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PARTIAL LINEARITY AND PARTIAL NATURAL MAL'TSEVNESS

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Abstract. We introduced in [7] a notion of Mal'tsevness relative to a specific class Σ of split epimorphisms. We investigate here the induced relative notion of natural Mal'tsevness, with a special attention to the example of quandles.

Introduction

In [7] we introduced a notion of Mal'tsevness which is only relative to a class Σ of split epimorphisms (stable under pullback and containing the isomorphisms), and we investigated what is remaining of the properties of the global Mal'tsev context (A. Carboni and all [11],[12]), after a first work about partial pointed protomodularity [10].

The Mal'tsev context contains, in particular, the naturally Mal'tsev one introduced by P.T. Johnstone [15] which corresponds to the "additive heart" of the theory.

So, we shall investigate here what is remaining of the properties of the global naturally Mal'tsev context inside the relative frame. The generic example for the partial Mal'tsev context in [8] was the category of quandles, an algebraic structure independently introduced in [16] and [18] for Knot theorists, since it formalized the Reidemeister moves on oriented link diagrams, see also [13]; so here will be a special attention to the notion of autonomous quandle which retains the partial naturally Mal'tsev part of the theory.

In [6], the author specified that the non-pointed additive context was actually structured by a subtle hierarchy of notions. It is not unexpected that, in the relative context, the previous subtleties grow up in complexity: for instance, there will appear examples of Mal'tsev (or Σ-Mal'tsev) categories which become naturally Mal'tsev for a certain subclass Σ'. This gives rise to the beginning of a kind of cartography for the linear and additive parts in Categorical Algebra.

Notice that our notion of partial Mal'tsevness is different from the relative Mal'tsevness studied in [14].

This article is organized along the following lines:
Section 1 is devoted to some recalls and to the definition of the partial naturally Mal'tsev context while Section 2 is devoted to what remains of the results of from [15] and [4] in this relative context. Section 3 investigates the particular case of the point-congruous

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classes Σ which produces *a global naturally Mal'tsev core*. In Section 4 we briefly show that, in the same way as in the global situation, the regular context allows us to extend some properties from the split epimorphisms to the regular epimorphisms. In Section 5, on the model of [6], we show that when the category $\mathbb{E}$ is a Σ-Mal'tsev category, any fibre $Grd_Y\mathbb{E}$ of the fibration of internal groupoids $Grd\mathbb{E} \to \mathbb{E}$ (and in particular the category $Gp\mathbb{E}$ of internal groups in $\mathbb{E}$) is a Σ-naturally Mal'tsev category. This leads to Section 6 where the stronger notion of Σ-affine category is introduced, since, under the previous condition, any fibre $Grd_Y\mathbb{E}$ is actually of this kind. Section 7 applies part of the previous results to the example of the category Qnd of quandles.

1. Partial Mal'tsevness

From now on, any category $\mathbb{E}$ will be supposed to be finitely complete, and split epimorphism will mean split epimorphism with a given splitting. Recall from [3] that, for any category $\mathbb{E}$, $\mathrm{Pt}(\mathbb{E})$ denotes the category whose objects are the split epimorphisms (=the "genereralized points") of $\mathbb{E}$ and whose arrows are the commuting squares between such split epimorphisms, and that $\P_{\mathbb{E}} : \mathrm{Pt}(\mathbb{E}) \to \mathbb{E}$ denotes the functor associating with each split epimorphism its codomain. It is a fibration called the *fibration of points*.

1.1. Definition. *Let Σ be a class of split epimorphisms; denote by $\Sigma(\mathbb{E})$ the full subcategory of $\mathrm{Pt}(\mathbb{E})$ whose objects are in Σ. This class is said to be:*
1) fibrational when Σ is stable under pullback and contains the isomorphisms
2) point-congruous when, in addition, $\Sigma(\mathbb{E})$ is stable under finite limits in $\mathrm{Pt}(\mathbb{E})$.

When Σ is fibrational, it determines a pointed subfibration $\P^{\Sigma}_{\mathbb{E}} : \Sigma(\mathbb{E}) \to \mathbb{E}$ of the fibration of points. Recall from [7]:

1.2. Definition. *Let Σ be a fibrational class of split epimorphisms in $\mathbb{E}$. Then $\mathbb{E}$ is said to be a Σ-Mal'tsev category when, for any leftward pullback of split epimorphisms:*

$$\begin{tikzcd}
X \times_Y Z \arrow[d, shift right, "p_Z"'] \arrow[r, shift right, "p_X"'] & X \arrow[l, shift right, "\iota_X"'] \arrow[d, shift right, "f"'] \\
Z \arrow[u, shift right, "\iota_Z"'] \arrow[r, shift right, "g"'] & Y \arrow[l, shift right, "t"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

the pair (i_Z, i_X) is jointly extremally epic whenever the split epimorphism (f, s) belongs to Σ.

In [4], a finitely complete category $\mathbb{E}$ was shown to be a Mal'tsev one (i.e. a category in which any reflexive relation is an equivalence relation [12]) when the previous condition holds for any split epimorphism (f, s); and a pointed category $\mathbb{D}$ was defined to be unital when the previous condition holds for the class $\Pi_{\mathbb{D}}$ of the canonically split product projections (which becomes a point-congruous class in this case). Recall the following definition from [10]:

1.3. Definition. *Let $\mathbb{C}'$ be a full subcategory of a pointed category $\mathbb{C}$. The category $\mathbb{C}$ is said to be* $\mathbb{C}'$-unital *when, for any object $A \in \mathbb{C}'$ and any object $B \in \mathbb{C}$, the canonical injections i_A and i_B in the following diagram are jointly strongly epimorphic:*

$$A \underset{i_A}{\overset{p_A}{\leftrightarrows}} A \times B \underset{i_B}{\overset{p_B}{\rightleftarrows}} B.$$

Now let Σ be a fibrational class in $\mathbb{E}$ and denote by Σ_Y the full subcategory of the fibre $\mathrm{Pt}_Y(\mathbb{E})$ whose objects are in Σ. We get immediately:

1.4. Proposition. *The category $\mathbb{E}$ is a Σ-Mal'tsev category if and only if any (pointed) fibre $\mathrm{Pt}_Y(\mathbb{E})$ is Σ_Y-unital.*

1.5. Examples. **1)** Let Mon be the category of monoids. It is unital. But it actually fulfils the partial Mal'tsev condition for a much larger class of split epimorphisms. A split epimorphism $(f, s) : X \rightleftarrows Y$ will be called a *weakly Schreier* split epimorphism when, for any element $y \in Y$, the map $\mu_y : Kerf \to f^{-1}(y)$ defined by $\mu_y(k) = k \cdot s(y)$ is surjective. The class Σ of weakly Schreier split epimorphisms is fibrational (but not point-congruous) and *the category Mon is a Σ-Mal'tsev category.*

Proof. Stability under pullback is straightforward. Let be given a submonoid $W \subset X \times_Y Z$ containing the elements $(sg(z), z)$ and $(x, tf(x))$. Suppose (f, s) is a weakly Schreier split epimorphism, taking any $(x, z) \in X \times_Y Z$, i.e. such that $f(x) = g(z)$, there is some $k \in Kerf$ such that:

$$(x, z) = (k \cdot sf(x), z) = (k \cdot sg(z), z) = (k, 1) \cdot (sg(z), z)$$

so we get: $(x, z) \in W$. ∎

1′) In [17] a split epimorphism $(f, s) : X \rightleftarrows Y$ in Mon was called a *Schreier* split epimorphism *when the map μ_y is bijective.* This defines a sub-class $\Sigma' \subset \Sigma$ which was shown to be point-congruous in [10]; by Theorem 2.4.2 in this same monograph, the category Mon is a Σ'-Mal'tsev category according to the present definition.
2) Suppose that $U : \mathbb{C} \to \mathbb{D}$ is a left exact functor. It is clear that if Σ is a fibrational (resp. point-congruous) class of split epimorphisms in $\mathbb{D}$, so is the class $\bar{\Sigma} = U^{-1}\Sigma$ in $\mathbb{C}$. When, in addition, the functor U is conservative (i.e. reflects the isomorphisms), then $\mathbb{C}$ is a $\bar{\Sigma}$-Mal'tsev category as soon as $\mathbb{D}$ is a Σ-Mal'tsev one.
3) Let SRg be the category of semi-rings. The functor $U : SRg \to CoM$ towards the category of commutative monoids is left exact and conservative. We call *weakly Schreier* a split epimorphism in $\bar{\Sigma} = U^{-1}\Sigma$. In [10] a split epimorphism in $\bar{\Sigma}' = U^{-1}\Sigma'$ was called a Scheier one. Thanks to the point 2), this gives rise to two partial Mal'tsev structures on SRg, the first one not being point-congruous.
4) A quandle is a set X endowed with a binary operation $\triangleright : X \times X \to X$ which is idempotent and such that for any object x the translation $- \triangleright x : X \to X$ is an automorphism with respect to the binary operation $\triangleright$ whose inverse is denoted by $- \triangleright^{-1} x$. A homomorphism of quandles is a map $f : (X, \triangleright) \to (Y, \triangleright)$ which respects the binary

operation. This defines the category *Qnd* of quandles. Quandles recapture the formal aspects of group conjugation: starting from a group $(G,\cdot)$, the binary operation $x \triangleright_G y = y \cdot x \cdot y^{-1}$ is a quandle operation.

In [8] a split epimorphism $(f,s) : X \rightleftarrows Y$ in the category *Qnd* was called a *puncturing* (resp. *acupuncturing*) split epimorphism when, for any element $y \in Y$, the map $s(y) \triangleleft - : f^{-1}(y) \to f^{-1}(y)$ is surjective (resp. bijective). The class Σ of puncturing (resp. Σ' of acupuncturing) split epimorphisms was shown to be fibrational (resp. point-congruous), and the category *Qnd* was shown to be a Σ-Mal'tsev (and a fortiori a Σ'-Mal'tsev) category.

