Theorem 4.1, **Pre**$^{\bar{\beta}}$ is concretely isomorphic to **PNach**. By Theorem 4.15, **Top**$^{\bar{\beta}}$ is concretely isomorphic to **BQStKSp**. The result follows. ∎

5. More on the isomorphism of **BQStKSp** and **PNach**

In the previous section we deduced the isomorphism of **BQStKSp** and **PNach** from a general result in [HST14, Cor. III.5.4.2]. But this isomorphism can also be seen as a direct generalization of the isomorphism of **StKSp** and **Nach** [GHKLMS03, Prop. VI-6.23]. In this section we give the details of how such a generalization works.

5.1. LEMMA. *Let* (X, τ, π) *be a bitopological quasi stably compact space and let* $\leq$ *be the specialization preorder of* τ. *Then* $(X, \pi, \leq)$ *is a preordered Nachbin space.*

PROOF. Suppose $x \not\leq y$. Then there is $W \in \tau$ with $x \in W$ and $y \notin W$. As (X, τ) is locally compact, there are an open set U and a compact set K in (X, τ) such that $x \in U \subseteq K \subseteq W$. Since the saturation of K is compact and is contained in W, without loss of generality we may assume that K is compact saturated, hence closed in (X, π). Let $V = X - K$. Then $x \in U$, $y \in V$, U is an open upset, V is an open downset in $(X, \pi, \leq)$, and $U \cap V = \varnothing$. Therefore, $U \times V$ is an open neighborhood of (x, y) missing $\leq$. Thus, $\leq$ is closed, and hence $(X, \pi, \leq)$ is a preordered Nachbin space. ∎

5.2. LEMMA. *Suppose $(X, \pi, \leq)$ is a preordered Nachbin space.*

(1) *The specialization preorder of π^u coincides with $\leq$.*

(2) *Compact saturated subsets of (X, π^u) are exactly the closed upsets of $(X, \pi, \leq)$.*

(3) *Compact saturated subsets of (X, π^d) are exactly the closed downsets of $(X, \pi, \leq)$.*

PROOF. (1) Since $\leq$ is closed in $X \times X$, it is clear that $x \leq y$ iff for each open upset U, we have $x \in U$ implies $y \in U$. Thus, the specialization preorder of π^u coincides with $\leq$.

(2) If K is a closed upset in $(X, \pi, \leq)$, then K is compact in (X, π), hence compact in (X, π^u). By (1), it is also saturated in (X, π^u). Therefore, K is compact saturated in (X, π^u). Conversely, suppose that K is compact saturated in (X, π^u). By (1), K is an upset in $(X, \pi, \leq)$. Let $x \notin K$. Then $y \not\leq x$ for each $y \in K$. Since $\leq$ is closed in $X \times X$, for each such y, there are an open upset U_y and an open downset V_y such that $y \in U_y$, $x \in V_y$, and $U_y \cap V_y = \varnothing$. The U_y provide an open cover of K in (X, π^u). Therefore, there is a finite subcover $U_{y_1}, \ldots, U_{y_n}$. Set $U := U_{y_1} \cup \cdots \cup U_{y_n}$ and $V := V_{y_1} \cap \cdots \cap V_{y_n}$. Then $K \subseteq U$, $U \cap V = \varnothing$, and $x \in V$. Thus, $x \in V \subseteq X - K$. This yields that $X - K$ is open, hence K is closed in (X, π).

(3) If $(X, \pi, \leq)$ is a preordered Nachbin space, then so is $(X, \pi, \leq^{\mathrm{op}})$. Now apply (2). ∎

Let $(X, \leq)$ be a preorder. For $S \subseteq X$, we call $m \in S$ a *maximal* point of S if $m \leq s$ and $s \in S$ imply $s \leq m$; *minimal* points are defined dually. Let $\mathsf{max}(S)$ be the set of maximal points and $\mathsf{min}(S)$ the set of minimal points of S. We call S a *chain* provided $s \leq t$ or $t \leq s$ for all $s, t \in S$. The next lemma generalizes [Esa85, Thm. III.2.1].