1.6. Partial linearity and partial natural Mal'tsevness. On the model of the previous definitions, let us introduce the following stricter ones:

1.7. Definition. *Let $\mathbb{C}'$ be a full subcategory of a pointed category $\mathbb{C}$. The category $\mathbb{C}$ is said to be* $\mathbb{C}'$-linear *when, for any object $A \in \mathbb{C}'$ and any object $B \in \mathbb{C}$, the canonical injections i_A and i_B in the following diagram define a binary sum:*

$$\begin{tikzcd} A \arrow[r, shift right, "i_A"'] & A \times B \arrow[l, shift right, "p_A"'] \arrow[r, shift left, "p_B"] & B. \arrow[l, shift left, "i_B"] \end{tikzcd}$$

It is clear that a pointed category $\mathbb{C}$ is linear in the classical sense when the subcategory $\mathbb{C}'$ coincides with $\mathbb{C}$.

1.8. Definition. *Let Σ be a fibrational class of split epimorphisms in $\mathbb{E}$. Then $\mathbb{E}$ will be said to be a Σ-naturally Mal'tsev category when, for any leftward pullback of split epimorphisms:*

$$\begin{tikzcd} X \times_Y Z \arrow[d, shift right, "p_Z"'] \arrow[r, shift right, "p_X"'] & X \arrow[l, shift right, "\iota_X"'] \arrow[d, shift right, "f"'] \\ Z \arrow[u, shift right, "\iota_Z"'] \arrow[r, shift right, "g"'] & Y \arrow[l, shift right, "t"'] \arrow[u, shift right, "s"'] \end{tikzcd}$$

the upward and rightward square is a pushout whenever the split epimorphism (f,s) belongs to Σ.

Clearly a Σ-naturally Mal'tsev category is a Σ-Mal'tsev one. Recall that a finitely complete category $\mathbb{E}$ is a naturally Mal'tsev one (i.e a category in which any object is equipped with a natural Mal'sev operation [15]) if and only if the previous condition holds for any split epimorphism (f,s), see [4]. An additive category is just a pointed naturally Mal'tsev category. Let Σ be a fibrational class in $\mathbb{E}$; we get immediately as above:

1.9. Proposition. *The category $\mathbb{E}$ is a Σ-naturally Mal'tsev category if and only if any (pointed) fibre $\mathrm{Pt}_Y(\mathbb{E})$ is Σ_Y-linear.*

1.10. Examples. **1)** Any linear category $\mathbb{D}$ is a $\Pi_{\mathbb{D}}$-linear category.
2) Let CoM be the pointed category of commutative monoids. It is linear. A split epimorphism $(f,s): X \rightleftarrows Y$ is a (resp. weakly) Schreier one if and only if the canonical comparison map $Y \times Kerf \to X$ is bijective (resp. surjective). In other words, in CoM, the class Σ' of the Schreier split epimorphisms coincide with the class Π of the canonically split product projections. The category CoM of commutative monoids provides us with a situation where a Σ-Mal'tsev category is a Σ'-naturally Mal'tsev category as well for a certain subclass $\Sigma' \subset \Sigma$.
3) A quandle X is said to be *autonomous* when the binary operation $\triangleright$ is a quandle homomorphism. Let us denote by AQd the full subcategory of Qnd whose objects are the autonomous quandles. Let (f,s) be an acupunturing split epimorphism and let us denote by $\rho(x)$ the unique element of $f^{-1}(f(x))$ of such that $sf(x) \triangleright \rho(x) = x$. *When X is autonomous, the function $\rho: X \to X$ is a homomorphism of quandles.* From:
$sf(x \triangleright x') \triangleright (\rho(x) \triangleright \rho(x')) = (sf(x) \triangleright sf(x')) \triangleright (\rho(x) \triangleright \rho(x'))$
$= (sf(x) \triangleright \rho(x)) \triangleright (sf(x') \triangleright \rho(x')) = x \triangleright x'$
we get $\rho(x) \triangleright \rho(x') = \rho(x \triangleright x')$ by the uniqueness of the factorization property.

1.11. Proposition. *The category AQd is a Σ'-naturally Mal'tsev category where Σ' is the class of acupuncturing split epimorphisms.*

Proof. Consider any pullback of split epimorphisms in AQd with (f,s) in Σ':

$$\begin{tikzcd}
X \times_Y Z \arrow[r, shift right, "p_X"'] \arrow[d, shift right, "p_Z"'] & X \arrow[l, shift right, "\iota_X"'] \arrow[d, shift right, "f"'] \\
Z \arrow[u, shift right, "\iota_Z"'] \arrow[r, shift right, "g"'] & Y \arrow[l, shift right, "t"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

Suppose $(x,z) \in X \times_Y Z$. We have $z = k(z) \triangleright tg(z) = k(z) \triangleright tf(x)$, where the mapping k defined by $k(z) = z \triangleright^{-1} tg(z)$ is a quandle homomorphism since Z is autonomous. Since (f,s) is in Σ', we have $x = sf(x) \triangleright \rho(x) = sg(z) \triangleright \rho(x)$ where ρ is a quandle homomorphism as well. Whence:

$$(x,z) = (sg(z), k(z)) \triangleright (\rho(x), tf(\rho(x))) = \iota_Z(k(z)) \triangleright \iota_X(\rho(x))$$

Suppose now we have a pair $(m: Z \to T, n: X \to T)$ of quandle homomorphisms in AQd such that $m \circ t = n \circ s$. Then the unique desired quandle factorization $l: X \times_Y Z \to T$ is (necessarily) defined by $l(x,z) = m(k(z)) \triangleright n(\rho(x))$; this shows that the upward and rightward square is a pushout. ∎

We construct many further examples of Σ-natural Mal'tsevness in Section 5.7.

2. First properties of the Σ-naturally Mal'tsev categories

Recall the following characterizations from [15] and [4]: a finitely complete category $\mathbb{D}$ is a naturally Mal'tsev one if and only if any of the following conditions is satisfied:

1) any fibre $Pt_Y(\mathbb{E})$ of the fibration of points $¶_{\mathbb{D}}$ is linear
1′) any fibre $Pt_Y(\mathbb{E})$ of the fibration of points $¶_{\mathbb{D}}$ is additive
2) it is a Mal'tsev category in which any pair of equivalence relations centralizes each other, or equivalently any equivalence relation is central
3) any internal reflexive graph is a groupoid (*the Lawvere condition*)
4) any base change along a split epimorphism with respect to the fibration of points $¶_{\mathbb{D}}$ is an equivalence of categories.

In this section we shall investigate what is remaining of these characterizations in the partial context. The translation of the condition 1) is the characterization given by Proposition 1.9. Recall from [7] (see also [10]) the following definition and results concerning the Σ-Mal'tsev categories:

2.1. Definition. *A graph X_1 on an object X will be said to be a Σ-graph when it is reflexive:*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X \arrow[l, "s_0" description] \end{tikzcd}$$

and such that the split epimorphism (d_0, s_0) belongs to the class Σ. The same definition applies respectively to relations, internal categories, and internal groupoids.
A morphism $f : X \to Y$ is called Σ-special when its kernel relation $R[f]$ is a Σ-equivalence relation. An object X is said to be Σ-special when the terminal map $\tau_X : X \to 1$ is Σ-special.

2.2. Proposition. *Let $\mathbb{E}$ be a Σ-Mal'tsev category. Any Σ-relation S on an object X is necessarily transitive. A Σ-relation S is an equivalence relation if and only if the map $d_0 : S \to X$ is Σ-special. On a Σ-graph there is at most one structure of internal category. When a Σ-special map f is split by any map s, the split epimorphism (f, s) lies in Σ.*

The first two assertions allow to measure precisely the weakening of the partial context in comparison with the global one in which any reflexive relation is an equivalence relation.

Commutation in $Pt_Y(\mathbb{E})$

Consider two maps with same codomain in the fibre $Pt_Y\mathbb{E}$ as on the left hand side and suppose that the split epimorphism (f, s) is in Σ; then take, as on the right hand side, the pullback of f along g:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[dr, shift right, "g"'] & V \arrow[d, shift right] & X \arrow[l, "k"'] \arrow[dl, shift right, "f"'] \\ & Y \arrow[ul, shift right, "t"'] \arrow[u, shift right] \arrow[ur, shift right, "s"'] & \end{tikzcd} \qquad \begin{tikzcd} & X' \arrow[dl, shift left] \arrow[dr, shift right] \arrow[d, dotted, "\phi"'] & \\ U \arrow[ur, shift left, "s'"] \arrow[r, "h"] \arrow[dr, shift right, "g"'] & V \arrow[d, shift right] & X \arrow[l, "k"'] \arrow[ul, shift right, "t'"'] \arrow[dl, shift right, "f"'] \\ & Y \arrow[ul, shift right, "t"'] \arrow[u, shift right] \arrow[ur, shift right, "s"'] & \end{tikzcd}$$

2.3. Definition. *Let* $\mathbb{E}$ *be a* Σ*-Mal'tsev category and* (f, s) *a split epimorphism in* Σ*. The pair* (h, k) *of morphisms is said to commute in the fibre* $Pt_Y\mathbb{E}$ *when there is a (necessarily unique) map* $\phi : X' \to V$ *such that* $\phi.t' = k$ *and* $\phi.s' = h$*. The map* ϕ *is called the* cooperator *of this pair.*

Immediately, we get:

2.4. Proposition. *Let* $\mathbb{E}$ *be a* Σ*-naturally Mal'tsev category. Then any pair of the previous kind commutes.*

Proof. The desired factorization is a straightforward consequence of the fact that the quadrangle with X' is underlying a binary sum in the fibre $\mathrm{Pt}_Y(\mathbb{E})$. ■

From that, we get a part of condition 3), namely a weakening of the Lawvere condition:

2.5. Proposition. *Let* $\mathbb{E}$ *be a* Σ*-naturally Mal'tsev category. Any* Σ*-reflexive graph:*

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X_0 \arrow[l, "s_0"' description] \end{tikzcd}$$

is underlying a (unique) internal category structure. In particular any split epimorphism $(f, s) : X \rightleftarrows Y$ *in* Σ *is underlying a (unique) structure of commutative monoid in the fibre* $Pt_Y(\mathbb{E})$*.*

Proof. It was shown in [7], that, in a Σ-Mal'tsev category, a Σ-reflexive graph is an internal category if and only if the following subobjects commute in $Pt_{X_0}\mathbb{E}$:

$$\begin{tikzcd} X_1 \arrow[r, tail, "{(d_0,1_{X_1})}"] \arrow[dr, shift right, "d_0"'] & X_0 \times X_1 \arrow[d, shift right, "p_{X_0}"'] & X_1 \arrow[l, tail, "{(d_1,1_{X_1})}"'] \arrow[dl, shift left, "d_1"] \\ & X_0 \arrow[ul, shift right, "s_0"'] \arrow[u, shift right, "{(1_{X_0},s_0)}"'] \arrow[ur, shift left, "s_0"] & \end{tikzcd}$$

which is necessarily true here when (d_0, s_0) is in Σ, according to the previous proposition. ■

Similarly, let us recall the following definition and results generalizing the global Mal'tsev context ([19], [9]):

2.6. Definition. *Let* $\mathbb{E}$ *be a* Σ*-Mal'tsev category and* (R, S) *a pair of a reflexive relation* R *and a* Σ*-relation* S *on the object* X*. We say that the two reflexive relations* R *and* S *centralize each other (which we shall denote by* $[R, S] = 0$ *as usual) when the two following subobjects commute in the fibre* $Pt_X(\mathbb{E})$*:*

$$\begin{tikzcd} R \arrow[r, tail, "{(d_1^R,d_0^R)}"] \arrow[dr, shift right, "d_1^R"'] & X \times X \arrow[d, shift right, "p_0"'] & S \arrow[l, tail, "{(d_0^S,d_1^S)}"'] \arrow[dl, shift right, "d_0^S"'] \\ & X \arrow[ul, shift right, "s_0^R"'] \arrow[u, shift right, "s_0"'] \arrow[ur, shift right, "s_0^S"'] & \end{tikzcd}$$

It is the case if and only if there is a (unique) map $p : R \times_X S \to X$ (called the connector of the pair (R, S)) such that $p(xRx'Sx') = x$ and $p(xRxSx') = x'$.

Again, immediately we get a part of condition 2):

2.7. Proposition. *Let $\mathbb{E}$ be a Σ-naturally Mal'tsev category. Then any pair (R, S) of a reflexive relation R and a Σ-relation S on the object X centralize each other. In particular, any Σ-equivalence relation is central. Any Σ-special morphism $f : X \to Y$ has a central kernel relation.*

2.8. Corollary. *Let $\mathbb{E}$ be a Σ-naturally Mal'tsev category. Then any Σ-special object X of $\mathbb{E}$ is endowed with a (unique) natural Mal'tsev operation in $\mathbb{E}$.*

Proof. An object X is Σ-special if and only if the indiscrete equivalence relation ∇_X is a Σ-relation. In the Σ-naturally Mal'tsev context we get $[\nabla_X, \nabla_X] = 0$ and a connector p which produces the Mal'tsev operation:

$$\begin{tikzcd}
X \times X \times X \arrow[r, shift left, "p_2"] \arrow[d, shift right, "p_0"'] \arrow[dr, dotted, "p"'] & X \times X \arrow[l, shift left, "s_1"] \arrow[d, shift right, "p_0"'] \\
X \times X \arrow[u, shift right, "s_0"'] \arrow[r, shift left, "p_1"] & X \arrow[l, shift left, "s_0"] \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

The naturality of this operation follows from the fact that the pair (s_0, s_1) of the previous diagram is jointly strongly epic. ■

When Σ is not point-congruous, this Mal'tsev operation on X does not necessarily belong to the full subcategory of Σ-special objects, since $X \times X$ is even not necessarily Σ-special when X is so; as for the point-congruous context, see Proposition 3.3 in the next section. We get also another part of condition 3):

2.9. Corollary. *Let $\mathbb{E}$ be a Σ-naturally Mal'tsev category. Then a Σ-reflexive graph is a Σ-groupoid if and only if the map d_0 is Σ-special. In particular:*
1) any Σ-special morphism $f : X \to Y$ split by s gives it an abelian group structure in $Pt_Y(\mathbb{E})$
2) a groupoid is a Σ-groupoid if and only if its underlying graph is a Σ-graph.

Finally let us investigate condition 4):

2.10. Proposition. *Let $\mathbb{E}$ be a Σ-naturally Mal'tsev category. Then any split epimorphism $(g, t) : Y' \rightleftarrows Y$ makes the base change $g^* : \Sigma_Y \to \Sigma_{Y'}$ an equivalence of categories.*

Proof. In a Σ-Mal'tsev category we know that this base change g^* is fully faithful. It remains to show that when $\mathbb{E}$ is a Σ-naturally Mal'tsev one, it is essentially surjective. Let us start with any split epimorphism $(f', s') : X' \rightleftarrows Y'$ in Σ. Complete the lower row with the kernel equivalence relation of g and denote by t_1 the unique map such that $d_1.t_1 = 1_{Y'}$ and $d_0.t_1 = t.g$. Then consider the following diagram where $(\check{f}', \check{s}')$ is $d_0^*(f', s')$, in other words where the non dotted left hand side square indexed by 0 is a pullback of split epimorphisms with a map $\sigma_0 : X' \to \check{X}$ above $s_0 : Y' \to R[g]$:

$$\begin{tikzcd}[column sep=large, row sep=large]
\check{X} \arrow[r, shift left=3, "\delta_0"] \arrow[r, dotted, shift right=1] \arrow[d, shift right, "\check{f}'"'] & X' \arrow[l, shift left=1] \arrow[l, dotted, shift left=3, "\delta_1"', "\tau_1"] \arrow[r, dotted, shift left, "g'"] \arrow[d, shift right, "f'"'] & X \arrow[l, shift left, "\tau"] \arrow[d, shift right, "f"'] \\
R[g] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[u, shift right, "\check{s}'"'] & Y' \arrow[l] \arrow[r, shift left, "g"] \arrow[u, shift right, "s'"'] \arrow[l, bend left=60, "t_1"] & Y \arrow[l, shift left, "t"] \arrow[u, shift right, "s"']
\end{tikzcd}$$

Since (f', s') is in Σ and $\mathbb{E}$ is a Σ-naturally Mal'tsev category, the upward and rightward left hand side square is a pushout which produces a map δ_1 above d_1 giving rise to the upper reflexive graph. The square indexed by 1 is a pullback as well since $d_1^*(f', s')$ is produced by the pushout along the common splitting s_0 of d_0 and d_1. This pullback indexed by 1 in turn produces the splitting τ_1 above the splitting t_1 and makes $(f', s') = t_1^*(\check{f}', \check{s}') = t_1^* d_0^*(f', s') = g^* t^*(f', s')$ with $t^*(f', s')$ in Σ since (f', s') is in Σ. ■

A unital category provides an example of a Σ-Mal'tsev category with fulfills the previous property with respect to the class $\Sigma = \Pi$ of canonically split product projections without being Σ-naturally Mal'tsev. We shall finish this section by

2.11. Proposition. *Let $\mathbb{E}$ be a Σ-naturally Mal'tsev category. When the split epimorphism (f, s) is in Σ, then the monomorphism s is canonically and naturally normal to an equivalence relation R_s.*

Proof. Consider the following leftward pullback of split epimorphisms:

$$\begin{tikzcd}[column sep=large, row sep=large]
X \times Y \arrow[r, shift right, "f \times 1"'] \arrow[d, shift right, "p_X"'] \arrow[d, dotted, bend left=40, "\psi_s"] & Y \times Y \arrow[l, shift right, "s \times 1"'] \arrow[d, shift right, "p_0"'] \arrow[d, dotted, bend left=60, "p_1"] \\
X \arrow[r, shift right, "f"'] \arrow[u, shift right, "(1,f)"'] & Y \arrow[l, shift right, "s"'] \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

When (f, s) is in Σ, the rightward and upward square is a pushout. So the map $p_1 : Y \times Y \to Y$ produces a factorization $\psi_s : X \times Y \to X$ such that $\psi_s.(1, f) = 1_X$ and $\psi_s.(s \times 1) = s.p_1$. Whence a reflexive relation $(p_X, \psi) : X \times Y \rightrightarrows X$ on X. It is actually an equivalence relation R_s since (f, s) is in Σ. The fact that $(s, s \times 1)$ determines a discrete fibration between ∇_Y and R_s (since so does $(f, f \times 1)$ in the inverse direction) makes the monomorphism s normal to the equivalence relation R_s.