5.3. LEMMA. *Let $(X, \pi, \leq)$ be a preordered Nachbin space. If F is a closed subset of (X, π), then for each $x \in F$ there exist $m \in \mathsf{max}(F)$ and $l \in \mathsf{min}(F)$ such that $l \leq x \leq m$.*

PROOF. Let $x \in F$. Since $\leq$ is closed in $X \times X$, by [Nac65, Prop. 1], $\uparrow x := \{y \in X \mid x \leq y\}$ is closed, and hence $\uparrow x \cap F$ is closed. Let C be a maximal chain in $\uparrow x \cap F$ starting at x. Then $\{\uparrow c \cap F \mid c \in C\}$ is a family of closed sets with the finite intersection property. Since (X, π) is compact, there is $m \in \bigcap \{\uparrow c \cap F \mid c \in C\}$. Clearly $x \leq m$, and since C is a maximal chain, it follows that $m \in \mathsf{max}(F)$. That there is $l \in \mathsf{min}(F)$ with $l \leq x$ is proved similarly. ∎

5.4. LEMMA. *If $(X, \pi, \leq)$ is a preordered Nachbin space, then (X, π^u) and (X, π^d) are quasi stably compact spaces.*

PROOF. We prove that (X, π^u) is a quasi stably compact space. That (X, π^d) is quasi stably compact is proved by switching to $\leq^{\mathrm{op}}$. Clearly (X, π^u) is compact. By Lemma 5.2(2), compact saturated sets in (X, π^u) are closed upsets in $(X, \pi, \leq)$. From this it is immediate that (X, π^u) is stable. It is well known that in a preordered Nachbin space, if $U \in \pi^u$, then $U = \bigcup\{V \in \pi^u \mid {\uparrow}\mathsf{cl}_\pi(V) \subseteq U\}$, where for $A \subseteq X$, we recall that ${\uparrow}A := \{x \in X \mid a \leq x \text{ for some } a \in A\}$. Since ${\uparrow}\mathsf{cl}_\pi(V)$ is a closed upset (see [Nac65, Prop. 4] or [BMM02, Prop. 2.3]), it is compact saturated in (X, π^u) by Lemma 5.2(2). This yields that (X, π^u) is locally compact. It remains to show that (X, π^u) is sober.

Let F be an irreducible closed set in (X, π^u). Since F is closed in (X, π^u), we see that F is a closed downset in $(X, \pi, \leq)$. By Lemma 5.3, for each $x \in F$, there is $m \in \mathsf{max}(F)$ with $x \leq m$. We show that $F = {\downarrow}m$ for some (and hence all) $m \in \mathsf{max}(F)$. If not, then for each $m \in \mathsf{max}(F)$ there is $n \in \mathsf{max}(F)$ with $n \not\leq m$. As $\leq$ is closed in $X \times X$, there are an open upset U_n and an open downset V_m such that $n \in U_n$, $m \in V_m$, and $U_n \cap V_m = \varnothing$. This yields $U_n \cap {\downarrow}\mathsf{cl}_\pi(V_m) = \varnothing$. The V_m cover F, so by compactness, there are finitely many V_m covering F. Therefore, finitely many ${\downarrow}\mathsf{cl}_\pi(V_m)$ cover F. Since the ${\downarrow}\mathsf{cl}_\pi(V_m)$ are closed in (X, π^u) and F is irreducible, there is one $m \in \mathsf{max}(F)$ with $F \subseteq {\downarrow}\mathsf{cl}_\pi(V_m)$. The obtained contradiction proves that $F = {\downarrow}m$ for each $m \in \mathsf{max}(F)$. Thus, $F = \mathsf{cl}_{\pi^u}(m)$, and hence (X, π^u) is sober. ∎

5.5. THEOREM. *The categories **BQStKSp** and **PNach** are concretely isomorphic.*

PROOF. Define a functor $F : \mathbf{BQStKSp} \to \mathbf{PNach}$ as follows. If (X, τ, π) is a bitopological quasi stably compact space, then set $F(X, \tau, \pi) = (X, \pi, \leq)$, where $\leq$ is the specialization preorder of τ; and if $f : (X, \tau, \pi) \to (X', \tau', \pi')$ is a bicontinuous map between bitopological quasi stably compact spaces, then set $F(f) = f$. By Lemma 5.1, $F(X, \tau, \pi)$ is a preordered Nachbin space. It is obvious that $F(f)$ is continuous and preorder preserving. Thus, F is well-defined.