To check the naturality of this construction, start with a commutative diagram of split epimorphisms in Σ:

$$\begin{tikzcd} \bar{X} \arrow[r, "x"] \arrow[d, shift right, "\bar{f}"'] & X \arrow[d, shift right, "f"'] \\ \bar{Y} \arrow[r, "y"'] \arrow[u, shift right, "\bar{s}"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

We have to show that $x : \bar{X} \to X$ induces a morphism $R_{\bar{s}} \to R_s$ between the canonical equivalence relations, namely that $\psi_s.(x \times y) = x.\psi_{\bar{s}}$. It is checked by composition with the jointly strongly epic pair $((1, \bar{f}), (\bar{s} \times 1))$. ■

3. The case of the point-congruous classes

When the class Σ is fibrational, the Σ-special morphisms are stable under pullback. It is also clear that any isomorphism is Σ-special. We shall denote by $\Sigma l(\mathbb{E})$ the category whose objects are the Σ-special morphisms and whose morphisms are the commutative squares between them. When Σ is point-congruous, $\Sigma l(\mathbb{E})$ is stable under finite limit in $\mathbb{E}^2$. Similarly we shall denote by $\Sigma l_Y\mathbb{E}$ the full subcategory of the slice category $\mathbb{E}/Y$ whose objects are the Σ-special morphisms. Recall from [8] the following:

3.1. Lemma. *Let $\mathbb{E}$ be a point-congruous Σ-Mal'tsev category. If $g.f$ and g are Σ-special, so is $f : X \to Y$. In particular, any splitting s of f gives rise to a split epimorphism (f, s) in Σ. The subcategory $\Sigma l_Y\mathbb{E}$ of the slice category $\mathbb{E}/Y$ is a Mal'tsev category.*

Proof. The kernel congruence $R[f]$ is given by the following pullback in the category $Equ\mathbb{E}$ of equivalence relations in $\mathbb{E}$:

$$\begin{tikzcd} R[f] \arrow[r] \arrow[d, tail, "j"'] & \Delta_Y \arrow[d, tail, "s_0"] \\ R[g.f] \arrow[r, "R(f)"'] & R[g] \end{tikzcd}$$

where Δ_Y is the discrete equivalence relation on Y. The equivalence relations $R[g]$ and $R[g.f]$ are Σ-relations. Since the pullbacks in $Equ\mathbb{E}$ are levelwise, and the class Σ is point-congruous, the relation $R[f]$ is a Σ-relation as well. In particular, any morphism in $\Sigma l_Y\mathbb{E}$ is Σ-special, and so any reflexive relation in $\Sigma l_Y\mathbb{E}$ is an equivalence relation. Accordingly the subcategory $\Sigma l_Y\mathbb{E}$ of the slice category $\mathbb{E}/Y$ is a Mal'tsev category. ■

In particular, if we denote by $\Sigma\mathbb{E}_\sharp = \Sigma l_1\mathbb{E}$ the full subcategory of $\mathbb{E}$ whose objects are the Σ-special objects, it is a Mal'tsev category, called the *Mal'tsev core* of the point-congruous Σ-Mal'tsev category $\mathbb{E}$; any of its morphisms is Σ-special.

3.2. Example. ***1)*** *The Mal'tsev core of the Σ'-Mal'tsev category Mon of monoids is the category Gp of groups, see [10].*
2) *The Mal'tsev core of the $\bar{\Sigma}'$-Mal'tsev category SRg of semi-rings is the category Rg of rings, see [10].*

3) The Mal'tsev core of the Σ'-Mal'tsev category Qnd of quandles is the category LQd of latin quandles, namely those quandles X which are such that, for any element x, the function $x \triangleright -$ is bijective, see [8].

Now we get:

3.3. Proposition. *Let $\mathbb{E}$ be a point-congruous Σ-naturally Mal'tsev category. The subcategory $\Sigma l_Y\mathbb{E}$ of the slice category $\mathbb{E}/Y$ is a naturally Mal'tsev category. This is the case, in particular, of its core $\Sigma\mathbb{E}_\sharp$.*

Proof. The quickest way to show it is to prove that it satisfies the Lawvere condition (condition 3) of the beginning of Section 2. Consider any reflexive graph in $\Sigma l_Y\mathbb{E}$:

$$\begin{tikzcd} X_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[d, "g_1"'] & X_0 \arrow[l, "s_0"' description] \arrow[d, "g_0"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

By Lemma 3.1 any map in $\Sigma l_Y\mathbb{E}$ is Σ-special and by Corollary 2.9, the map d_0 being Σ-special, the reflexive graph is underlying a groupoid structure. ■

3.4. Example. ***1)*** *The core of the Σ'-naturally Mal'tsev category CoM of commutative monoids is the additive category Ab of abelian groups.*
2) *The core of the Σ'-naturally Mal'tsev category AQd of autonomous quandles is the naturally Mal'tsev category $LAQd$ of latin autonomous quandles, namely sets X endowed with an idempotent binary operation $\triangleright$ which is a homomorphism for this law and is such that, for any element x, both $x \triangleright -$ and $- \triangleright x$ are bijective.*

4. The regular and exact contexts

In a regular category [1], relations can be composed. A regular category is a Mal'tsev one if and only if any pair of reflexive relations does permute [11]. Recall from [7]:

4.1. Proposition. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category. Given any pair of a reflexive relation R and a Σ-equivalence relation S on a object X, the two relations do permute.*

So, this result holds in any regular Σ-naturally Mal'tsev category.

4.2. Lemma. *Let $\mathbb{E}$ be a regular Σ-Mal'tsev category and the following square be a pullback of split epimorphisms along the regular epimorphism y:*

$$\begin{tikzcd} \bar{X} \arrow[r, two heads, "x"] \arrow[d, shift right, "\bar{f}"'] & X \arrow[d, shift right, "f"'] \\ \bar{Y} \arrow[r, two heads, "y"'] \arrow[u, shift right, "\bar{s}"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

When its domain $(\bar{f}, \bar{s})$ belongs to Σ, the upward square is a pushout. Accordingly the base change functor: $y^* : \Sigma_Y \to \Sigma_{Y'}$ is fully faithful.

PROOF. Consider any pair (ϕ, σ) of morphisms such that $\phi.\bar{s} = \sigma.y$ $(*)$:

$$\begin{tikzcd}
R[x] \arrow[r, shift left=2, "d_0^x"] \arrow[r, shift right=2, "d_1^x"'] \arrow[d, shift right, "R(\bar{f})"'] & \bar{X} \arrow[l] \arrow[r, two heads, "x"] \arrow[rr, bend left=10, "\phi"] \arrow[d, shift right, "\bar{f}"'] & T \arrow[d, shift right, "f"'] & \bar{X} \\
R[y] \arrow[r, shift left=2, "d_0^y"] \arrow[r, shift right=2, "d_1^y"'] \arrow[u, shift right, "R(\bar{s})"'] & \bar{Y} \arrow[l] \arrow[r, two heads, "y"'] \arrow[u, shift right, "\bar{s}"'] & Y \arrow[u, shift right, "s"'] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

and complete the diagram by the kernel relations $R[y]$ and $R[x]$ which produce the left hand side pullbacks above. Since the regular epimorphism x is the quotient of its kernel relation, we shall obtain the desired factorization by showing that ϕ coequalizes the pair (d_0^x, d_1^x). Now the left hand side squares being pullbacks and the split epimorphism $(\bar{f}, \bar{s})$ being in Σ, the coequalization can be checked by composition with the jointly extremally epic pair $(R(\bar{s}), s_0^x)$. This is trivial for the composition by s_0^x, and a consequence of the equality $(*)$ for the composition by $R(\bar{s})$.
Full faithfulness. Consider the following diagram:

$$\begin{tikzcd}
\bar{X} \arrow[rr, two heads, "x"] \arrow[dr, "m'"] \arrow[dd, shift right, "\bar{f}"'] & & X \arrow[dr, dotted, "m"] \arrow[dd, shift right, "f"'] & \\
& \bar{X}' \arrow[rr, two heads, "x'"] \arrow[dl, shift right, "\bar{f}'"'] & & X' \arrow[dl, shift right, "f'"'] \\
Y' \arrow[rr, two heads, "y"'] \arrow[uu, shift right, "\bar{s}"'] \arrow[ur, shift right, "\bar{s}'"'] & & Y \arrow[uu, shift right, "s"'] \arrow[ur, shift right, "s'"'] &
\end{tikzcd}$$

where the downward squares are pullback, (f, s) is in Σ and m' a morphism in $Pt_{Y'}(\mathbb{E})$. Since (f, s) is in Σ, so is $(\bar{f}, \bar{s})$ and the upward vertical square is a pushout; whence a unique map $m : X \to \bar{X}$ such that $m.x = x'.m'$ and $m.s = s'$; we get also $f'.m = f$ since x is a regular epimorphism; so m is a map in the fibre $Pt_Y(\mathbb{E})$ such that $y^*(m) = m'$. ■

Similarly to the global Mal'tsev situation, the exact context (and more generally the efficiently regular one, i.e. a context in which a *regular sub-equivalence relation* of an effective equivalence relation (X, R), is itself effective) allows us, here, to extend some properties from the split epimorphisms to the regular epimorphisms. For instance, from Proposition 2.10, we get:

4.3. PROPOSITION. *When $\mathbb{E}$ is an efficiently regular (and a fortiori exact) Σ-naturally Mal'tsev category such that pulling back along regular epimorphisms reflects the split epimorphisms in Σ, the base change functors along regular epimorphisms are equivalence of categories.*

PROOF. Suppose $\mathbb{E}$ is a Σ-naturally Mal'tsev and $g : Y' \twoheadrightarrow Y$ a regular epimorphism. From the previous proposition, it remains to show that g^* is essentially surjective. On the model of Proposition 2.10 let us start with any split epimorphism $(f', s') : X' \rightleftarrows Y'$ in Σ and complete the lower row with the kernel equivalence relation. Then consider the following diagram where $(\check{f}', \check{s}')$ is $d_0^*(f', s')$, in other words where the non dotted left hand side square indexed by 0 is a pullback of split epimorphisms with a map $\sigma_0 : X' \to \check{X}$ above $s_0 : Y' \to R[g]$:

$$\begin{tikzcd}
\check{X} \arrow[r, shift left=2, "\delta_0"] \arrow[r, shift right=2, dotted, "\delta_1"'] \arrow[d, shift right, "\check{f}'"'] & X' \arrow[l] \arrow[r, dotted, two heads, "g'"] \arrow[d, shift right, "f'"'] & X \arrow[d, dotted, shift right, "f"'] \\
R[g] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[u, shift right, "\check{s}'"'] & Y' \arrow[l] \arrow[r, two heads, "g"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, dotted, shift right, "s"']
\end{tikzcd}$$