Define a functor $G : \mathbf{PNach} \to \mathbf{BQStKSp}$ as follows. If $(X, \pi, \leq)$ is a preordered Nachbin space, then set $G(X, \pi, \leq) = (X, \pi^u, \pi)$; and if $f : (X, \pi, \leq) \to (X', \pi', \leq')$ is a continuous preorder preserving map, then set $G(f) = f$. By Lemmas 5.2 and 5.4, $G(X, \pi, \leq)$ is a bitopological quasi stably compact space. It is also obvious that $G(f)$ is bicontinuous. Thus, G is well-defined.

We show that if (X, τ, π) is a bitopological quasi stably compact space, then $\tau = \pi^u$. Since $\tau \subseteq \pi$ and $\leq$ is the specialization preorder of τ, it is clear that $\tau \subseteq \pi^u$. Conversely, suppose $U \in \pi^u$. Let $x \in U$. Then for each $y \notin U$, we have $x \not\leq y$. Therefore, there is $U_y \in \tau$ with $x \in U_y$ and $y \notin U_y$. Since (X, τ) is locally compact, there are open sets V_y and compact saturated sets K_y in (X, τ) such that $x \in V_y \subseteq K_y \subseteq U_y$. Clearly $\bigcap\{K_y \mid y \notin U\} \cap U^c = \varnothing$. As each K_y is closed in (X, π) and (X, π) is compact, there are $V_{y_1}, \ldots, V_{y_n}$ and $K_{y_1}, \ldots, K_{y_n}$ with $x \in V_{y_1} \cap \cdots \cap V_{y_n} \subseteq K_{y_1} \cap \cdots \cap K_{y_n} \subseteq U$. Thus, we found an open neighborhood $V := V_{y_1} \cap \cdots \cap V_{y_n}$ of x in (X, τ) contained in U, so $U \in \tau$, which completes the proof.

It follows that for a bitopological quasi stably compact space (X, τ, π), we have $GF(X, \tau, \pi) = (X, \tau, \pi)$. It also follows from Lemma 5.2(1) that if $(X, \pi, \leq)$ is a preordered Nachbin space, then the specialization preorder of π^u coincides with $\leq$, so $FG(X, \pi, \leq) = (X, \pi, \leq)$. Consequently, the functors F and G establish a concrete isomorphism of **BQStKSp** and **PNach**. ■

5.6. REMARK. If $(X, \pi, \leq)$ is a Nachbin space, then the corresponding bitopological quasi stably compact space (X, π^u, π) yields a stably compact space (X, π^u). Moreover, π is uniquely determined from π^u as the patch topology. Conversely, if (X, τ) is stably compact and π is the patch topology, then (X, τ, π) is a bitopological quasi stably compact space, and the corresponding preordered Nachbin space $(X, \pi, \leq)$ is a Nachbin space since (X, τ) is a T_0-space, so $\leq$ is a partial order. Thus, the isomorphism of Theorem 5.5 restricts to the well-known isomorphism between **StKSp** and **Nach** [GHKLMS03, Prop. VI-6.23].

6. Quasi spectral spaces

Spectral spaces were generalized to quasi spectral spaces in [BMM08]. A topological space is *quasi spectral* if it is a coherent supersober space. Equivalently, X is quasi spectral provided X is compact, coherent, and sober. Consequently, quasi spectral spaces generalize spectral spaces by dropping the T_0-separation axiom. Not surprisingly, a topological space X is quasi spectral iff its T_0-reflection is spectral [BMM08, Thm. 4.6].

Let **QSpec** be the category of quasi spectral spaces and spectral maps. Since each coherent space is locally compact, each quasi spectral space is quasi stably compact. Moreover, the same argument as in the case of spectral spaces gives that a continuous map between quasi spectral spaces is proper iff it is spectral. Therefore, **QSpec** is a full subcategory of **QStKSp**.

An important example of a quasi spectral space is given by $(\beta X, \bar{\tau})$ from Definition 4.2. Indeed, as follows from [Sal00, Sec. 2] (see also [BMM08, Prop. 4.2]), for an arbitrary topological space (X, τ), the space $(\beta X, \bar{\tau})$ is quasi spectral and $e_X : X \to \beta X$ is an embedding. By [Sal00, Prop. 3], if (X, τ) is quasi stably compact, then there is a (not necessarily unique) retraction $r_X : \beta X \to X$ such that $r_X \circ e_X = \mathrm{Id}_X$. Consequently, quasi stably compact spaces are precisely the retracts of quasi spectral spaces. This provides a generalization of the well-known characterization of stably compact spaces as retracts of spectral spaces; see [Joh82, Thm. VII.4.6] or [Sim82, Lem. 3.13].

We next introduce bitopological analogues of quasi spectral spaces.

6.1. DEFINITION. *Let (X, τ, π) be a bitopological space. We call (X, τ, π) a* bitopological quasi spectral space *if*

(1) (X, τ) *is a quasi spectral space;*

(2) (X, π) *is a Stone space;*

(3) $\tau \subseteq \pi$;

(4) *compact opens in (X, τ) are clopen in (X, π).*

Let **BQSpec** *be the category of bitopological quasi spectral spaces and bicontinuous maps.*

6.2. REMARK. If (X, τ, π) and (X', τ', π') are bitopological quasi spectral spaces and $f : (X, \tau, \pi) \to (X', \tau', \pi')$ is bicontinuous, then $f : (X, \tau) \to (X', \tau')$ is spectral. To see this, let U be compact open in (X', τ'). Then U is clopen in (X', π'). Since $f : (X, \pi) \to (X', \pi')$ is continuous, $f^{-1}(U)$ is clopen in (X, π). Therefore, $f^{-1}(U)$ is compact in (X, π), hence compact in (X, τ). Thus, f is spectral.

6.3. REMARK. The category **BQSpec** is a full subcategory of **BQStKSp**. To see this it is sufficient to observe that each bitopological quasi spectral space is a bitopological quasi stably compact space. Let (X, τ, π) be a bitopological quasi spectral space. Since each quasi spectral space is quasi stably compact, (X, τ) is quasi stably compact. It is also clear that (X, π) is compact Hausdorff and $\tau \subseteq \pi$. It is well known that in a spectral space compact saturated sets are intersections of compact opens. The same is true in quasi spectral spaces. Therefore, since compact opens in (X, τ) are clopen in (X, π), compact saturated sets in (X, τ) are closed in (X, π). Thus, (X, τ, π) is a bitopological quasi stably compact space.

We next show that the isomorphism between **BQStKSp** and **PNach** restricts to an isomorphism of **BQSpec** and **PPries**.

6.4. LEMMA. *Let (X, τ, π) be a bitopological quasi spectral space and let $\le$ be the specialization preorder of τ. Then $(X, \pi, \le)$ is a preordered Priestley space.*

PROOF. If (X, τ, π) is a bitopological quasi spectral space, then (X, π) is a Stone space. Moreover, if $x \not\le y$, then there is a compact open U in (X, τ) with $x \in U$ and $y \notin U$. Therefore, U is a clopen upset in $(X, \pi, \le)$ containing x but not y. Thus, $(X, \pi, \le)$ is a preordered Priestley space. ∎

6.5. LEMMA. *If $(X, \pi, \le)$ is a preordered Priestley space, then compact opens of (X, π^u) are exactly the clopen upsets and compact opens of (X, π^d) are exactly the clopen downsets of $(X, \pi, \le)$.*

PROOF. We prove that compact opens in (X, π^u) are clopen upsets in $(X, \pi, \le)$. That compact opens in (X, π^d) are clopen downsets is proved by switching to $\le^{op}$. Let U be a clopen upset in $(X, \pi, \le)$. Then U is open in (X, π^u). Moreover, U is compact in (X, π), which makes it compact in (X, π^u). Thus, U is compact open in (X, π^u). Conversely, let U be compact open in (X, π^u). Then it is open in (X, π) and an upset. Take $y \notin U$. For each $x \in U$, since U is an upset, $x \not\le y$. By the Priestley separation axiom, there is a clopen upset V_x with $x \in V_x$ and $y \notin V_x$. The V_x cover U and are open in (X, π^u). Therefore, compactness of U implies that there are $x_1, \ldots, x_n \in U$ with $U \subseteq V_{x_1} \cup \cdots \cup V_{x_n} := V_y$. Thus, we have a clopen upset V_y containing U and missing y. This yields $U^c \cap \bigcap \{V_y \mid y \notin U\} = \varnothing$. Since U^c is closed in (X, π) and (X, π) is compact, there are $y_1, \ldots, y_m \notin U$ with $U^c \cap V_{y_1} \cap \cdots \cap V_{y_m} = \varnothing$. If W is the intersection of the V_{y_i},

then W is a clopen upset, W contains U, and W misses U^c. Thus, $W = U$, and hence U is a clopen upset. ∎