Since (f', s') is in Σ and $\mathbb{E}$ is a Σ-naturally Mal'tsev category, the upward and rightward left hand side square is a pushout which produces a map δ_1 above d_1 giving rise to the upper reflexive graph. It is a reflexive relation since so is $R[g]$. It is an equivalence relation since (f', s') is in Σ and $R[g]$ is so. Accordingly the pair $(f', \check{f}')$ is underlying a discrete fibration between equivalence relations. Now, when $\mathbb{E}$ is efficiently regular, the upper equivalence relation is effective as soon as the lower one is so. Take g' the quotient of this upper equivalence relation. It produces a split epimorphism (f, s) such that the right hand side square is a pullback since so are the left hand side ones. If pulling back along regular epimorphisms reflects the split epimorphisms in Σ, the split epimorphism (f, s) belongs to Σ. ∎

Any regular linear category is such that pulling back along regular epimorphisms reflects the canonically split product projections. This is in particular the case of the category CoM of commutative monoids. Proposition 2.6 in [8] asserts that the category Qnd of quandles is such that pulling back along regular epimorphisms reflects the puncturing and acupuncturing split epimorphisms; so it is still the case for the category AQd of autonomous quandles.

5. Internal groupoids

5.1. Internal groupoids and abelian groupoids. Let $\mathbb{E}$ be a finitely complete category, and $Grd\mathbb{E}$ denote the category of internal groupoids in $\mathbb{E}$. An internal groupoid $\underline{Z}_1$ in $\mathbb{E}$ will be presented (see [2]) as a reflexive graph $Z_1 \rightrightarrows Z_0$ endowed with an operation π_2:

$$\begin{tikzcd}
R^2[d_0] \arrow[r, shift left=3, "p_2"] \arrow[r, "p_1"] \arrow[r, shift right=3, "p_0"'] \arrow[r, bend left=60, "R(\pi_2)"] & R[z_0] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[r, bend left=60, "\pi_2"] & Z_1 \arrow[r, shift left=3, "d_1"] \arrow[r, shift right=3, "d_0"'] & Z_0 \arrow[l, "s_0"']
\end{tikzcd}$$

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies), where $R[d_0]$ is the kernel equivalence relation of the map d_0. In the set theoretical context, this operation π_2 associates the composite $g.f^{-1}$ with any pair (f,g) of arrows with same domain. We denote by $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$ the forgetful functor which is a fibration. Any fibre $Grd_X\mathbb{E}$ above an object X has an initial object ΔX, namely the discrete equivalence relation on X, and a final object ∇X, namely the indiscrete equivalence relation on X. This fibre is *quasi-pointed* in the sense that the unique map

$$\varpi : 0 \to 1 \ = \ \Delta X \rightarrowtail \nabla X$$

is a monomorphism; this implies that any initial map is a monomorphism, and we can define the kernel of any map as its pullback along the initial map of the codomain. Recall from [6] the following:

5.2. Definition. *In a finitely complete quasi-pointed category, we shall call endosome of an object X the (unique) split epimorphism defined by the following pullback:*

$$\begin{tikzcd}
EnX \arrow[r, "\epsilon X"] \arrow[d, shift right] & X \arrow[d] \\
0 \arrow[r, "\varpi"'] \arrow[u, shift right] & 1
\end{tikzcd}$$

The fibre $Grd_1\mathbb{E}$ is nothing but the category $Gp\mathbb{E}$ of internal groups in $\mathbb{E}$ which is necessarily pointed protomodular. It was shown in [3] that any fibre $Grd_X\mathbb{E}$ is still protomodular although non-pointed. This involves an intrinsic notion of normal subobject and abelian object. They both have been characterized in [5]. Let us recall that:

5.3. Proposition. *The groupoid $\underline{Z}_1$ is abelian in the fibre $Grd_{Z_0}\mathbb{E}$ if and only if its endosome:*

$$\begin{tikzcd}
\underline{En}_1\underline{Z}_1 \arrow[r, tail, "\epsilon_1\underline{Z}_1"] \arrow[d, shift right, "\underline{e}_1\underline{Z}_1"'] & \underline{Z}_1 \arrow[d, "\underline{\omega}_1\underline{Z}_1"] \\
\Delta Z_0 \arrow[r, tail] \arrow[u, shift right] & \nabla Z_0
\end{tikzcd}$$

is abelian; in other words if and only if the group $\epsilon_1 : En_1\underline{Z}_1 \rightleftarrows Z_0$ of the "endomorphisms" of $\underline{Z}_1$ in the slice category $\mathbb{E}/Z_0$ is abelian.

In the set theoretical context, this means that any group of endomaps in $\underline{Z}_1$ is abelian. We shall denote by $AbGrd_X\mathbb{E}$ the full subcategory of $Grd_X\mathbb{E}$ whose objects are the abelian groupoids.

Now consider any internal functor $\underline{f}_1 : \underline{W}_1 \to \underline{Z}_1$ in $AbGrd_X\mathbb{E}$. Suppose it is split by a functor $\underline{s}_1$, and consider the following pullback determining the kernel of $\underline{f}_1$:

$$\begin{tikzcd}
\underline{K}_1[\underline{f}_1] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\
\Delta X \arrow[r, tail, "\underline{\alpha}_1\underline{Z}_1"'] \arrow[u, shift right] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"']
\end{tikzcd}$$

In the case $X = 1$, the upward square is actually a pushout in $AbGrd_1\mathbb{E} = Ab\mathbb{E}$ the category of abelian groups in $\mathbb{E}$. It was shown in [6] that this is no longer the case in

general in the fibers $AbGrd_X\mathbb{E}$. However, in this same article it was shown that *when* $\mathbb{E}$ *is a Mal'tsev category, any groupoid is abelian* and that *any pullback of split epimorphisms in* $Grd_X\mathbb{E}$ *produces an upward pushout in* $Grd_X\mathbb{E}$; this implies that *any fibre* $Grd_X\mathbb{E}$ *is naturally Mal'tsev*. The purpose of this section is to investigate what is remaining of these results in the partial Σ-Mal'tsev context.
For that, let us point out the following observation; let a split epimorphism $(\underline{f}_1, \underline{s}_1)$ in $Grd_X\mathbb{E}$ be given as above:
When $\mathbb{E} = Set$ there is a mapping $l : W_1 \to K_1[\underline{f}_1]$ defined by $l(x \stackrel{w}{\to} x') = w.s_1f_1(w^{-1})$ which is a retraction of k_1 and makes the following rightward diagram a pullback of split epimorphisms in Set:

$$\begin{tikzcd}
K_1[\underline{f}_1] \arrow[r, tail, "k_1"] \arrow[d, shift right] & W_1 \arrow[l, bend right=60, "l"'] \arrow[d, shift right, "f_1"'] \\
X \arrow[u, shift right] \arrow[r, tail, "\alpha_1\underline{Z}_1 = s_0^{Z_1}"'] & Z_1 \arrow[u, shift right, "s_1"'] \arrow[l, bend left=60, "d_0^{Z_1}"]
\end{tikzcd} \tag{1}$$

We have: $l(w'.w) = (s_1f_1(w^{-1}).l(w').s_1f_1(w)).l(w)$
while: $l(w^{-1}) = s_1f_1(w).l(w)^{-1}.s_1f_1(w^{-1})$;
this map l is the unique one such that $1_{W_1} = \pi_2^{W_1}(()^{-1}k_1l, s_1.f_1)$. Finally it is worth noticing that the split epimorphism (l, k_1) actually lies in the fibre $Pt_X(\mathbb{E})$:

$$\begin{tikzcd}
K_1[\underline{f}_1] \arrow[rr, tail, "k_1"] \arrow[dr, shift left] & & W_1 \arrow[ll, bend right=40, "l"'] \arrow[dl, shift left, "d_0^{W_1}"'] \\
& X \arrow[ul, shift left] \arrow[ur, shift left, "s_0^{W_1}"'] &
\end{tikzcd}$$

Proof. Straightforward calculation based on the fact that $w = s_1f_1(w).l(w)$. ■

Now we get the following lemma:

5.4. Lemma. *Let* $\mathbb{E}$ *be finitely complete category and* $(\underline{f}_1, \underline{s}_1)$ *a split epimorphism in the fibre* $Grd_X\mathbb{E}$*. Then there is a unique natural map* $l : W_1 \to K_1[\underline{f}_1]$ *in* $\mathbb{E}$ *such that* $1_{W_1} = \pi_2^{W_1}(()^{-1}k_1l, s_1.f_1)$*. It is a retraction of* k_1 *and makes the rightward part in diagram (1) a pullback of split epimorphisms in* $\mathbb{E}$.

Proof. It is straightforward from the previous observation and from the Yoneda embedding that the map l described above in Set is representable in $\mathbb{E}$ as soon as it is finitely complete. ■

Now suppose given a pair $\underline{h}_1 : \underline{K}_1[\underline{f}_1] \to \underline{V}_1, \underline{t}_1 : \underline{Z}_1 \to \underline{V}_1$ of internal functors in $Grd_X\mathbb{E}$. Reformulating Lemma 1.3 from [6] we get:

5.5. Lemma. *When $\mathbb{E} = Set$, there is a (necessarily unique) factorization $\underline{g}_1 : \underline{W}_1 \to \underline{V}_1$ such that $\underline{g}_1.\underline{k}_1 = \underline{h}_1$ and $\underline{g}_1.\underline{s}_1 = \underline{t}_1$ if and only if, for any arrow w in W_1, we have $h_1l(w.s_1f_1(w^{-1}) = t_1f_1(w).h_1l(w).t_1f_1(w^{-1})$.*
If we denote by $\check{h}_1 : W_1 \to V_1$ the mapping defined by $\check{h}_1(w) = h_1l(w.s_1f_1(w^{-1})$ and $\check{t}_1 : W_1 \to V_1$ the mapping defined by $\check{t}_1(w) = t_1f_1(w).h_1l(w).t_1f_1(w^{-1})$, the pair $(\check{h}_1, \check{t}_1)$ is equalized by k_1 (1) and by s_1 (2).