6.6. LEMMA. *If $(X, \pi, \leq)$ is a preordered Priestley space, then (X, π^u) and (X, π^d) are quasi spectral spaces.*

PROOF. Clearly (X, π^u) is compact. By Lemma 5.4, (X, π^u) is sober. By Lemma 6.5, compact opens of (X, π^u) are clopen upsets of $(X, \pi, \leq)$. Since in a preordered Priestley space, each open upset is the union of clopen upsets contained in it and finite intersections of clopen upsets are clopen upsets, (X, π^u) is coherent. Thus, (X, π^u) is a quasi spectral space. That (X, π^d) is also quasi spectral is proved by switching to $\leq^{\mathrm{op}}$. ∎

6.7. THEOREM. *The categories **BQSpec** and **PPries** are concretely isomorphic.*

PROOF. Let $F : \mathbf{BQStKSp} \to \mathbf{PNach}$ and $G : \mathbf{PNach} \to \mathbf{BQStKSp}$ be the functors from Theorem 5.5. By Lemmas 6.4–6.6, F restricts to a functor $\mathbf{BQSpec} \to \mathbf{PPries}$ and G restricts to a functor $\mathbf{PPries} \to \mathbf{BQSpec}$. Thus, by Theorem 5.5, the restrictions yield a concrete isomorphism of $\mathbf{BQSpec}$ and $\mathbf{PPries}$. ∎

Thus, we obtain the following commutative diagram, which generalizes the commutative diagram of Section 2.

$$
\begin{array}{ccc}
\mathbf{BQStKSp} & \longleftrightarrow & \mathbf{PNach} \\
\uparrow & & \uparrow \\
\mathbf{BQSpec} & \longleftrightarrow & \mathbf{PPries}
\end{array}
$$

6.8. REMARK. Theorem 6.7 generalizes the well-known isomorphism of **Spec** and **Pries** [Cor75].

6.9. REMARK. An alternate approach to Theorem 6.7 was pointed out to us by the referee. Let $(X, \pi, \leq)$ be a preordered Nachbin space. The cone $(f_i : X \to X_i)$ of **PNach**-morphisms from $(X, \pi, \leq)$ to finite **PNach**-objects $(X_i, \leq_i)$ is *initial* provided both π and $\leq$ are determined by the cone; that is, π is generated by $f_i^{-1}(U)$, where $U \subseteq X_i$, and $x \leq y$ iff $f_i(x) \leq_i f_i(y)$ for all f_i in the cone.[2] It is immediate from the definition of a preordered Priestley space that the cone $(f_i : X \to X_i)$ is initial iff $(X, \pi, \leq)$ is a preordered Priestley space.

Similarly, for $(X, \tau, \pi) \in \mathbf{BQStKSp}$, the cone $(f_i : X \to X_i)$ of **BQStKSp**-morphisms from (X, τ, π) to finite **BQStKSp**-objects is initial (that is, both τ and π are determined by the cone) iff $(X, \tau, \pi) \in \mathbf{BQSpec}$.

To obtain Theorem 6.7 it now suffices to observe that the concrete isomorphism between **BQStKSp** and **PNach** of Corollary 4.16 preserves and reflects initial cones, and restricts to an isomorphism between the subcategories of finite objects of **BQStKSp** and

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[2]Since X_i is finite, π_i is discrete.

PNach, respectively. Therefore, it carries bitopological quasi spectral spaces to preordered Priestley spaces and vice versa. Thus, it restricts to an isomorphism between **BQSpec** and **PPries**.

This approach also permits to deduce Lemma 6.6 from a result of [HN14]. Similarly to the above, an object (X, τ) of **QStKSp** is in **QSpec** iff the cone $(f_i : X \to X_i)$ of **QStKSp**-morphisms from (X, τ) to finite **QStKSp**-objects (that is, finite topological spaces) is initial. Now apply [HN14, Prop. 2.1] to obtain that if $(X, \pi, \leq) \in$ **PPries**, then $K(X, \pi, \leq) \in$ **QSpec**.

6.10. REMARK. It follows from Remark 6.9 that the initial cones $(f_i : X \to X_i)$ in **PNach** and **BQStKSp** are in fact limiting cones. In other words, each object in **PPries** and **BQSpec** is the inverse limit of the inverse system of all its finite images. This is not true in **QStKSp** or **Top**.

For a simple example in **QStKSp**, let X be an infinite set with the trivial topology τ. Then $(X, \tau) \in$ **QSpec**. Therefore, as we will see in Theorem 6.16, (X, τ) is homeomorphic to the inverse limit of an inverse system of finite spaces. However, the inverse limit of all finite images of (X, τ) is homeomorphic to βX with the trivial topology, hence is not homeomorphic to (X, τ).

For an example in **Top**, let X be the ordinal $\omega + 1$ and let τ be the topology of open downsets. It is easy to see that (X, τ) is a spectral space. In particular, (X, τ) is a T_0-space. By [Hoc69, Joy71], (X, τ) is homeomorphic to the inverse limit of an inverse system of finite T_0-spaces. In fact, (X, τ) is homeomorphic to the inverse limit of all finite T_0-spaces that are spectral images of (X, τ). However, (X, τ) is not homeomorphic to the inverse limit of all finite T_0-images of (X, τ). To see this, observe that finite T_0-quotients of (X, τ) are obtained by breaking X into finitely many intervals, exactly one of which is infinite. The quotient map is spectral iff the infinite interval contains ω. For example, if we break X into $[0, \omega)$ and $\{\omega\}$, then the corresponding quotient map is not spectral. Because of this, choosing the point corresponding to the infinite interval in each quotient space produces a new point in the inverse limit. In fact, the inverse limit is homeomorphic to (X', τ'), where $X' = \omega + 2$ and τ' is the topology of open downsets in $\omega + 2$. Clearly the same example works also for the category of T_0-spaces.

The next corollary is an immediate consequence of [Dia07, Sec. 2] and Theorems 5.5 and 6.7.

6.11. COROLLARY. *Profinite objects in* **BQStKSp** *are exactly the bitopological quasi spectral spaces.*

Moreover, since the functor $K :$ **PNach** $\to$ **Top** factors through **BQStKSp**, from Theorems 3.2 and 6.7 we conclude:

6.12. COROLLARY. *A topological space* (X, τ) *is profinite iff it admits a stronger Stone topology* π *such that* (X, τ, π) *is a bitopological quasi spectral space.*

Next comes our key result, that each quasi stably compact space (X, τ) admits a stronger compact Hausdorff topology π such that $(X, \tau, \pi) \in$ **BQStKSp**, and that if (X, τ)

is quasi spectral, then $(X, \tau, \pi) \in$ **BQSpec**.

6.13. LEMMA. *Let (Y, σ) be a compact Hausdorff space, X be a set, and $p : X \to Y$, $s : Y \to X$ be maps with the composite ps equal to the identity map on Y. Then there is a compact Hausdorff topology π on X such that both p and s are continuous. Moreover, for each $y \in Y$, the subspace topology on $p^{-1}(y)$ is the same as the one-point compactification of the discrete space $p^{-1}(y) \smallsetminus \{s(y)\}$. Furthermore, if Y is a Stone space, then (X, π) is a Stone space.*

PROOF. Let $\mathcal{B}$ be the collection of subsets of X of the form $p^{-1}(U) + F$, where $U \in \sigma$, F is a finite subset of $X \smallsetminus s(Y)$, and $+$ denotes symmetric difference of sets. Since

$$\left(p^{-1}(U) + F\right) \cap \left(p^{-1}(U') + F'\right) = p^{-1}(U \cap U') + \left[\left(p^{-1}(U) \cap F'\right) + \left(p^{-1}(U') \cap F\right) + (F \cap F')\right]$$

and $(p^{-1}(U) \cap F') + (p^{-1}(U') \cap F) + (F \cap F') \subseteq F \cap F'$ is a finite subset of $X \smallsetminus s(Y)$, we see that $\mathcal{B}$ is closed under finite intersections. In addition, $X = p^{-1}(Y) + \varnothing \in \mathcal{B}$. Therefore, $\mathcal{B}$ generates a topology π on X.