Proof. For any $\delta : x \to x$ in $\underline{K}_1[\underline{f}_1]$, we must have $g_1(\delta) = h_1(\delta)$, and for any $\phi : x \to x'$ in $\underline{Z}_1$, we must have $g_1.s_1(\phi) = t_1(\phi)$. So for any $w : x \to x'$ in $\underline{W}_1$, we must have:
$g_1(w) = g_1(s_1f_1(w).l(w)) = g_1(s_1f_1(w)).g_1(l(w)) = t_1f_1(w).h_1l(w)$
and in the same way:
$g_1(w) = g_1(w.s_1f_1(w^{-1})).g_1(s_1f_1(w)) = h_1(w.s_1f_1(w^{-1})).t_1(f_1(w))$
Whence our condition. It remains to check that this condition is sufficient to show that this definition of g_1 is functorial, which is a straightforward calculation. Finally we check: $\check{h}_1(\delta) = h_1(\delta) = \check{t}_1(\delta)$ (1) and $\check{h}_1(s_1(\phi)) = h_1(1_{x'}) = t_1(1_{x'}) = \check{t}_1(\phi)$ (2). ∎

5.6. Lemma. *When $\mathbb{E}$ is a finitely complete category, the functions $\check{h}_1$ and $\check{t}_1$ as above are representable in $\mathbb{E}$. The pair $(\check{h}_1, \check{t}_1)$ is equalized by k_1 and by s_1. There is a (necessarily unique) factorization $\underline{g}_1 : \underline{W}_1 \to \underline{V}_1$ such that $\underline{g}_1.\underline{k}_1 = \underline{h}_1$ and $\underline{g}_1.\underline{s}_1 = \underline{t}_1$ if and only if we have $\check{h}_1 = \check{t}_1$.*

Proof. The first point is straightforward; the second and third points are obtained by the Yoneda embedding from the previous lemma. ∎

Starting with any internal groupoid $\underline{Z}_1$ in $\mathbb{E}$, let us consider the following diagram in $Grd_{Z_0}\mathbb{E}$ where the right hand side square is a pullback:

$$\begin{tikzcd}
\underline{En}_1\underline{Z}_1 \arrow[r, dotted, tail, "\bar{\underline{\epsilon}}_1\underline{Z}_1"'] \arrow[rr, tail, bend left, "\underline{\epsilon}_1\underline{Z}_1"] \arrow[d, shift right, "\underline{e}_1\underline{Z}_1"'] & \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, "p_1"] \arrow[d, shift right, "p_0"'] & \underline{Z}_1 \arrow[d, "\underline{\omega}_1\underline{Z}_1"] \\
\Delta Z_0 \arrow[r, tail] \arrow[u, shift right] & \underline{Z}_1 \arrow[r, "\underline{\omega}_1\underline{Z}_1"'] \arrow[u, shift right, "s_0"'] & \nabla Z_0
\end{tikzcd}$$

It produces a unique factorization $\bar{\epsilon}_1\underline{Z}_1$ and the left hand side pullback. From the previous Lemma, and reformulating Proposition 4.2 in [6], we get: there is a map l_{Z_1} in $\mathbb{E}$ making the rightward left hand side square a pullback of split epimorphism in $\mathbb{E}$:

$$
\begin{tikzcd}
En_1\underline{Z}_1 \arrow[r, tail, "\bar{\epsilon}_1\underline{Z}_1"'] \arrow[rr, tail, bend left=20] \arrow[d, shift right, "e_1\underline{Z}_1"'] & Z_1 \times_0 Z_1 \arrow[r, "p_1"] \arrow[l, bend right=60, "l_{Z_1}"'] \arrow[d, shift right, "p_0"'] \arrow[d, dotted, bend left=40, "p_1"] & Z_1 \arrow[d, "\omega_1\underline{Z}_1=(d_0^{Z_1},d_1^{Z_1})"] \\
Z_0 \arrow[r, tail, "s_0^{Z_1}"'] \arrow[u, shift right] & Z_1 \arrow[r, tail, "\omega_1\underline{Z}_1"'] \arrow[u, shift right, "s_0"'] \arrow[l, bend left=60, "d_0^{Z_1}"] & Z_0 \times Z_0
\end{tikzcd}
\tag{2}
$$

which, with the map p_1, produces an action of the vertical left hand side group in $\mathbb{E}/Z_0$ on the split epimorphism $(d_0^{Z_1}, s_0^{Z_1})$.

5.7. Groupoids in Σ-Mal'tsev categories. Suppose now $\mathbb{E}$ is a Σ-Mal'tsev category. Given any object Y, we shall denote by Σ^Y the class of those split epimorphisms $(\underline{f}_1, \underline{s}_1)$ in the fibre $Grd_Y\mathbb{E}$ which are such that the split epimorphism (f_1, s_1) in $\mathbb{E}$ belongs to Σ. It is fibrational (resp. point-congruous) as soon as Σ is so. A groupoid in $Grd_Y\mathbb{E}$ is Σ^Y-special when the map $(d_0^{Z_1}, d_1^{Z_1}) : Z_1 \to Z_0 \times Z_0$ is Σ-special in $\mathbb{E}$.
To take a step further, we shall need now the following definitions:

5.8. Definition. *Let $\mathbb{D}$ be a category equipped with a fibrational class Σ. It will be said to be Σ-antepenessentially affine when, for any square of split epimorphisms:*

$$
\begin{tikzcd}
X' \arrow[r, "x"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\
Y' \arrow[r, "y"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"']
\end{tikzcd}
$$

the upward square is a pushout as soon as the downward square is a pullback whenever (f, s) is in Σ. It is equivalent to saying that any change of base functor $y^* : \Sigma_Y \to \Sigma_{Y'}$ is fully faithful. This category will be said to be Σ-penessentially affine when moreover any of these (fully faithful) change of base functors y^* is saturated on subobjects (i.e. induces a bijection on subobjects).

Clearly *any Σ-antepenessentially affine category is a Σ-naturally Mal'tsev one.* Since the pair (x, s) above is jointly extremely epic, the split epimorphism (f, s) is strongly split in the sense of [7] and so *any Σ-antepenessentially affine category is Σ-protomodular* as well. The previous definitions generalize those from [6] where a category $\mathbb{E}$ was said antepenessentially affine (resp. penessentially affine) when the same properties hold for any split epimorphism in $\mathbb{E}$. Recall that any antepenessentially affine category is protomodular and naturally Mal'tsev; moreover in the same way as in an additive category, *in a penessentially affine category, any monomorphism is normal.* Now we can assert:

5.9. Proposition. *Let Σ be a point-congruous class of split epimorphisms. If $\mathbb{E}$ is Σ-antepenessentially affine, any fibre $\Sigma l_Y(\mathbb{E})$ is antepenessentially affine; in particular its core $\Sigma\mathbb{E}_\sharp$ is antepenessentially affine.*

Proof. Straightforward from Lemma 3.1, since any split epimorphism in these fibers belongs to Σ. ∎

When $\mathbb{E}$ is Mal'tsev category, any fibre is $Grd_Y\mathbb{E}$ is penessentially affine. In this section we shall review what is remaining of this observation in the partial context of the Σ-Mal'tsev categories.

5.10. Theorem. *Let $\mathbb{E}$ be a Σ-Mal'tsev category. Then any fibre $Grd_Y\mathbb{E}$ above Y is Σ^Y-penessentially affine. A groupoid is a Σ^Y-special groupoid if and only if the underlying split epimorphism of its endosome is in Σ; accordingly any Σ^Y-special groupoid is an abelian groupoid. When, in addition, Σ is point-congruous, the core $\Sigma^Y(Grd_Y\mathbb{E})_\sharp$ is antepenessentially affine.*

Proof. Let us show first $Grd_Y\mathbb{E}$ is Σ^Y-antepenessentially affine. Since $Grd_Y\mathbb{E}$ has a initial object, it is sufficient to check the property for the initial pullbacks. So let $(\underline{f}_1, \underline{s}_1)$ be a split epimorphism in Σ^Y. We have to show that the following upward square is a pushout in $Grd_Y\mathbb{E}$:

$$\begin{tikzcd}
\underline{K}_1[\underline{f}_1] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\
\Delta Y \arrow[r, tail, "\underline{\alpha}_1\underline{Z}_1"'] \arrow[u, shift right] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"']
\end{tikzcd}$$

According to Lemma 5.4 we get the following pullback in $\mathbb{E}$:

$$\begin{tikzcd}
K_1[\underline{f}_1] \arrow[r, tail, "k_1"] \arrow[d, shift right] & W_1 \arrow[d, shift right, "f_1"'] \arrow[l, bend right, "l"'] \\
Y \arrow[r, tail, "\alpha_1\underline{Z}_1 = s_0^{Z_1}"'] \arrow[u, shift right] & Z_1 \arrow[u, shift right, "s_1"'] \arrow[l, bend left, "d_0^{Z_1}"]
\end{tikzcd}$$

Suppose given a pair $\underline{h}_1 : \underline{K}_1[\underline{f}_1] \to \underline{V}_1, \underline{t}_1 : \underline{Z}_1 \to \underline{V}_1$ of internal functors in $Grd_Y\mathbb{E}$. Since (f_1, s_1) is in Σ, the pair (k_1, s_1) is jointly strongly epic in $\mathbb{E}$. So, according to Lemma 5.6, we have then $\check{h}_1 = \check{t}_1$, and we get the desired (unique) factorization $\underline{g}_1 : \underline{W}_1 \to \underline{V}_1$.