To see that (X, π) is Hausdorff, let $x, x' \in X$ with $x \neq x'$. If $p(x) \neq p(x')$, then as (Y, σ) is Hausdorff, there are disjoint $U, U' \in \sigma$ separating $p(x), p(x')$, so $p^{-1}(U), p^{-1}(U') \in \mathcal{B}$ are disjoint and separate x, x'. On the other hand, if $p(x) = p(x')$, then one of x, x' does not belong to the s-image of Y, and without loss of generality we may assume that $x \notin s(Y)$. Thus, $\{x\} \in \mathcal{B}$, so $\{x\}$ and $X \smallsetminus \{x\} = p^{-1}(Y) + \{x\}$ are disjoint open sets of (X, π) and separate x, x'.

To see that (X, π) is compact, let $\{p^{-1}(U_i) + F_i : i \in I\}$ be a cover of X with elements of $\mathcal{B}$. We show that the U_i cover Y. Let $y \in Y$. Then there is $i \in I$ with $s(y) \in p^{-1}(U_i) + F_i$. Since $s(y) \notin F_i$, we see that $s(y) \in p^{-1}(U_i)$, hence $y = ps(y) \in U_i$. As (Y, σ) is compact, there is a finite subcover $U_{i_1}, \ldots, U_{i_n}$. Therefore, $p^{-1}(U_{i_1}), \ldots, p^{-1}(U_{i_n})$ is a cover of X. Thus, $(p^{-1}(U_{i_1}) + F_{i_1}) \cup \cdots \cup (p^{-1}(U_{i_n}) + F_{i_n})$ misses at most finitely many points of X since it contains the complement of $F_{i_1} \cup \cdots \cup F_{i_n}$. Adding finitely many $p^{-1}(U_i) + F_i$ will then produce a finite subcover, yielding compactness of (X, π).

That p is continuous follows from the definition of π, and continuity of s follows since

$$s^{-1}\left(p^{-1}(U) + F\right) = s^{-1}p^{-1}(U) + s^{-1}(F) = U + \varnothing = U.$$

Next, let $y \in Y$. The subspace topology on $p^{-1}(y)$ is generated by the sets

$$p^{-1}(y) \cap \left(p^{-1}(U) + F\right) = p^{-1}(\{y\} \cap U) + \left(p^{-1}(y) \cap F\right).$$

If $y \in U$, then such sets are cofinite in $p^{-1}(y)$ and contain $s(y)$; and if $y \notin U$, then such sets are finite and do not contain $s(y)$. This is precisely the topology obtained by compactifying the discrete space $p^{-1}(y) \smallsetminus \{s(y)\}$ with the point $s(y)$.

Finally, if (Y, σ) is a Stone space, let $\mathcal{B}_0$ be the subset of $\mathcal{B}$ of those $p^{-1}(U) + F$, where U is clopen in (Y, σ). Then $\mathcal{B}_0$ is also closed under complements as

$$X \smallsetminus \left(p^{-1}(U) + F\right) = X + \left(p^{-1}(U) + F\right) = \left(X + p^{-1}(U)\right) + F$$
$$= \left(X \smallsetminus p^{-1}(U)\right) + F = p^{-1}(Y \smallsetminus U) + F.$$

Thus, $\mathcal{B}_0$ is a Boolean algebra. As (Y, σ) is a Stone space, it is clear that $\mathcal{B}_0$ and $\mathcal{B}$ generate the same topology. Hence, (X, π) is a Stone space. ∎

6.14. THEOREM. *The forgetful functors* **BQStKSp** → **QStKSp** *and* **BQSpec** → **QSpec** *are surjective on objects.*

PROOF. Let (X, τ) be quasi stably compact and let (X_0, τ_0) be its T_0-reflection with the reflection map $p : X \to X_0$. Then (X_0, τ_0) is stably compact. If π_0 is the patch topology of τ_0, then (X_0, π_0) is compact Hausdorff. Choose any section $s : X_0 \to X$ of $p : X \to X_0$. By Lemma 6.13, there is a compact Hausdorff topology π on X such that $p : (X, \pi) \to (X_0, \pi_0)$ is continuous. We claim that (X, τ, π) is a bitopological quasi stably compact space. Clearly (X, τ) is quasi stably compact and (X, π) is compact Hausdorff. Let $U \in \tau$. Then $p(U) \in \tau_0 \subseteq \pi_0$, so $U = p^{-1}p(U) \in \pi$. Therefore, $\tau \subseteq \pi$. Let K be compact saturated in (X, τ). Then $p(K)$ is compact saturated in (X_0, τ_0), so $p(K)$ is closed in (X_0, π_0). Thus, $K = p^{-1}p(K)$ is closed in (X, π), yielding that $(X, \tau, \pi) \in$ **BQStKSp**. Consequently, the forgetful functor **BQStKSp** → **QStKSp** is surjective on objects.