Now let us show it is Σ^Y-penessentially affine. So let $\underline{i}_1 : \underline{A}_1 \rightarrowtail \underline{K}_1[\underline{f}_1]$ a subobject in $\Sigma^Y_{\Delta_Y}$. Let us consider the following diagram in $\mathbb{E}$ where the upper parallelogram is a rightward pullback:

$$\begin{tikzcd}
A_1 \arrow[r, tail, dotted, shift right] \arrow[dr, "i_1"] \arrow[ddr, shift left] & W'_1 \arrow[l, shift right, "l'"'] \arrow[dr, "j_1"] & \\
& K_1[\underline{f}_1] \arrow[r, tail, shift right, "k_1"'] \arrow[d, shift right] & W_1 \arrow[l, shift right, "l"'] \arrow[d, shift right, "f_1"'] \\
& Y \arrow[r, tail, shift right, "s_0^{Z_1}"'] \arrow[u, shift right] \arrow[uul, shift left] & Z_1 \arrow[l, shift right, "d_0^{Z_1}"'] \arrow[u, shift right, "s_1"']
\end{tikzcd}$$

It produces a subobject W'_1 of W_1 in $\mathbb{E}$. Since the lower square is a rightward pullback as well, so is the external quadrangle which produces a split epimorphism $(f'_1, s'_1) : W'_1 \rightleftarrows Z_1$ which is in Σ, since so is $A_1 \rightleftarrows Y$. The pair (f'_1, s'_1) is actually underlying a split epimorphism between reflexive graphs. It remains to show that the subgraph W'_1 is actually a subgroupoid of $\underline{W}_1$. In *Set*, the subobject W'_1 is the set of those arrows $w : x \to x'$ in W_1 which are such that $l(w) = s_1 f_1(w^{-1}).w$ belongs to A_1. The formulae $l(w'.w) = (s_1 f_1(w^{-1}).l(w').s_1 f_1(w)).l(w)$ and $l(w^{-1}) = s_1 f_1(w).l(w)^{-1}.s_1 f_1(w^{-1})$ shows that, in order to check that W'_1 is underlying a subgroupoid, it is enough to check that $s_1(\phi).a.s_1(\phi^{-1})$ belongs to A_1 whenever a belongs to A_1 for any pair $(a, \phi) : x \xrightarrow{\phi} x' \xrightarrow{a} x'$ in the following pullback $A_1 \times_1 Z_1$:

$$\begin{tikzcd}
A_1 \arrow[r, tail, shift right, "\bar{k}_1"'] \arrow[d, shift right] & A_1 \times_1 Z_1 \arrow[l, shift right, "\bar{l}"'] \arrow[d, shift right, "\pi_1"'] \\
Y \arrow[r, tail, shift right, "s_0^{Z_1}"'] \arrow[u, shift right] & Z_1 \arrow[l, shift right, "d_1^{Z_1}"'] \arrow[u, shift right, "\sigma_1"']
\end{tikzcd}$$

Denote by $\chi : A_1 \times_1 Z_1 \to W_1$ the map defined by $\chi(a, \phi) = a.s_1(\phi)$. The condition described above is equivalent to the fact that $l.\chi : A_1 \times_1 Z_1 \to K_1[\underline{f}_1]$ factorizes through A_1, or to the fact that the pullback $\bar{i}_1 : \bar{A}_1 \rightarrowtail A_1 \times_1 Z$ of i_1 along $l.\chi$ is an isomorphism. Observe that $\bar{k}_1$ factorizes through $\bar{i}_1$ since $l.\chi(a, 1_{x'}) = a$, in the same way as σ_1 since $l.\chi(1_{x'}, \phi) = 1_x$. Now, since A_1 is a subobject in $\Sigma^Y_{\Delta_Y}$, the split epimorphism $A_1 \rightleftarrows Y$ is in Σ, and so is (π_1, σ_1). Accordingly the pair $(\bar{k}_1, \sigma_1)$ is jointly strongly epic in $\mathbb{E}$. Since both $\bar{k}_1$ and σ_1 factorizes through $\bar{i}_1$, it is an isomorphism.
The groupoid $\underline{Z}_1$ is Σ^Y-special if and only, in the diagram (2) in $\mathbb{E}$ above, the map $(d_0^{Z_1}, d_1^{Z_1}) : Z_1 \to Z_0 \times Z_0$ is Σ-special or equivalently if and only if the split epimorphism (p_0, s_0) in Σ. So it is the case if and only if the underlying split epimorphism of its endosome on the vertical left hand side is in Σ. When it is so, the induced group in $\mathbb{E}/Z_0$ is necessarily abelian, and the groupoid $\underline{Z}_1$ is an abelian groupoid. The last point is a consequence of the previous proposition. ∎

It does not seem possible to show that the construction of the groupoid $\underline{W}'_1$ of the first part of the theorem is stable in $\Sigma^Y(Grd_Y\mathbb{E})_\sharp$, or in other words to show that the core $\Sigma^Y(Grd_Y\mathbb{E})_\sharp$ is penessentially affine.

5.11. Proposition. *Let $\mathbb{E}$ be a Σ-Mal'tsev category. Let $\underline{h}_1 : \underline{Z}'_1 \to \underline{Z}_1$ be any internal functor in the fibre $Grd_Y\mathbb{E}$. If $\underline{Z}'_1$ and $\underline{Z}_1$ are Σ-groupoids, then any pullback of split epimorphism in this fibre:*

$$\begin{tikzcd} \underline{W}'_1 \arrow[r, "\underline{k}_1"] \arrow[d, shift right, "\underline{f}'_1"'] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\ \underline{Z}'_1 \arrow[r, "\underline{h}_1"'] \arrow[u, shift right, "\underline{s}'_1"'] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"'] \end{tikzcd}$$

produces an upward pushout, namely the change base $\underline{h}_1^* : Pt_{\underline{Z}_1} \to Pt_{\underline{Z}'_1}$ with respect to the fibration of points is fully faithful. It is moreover saturated on subobjects.

Proof. Since the initial groupoid Δ_Y is a Σ-groupoid, it is enough to check the property for the initial pullbacks:

$$\begin{tikzcd} \underline{K}_1[\underline{f}_1] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\ \Delta X \arrow[r, tail, "\alpha_1\underline{Z}_1"'] \arrow[u, shift right] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"'] \end{tikzcd}$$

The proof now is exactly the same as in Theorem 5.10 since, the groupoid $\underline{Z}_1$ being a Σ-groupoid (i.e. the split epimorphism $(d_0^{Z_1}, s_0^{Z_1})$ being in Σ), the pair (k_1, s_1) is jointly strongly epic in $\mathbb{E}$.

Again, the saturation on subobjects is checked exactly as in Theorem 5.10 since, the groupoid $\underline{Z}_1$ being a Σ-groupoid, the pair $(\bar{k}_1, \sigma_1)$ in this proof is jointly strongly epic in $\mathbb{E}$. ■

We get some precisions about the fibre $Grd_Y\mathbb{E}$ when the object Y is supposed to be Σ-special in $\mathbb{E}$:

5.12. Proposition. *Let $\mathbb{E}$ be a Σ-Mal'tsev category and Σ be point-congruous. Let us denote by $\Sigma Grd_Y\mathbb{E}$ the full subcategory of $Grd_Y\mathbb{E}$ whose objects are the Σ-groupoids. When the object Y is Σ-special in $\mathbb{E}$, any Σ-groupoid on Y is Σ^Y-special and therefore an abelian groupoid. This full subcategory $\Sigma Grd_Y\mathbb{E}$ of the core $\Sigma^Y(Grd_Y\mathbb{E})_\sharp$ is penessentially affine.*

Proof. When Y is Σ-special, then the groupoid ∇_Y (i.e. the terminal object of the fibre $Grd_Y\mathbb{E}$) is a Σ-groupoid. When Σ is point-congruous, the subcategory $\Sigma Grd_Y\mathbb{E}$ is stable under finite limits in $Grd_Y\mathbb{E}$. When $\underline{Z}_1$ is a Σ-groupoid, the following diagram in $\mathbb{E}$:

$$\begin{tikzcd} Z_1 \arrow[r, "(d_0^{Z_1}, d_1^{Z_1})"] \arrow[d, shift right, "d_0"'] & Z_0 \times Z_0 \arrow[d, shift right, "p_0"'] \\ Y \arrow[r, equal] \arrow[u, shift right, "s_0"'] & Y \arrow[u, shift right, "s_0"'] \end{tikzcd}$$

shows by Lemma 3.1 that the map $(d_0^{Z_1}, d_1^{Z_1}) : Z_1 \to Z_0 \times Z_0$ is Σ-special in $\mathbb{E}$ since so are d_0 and p_0. Accordingly $\underline{Z}_1$ is Σ^Y-special.

Then $\Sigma Grd_Y\mathbb{E}$ is antepenessentially affine since so is $\Sigma^Y(Grd_Y\mathbb{E})_\sharp$. As for the construction of the subgroupoid $\underline{W}'_1$ from the subobject A_1, it remains to check it lies in $\Sigma Grd_Y\mathbb{E}$. It is the case since, in the pullback defining it, we noticed that the map l:

$$\begin{tikzcd} K_1[\underline{f}_1] \arrow[dr, shift left] & & W_1 \arrow[ll, "l"'] \arrow[dl, shift left, "d_0^{W_1}"'] \\ & Y \arrow[ul, shift left] \arrow[ur, shift left, "s_0^{W_1}"'] & \end{tikzcd}$$

actually lies in $Pt_Y(\mathbb{E})$, and more precisely in Σ_Y in our case, and since, Σ being point-congruous, Σ_Y is stable under finite limits in $Pt_Y(\mathbb{E})$. ∎

With Theorem 5.10 we produced examples of Malt'sev categories (since they are protomodular) which are Σ-naturally Mal'tsev as well, for a certain class Σ of split epimorphisms.