If in addition $(X, \tau) \in$ **QSpec**, then (X_0, τ_0) is a spectral space, hence (X_0, π_0) is a Stone space. By Lemma 6.13, π is a Stone topology on X. Therefore, (X, τ) is a quasi spectral space, (X, π) is a Stone space, and $\tau \subseteq \pi$. Let U be compact open in (X, τ). Then $p(K)$ is compact open in (X_0, τ_0), so $p(K)$ is clopen in (X_0, π_0). Thus, $K = p^{-1}p(K)$ is clopen in (X, π), yielding that $(X, \tau, \pi) \in$ **BQSpec**. Consequently, the forgetful functor **BQSpec** → **QSpec** is also surjective on objects. ∎

6.15. REMARK. While the forgetful functors **BQStKSp** → **QStKSp** and **BQSpec** → **QSpec** are surjective on objects, neither is an equivalence as they map non-isomorphic objects to isomorphic objects. To see this, take any two Stone topologies π_1 and π_2 on the same set X so that (X, π_1) and (X, π_2) are not homeomorphic, and let τ be the trivial topology on X. Then (X, τ, π_1) and (X, τ, π_2) are non-isomorphic objects in **BQSpec**, but their images under the forgetful functor coincide with (X, τ).

In fact, **BQStKSp** and **QStKSp** are not equivalent, and neither are **BQSpec** and **QSpec**. This can be seen as follows. By Theorem 4.15, **BQStKSp** is isomorphic to **Top**$^{\beta}$, hence is complete. On the other hand, it is easy to see that **QStKSp** does not have equalizers, and the same is true about **BQSpec** and **QSpec**.[3] To see this, let $X = \mathbb{N} \cup \{\infty_1, \infty_2\}$ be the "double limit" space, where all points in $\mathbb{N}$ are isolated, while the neighborhoods of ∞_i are cofinite subsets of X containing both ∞_1 and ∞_2. Clearly the T_0-reflection of X is homeomorphic to the one-point compactification of $\mathbb{N}$, hence is a Stone space. In particular, it is a spectral space, and so X is a quasi spectral space. Let $f : X \to X$ be the identity on $\mathbb{N}$ and switch ∞_1 and ∞_2. It is easy to see that f is a homeomorphism, hence a spectral map. If f and Id_X have an equalizer $g : X' \to X$, where $X' \in$ **QStKSp** and g is a **QStKSp**-morphism, then $g(X')$ is a compact subset of X missing ∞_1 and ∞_2. Therefore, $g(X')$ is a finite subset of $\mathbb{N}$. But any finite subset of $X \smallsetminus \{\infty_1, \infty_2\}$ also belongs to **QStKSp** and equalizes f and Id_X. Therefore, not every

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[3]We would like to thank the referee for simplifying our original argument considerably.

QStKSp-morphism equalizing f and Id_X factors through g, so there is no equalizer for Id_X and f in **QStKSp**.

Putting Corollary 6.12 and Theorem 6.14 together yields the main result of the paper.

6.16. THEOREM. *For a topological space (X, τ), the following are equivalent:*

(1) *(X, τ) is a profinite topological space.*

(2) *(X, τ) is a quasi spectral space.*

(3) *(X, τ) admits a stronger Stone topology π such that (X, τ, π) is a bitopological quasi spectral space.*

The well-known theorem [Hoc69, Joy71] characterizing profinite T_0-spaces is now an immediate consequence of Theorem 6.16.

6.17. COROLLARY. *([Hoc69, Joy71]) A T_0-space (X, τ) is profinite iff (X, τ) is a spectral space.*

Since (X, τ) is quasi spectral iff its T_0-reflection is spectral, we conclude:

6.18. COROLLARY. *A topological space is profinite iff its T_0-reflection is a profinite T_0-space.*

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