6. Σ-affine categories

In this section we shall deal with what is remaining of both conditions 1′) and 4) of the characterization given in Section 2.

6.1. Definition and characterization. For that, it is worth introducing the following:

6.2. Definition. *Let $\mathbb{E}$ be a category endowed with a fibrational class Σ of split epimorphisms. We call it Σ-affine, if the base change $f^*: Pt_Y(\mathbb{E}) \to Pt_X(\mathbb{E})$ is an equivalence of categories whenever f is underlying a split epimorphism $(f,s): X \rightleftarrows Y$ in Σ.*

6.3. Proposition. *Let $\mathbb{E}$ be a finitely complete category and Σ a fibrational class of split epimorphisms. The following conditions are equivalent:*
1) $\mathbb{E}$ is Σ-affine
2) $\mathbb{E}$ is Σ-naturally Mal'tsev and any split epimorphism in Σ is endowed with an abelian group structure
3) $\mathbb{E}$ is Σ-naturally Mal'tsev and any split epimorphism in Σ is Σ-special.
When, in addition, Σ is point-congruous, the condition 2) is equivalent to:
2′) $\mathbb{E}$ is Σ-naturally Mal'tsev and any fibre Σ_Y is additive.

Proof. Since s^* is a left inverse to f^*, saying that f^* is an equivalence of categories is equivalent to saying that its inverse equivalence is s^*, or that the inverse equivalence of s^* is f^*. Here this means that pushing out along s coincides with pulling back along f. If the condition 1) holds for any split epimorphism in Σ, then the condition of Σ-natural Mal'tsevness is fulfilled and any split epimorphism (f,s) in Σ is given a canonical commutative monoid structure. Then consider the following rightward pullback of split epimorphism:

$$\begin{tikzcd}
R[f] \arrow[r, dotted, shift right, "d"'] \arrow[d, shift right, "d_0"'] & X \arrow[l, shift right, "s_1"'] \arrow[d, shift right, "f"'] \\
X \arrow[u, shift right, "s_0"'] \arrow[r, shift right, "f"'] & Y \arrow[l, shift right, "s"'] \arrow[u, shift right, "s"']
\end{tikzcd}$$

Since $s^* : Pt_X(\mathbb{E}) \to Pt_Y(\mathbb{E})$ is an equivalence of categories there is a unique map $d : R[f] \to X$ making the leftward square a pullback of split epimorphism. The commutativity of this square makes d a subtraction on (f, s) which gives a group structure to this canonical commutative monoid structure.

On the other hand any abelian group structure on a split epimorphism in Σ makes f Σ-special by Corollary 2.9.

Conversely suppose that $\mathbb{E}$ is a Σ-naturally Mal'tsev category and that any split epimorphism (f, s) in Σ makes f Σ-special. Take any split epimorphism $(g', t') : X' \rightleftarrows X$ and denote $(g, t) = s^*(g', t')$. Complete the lower row below with the kernel equivalence relation of f and denote by s_1 the unique map such that $d_1.s_1 = 1_X$ and $d_0.s_1 = s.f$. The split epimorphism (d_0, s_0) is in Σ since f is Σ-special. Then consider the following diagram where $(\check{g}', \check{t}')$ is $d_0^*(g', t')$, in other words where the non dotted left hand side square indexed by 0 is a pullback of split epimorphisms with a map $\sigma_0 : X' \to \check{R}$ above $s_0 : X \to R[f]$:

$$\begin{tikzcd}
\check{R} \arrow[r, shift left=3, "\delta_0"] \arrow[r, dotted, shift right=1, "\delta_1"'] \arrow[d, shift right, "\check{g}'"'] & X' \arrow[l, shift left=1] \arrow[l, dotted, shift left=3, "\sigma_1"] \arrow[r, dotted, two heads, shift left, "f'"] \arrow[d, shift right, "g'"'] & Y' \arrow[l, shift left, "s'"] \arrow[d, shift right, "g"'] \\
R[f] \arrow[u, shift right, "\check{t}'"'] \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X \arrow[u, shift right, "t'"'] \arrow[l] \arrow[r, two heads, shift left, "f"] \arrow[l, bend left=60, "s_1"] & Y \arrow[l, shift left, "s"] \arrow[u, shift right, "t"']
\end{tikzcd}$$

Since (d_0, s_0) is in Σ and $\mathbb{E}$ is a Σ-naturally Mal'tsev category, the upward and rightward left hand side square is a pushout which produces a map δ_1 above d_1 giving rise to the upper reflexive graph. The square indexed by 1 is a pullback as well since $d_1^*(g', t')$ is produced by the pushout along the common splitting s_0 of d_0 and d_1. This pullback indexed by 1 in turn produces the splitting σ_1 above the splitting s_1 and makes $(g', t') = s_1^*(\check{g}', \check{t}') = s_1^* d_0^*(g', t') = f^* s^*(g', t')$.

Suppose now Σ is point-congruous and $\mathbb{E}$ is Σ-naturally Mal'tsev. Then the fibre Σ_Y is stable under finite limits in $Pt_Y(\mathbb{E})$. So, if any split epimorphism $(f, s) : X \rightleftarrows Y$ in Σ is endowed with a natural abelian group structure, the fibre Σ_Y is an additive subcategory of $Ab(Pt_Y(\mathbb{E}))$. Conversely if Σ_Y is additive, the commutative monoid structure on any split epimorphism (f, s) in Σ is actually a group structure. ∎

The category CoM of commutative monoids is an example of a Π-naturally Mal'tsev category which is not Π-affine. Any protomodular Σ-naturally Mal'tsev category is Σ-

affine: any split epimorphism $(f, s) : X \rightleftarrows Y$ in Σ is endowed with a canonical monoid structure of (f, s) in $Pt_Y(\mathbb{E})$ (by Proposition 2.5) which is actually a group structure, since $\mathbb{E}$ is protomodular.

6.4. Corollary. *Let $\mathbb{E}$ be a Σ-Mal'tsev category, then any fibre $Grd_Y\mathbb{E}$ is a Σ^Y-affine category.*

6.5. Back to the autonomous quandles.

6.6. Proposition. *The category AQd of autonomous quandles is Σ'-affine, where Σ' is the class of acupuncturing split epimorphisms.*

Proof. Let $(f, s) : X \rightleftarrows Y$ a split epimorphism and $(g, t) : Y \rightleftarrows Z$ a split epimorphism in Σ' with its associated homomorphism $\rho : Y \to Y$ (see Example 1.10.3). Then consider the following rightward pullback along t, and complete the diagram with the universal quadrangle:

$$\begin{tikzcd}
Y \times_Z f^{-1}Z \arrow[dr, "\psi"] \arrow[drr, shift left, "p_Z"] \arrow[ddr, shift right, "p_Y"'] & & \\
& X \arrow[d, shift right, "f"'] & f^{-1}Z \arrow[l, "t'"'] \arrow[ull, shift left, "i_Z"] \arrow[d, shift right, "\bar{f}"'] \\
& Y \arrow[u, shift right, "s"'] \arrow[uul, shift right, "i_Y"'] \arrow[r, shift left, "g"] & Z \arrow[u, shift right, "\bar{s}"'] \arrow[l, shift left, "t"]
\end{tikzcd}$$

Since AQd is Σ'-naturally Mal'tsev, there is a natural factorization ψ defined by $\psi(y, a) = k(a) \triangleright s\rho(y)$. We have $f.\psi(y, a) = y$ and $a = k(a) \triangleright \bar{s}\bar{f}(a) = k(a) \triangleright stg(y)$. So ψ is injective. It remains to show it is surjective. Let us set:

$$g'(x) = (x \triangleright^{-1} s\rho f(x)) \triangleright stgf(x)$$

We check: $f(g'(x)) = (f(x) \triangleright^{-1} \rho f(x)) \triangleright tgf(x) = tgf(x) \triangleright tgf(x) = tgf(x)$. From $g'(x) = k(g'(x)) \triangleright sfg'(x) = k(g'(x)) \triangleright stgf(x)$, we get:

$$k(g'(x)) = g'(x) \triangleright^{-1} stgf(x) = x \triangleright^{-1} s\rho f(x)$$

whence: $\psi(f(x), g'(x)) = k(g'(x)) \triangleright s\rho f(x) = (x \triangleright^{-1} s\rho f(x)) \triangleright s\rho f(x) = x$. ∎

7. Internal groups in Qnd

The results of Section 5.7 apply in particular to the category $Gp\mathbb{E} = Grd_1\mathbb{E}$ of internal groups in a Σ-Mal'tsev category $\mathbb{E}$. This will allow us to elaborate on the results already given in [8] about the category $Gp(Qnd)$ of internal groups in the category of quandles. Denote $U : Gp\mathbb{E} \to \mathbb{E}$ the forgetful functor which is conservative. The class Σ^1 with respect to this same section coincides with the class $\bar{\Sigma} = U^{-1}\Sigma$. So we get:

7.1. Proposition. *Let $\mathbb{E}$ be a Σ-Mal'tsev category. A split epimorphism $(f,s) : X \rightleftarrows Y$ in $Gp\mathbb{E}$ is in $\bar{\Sigma}$ if and only if its kernel $K[f]$ is a Σ-special object in $\mathbb{E}$, which implies that it is an abelian group. The category $Gp\mathbb{E}$ is $\bar{\Sigma}$-penessentially affine. When Σ is point-congruous, the full subcategory of Σ-special (necessarily abelian) groups, i.e. the core $\bar{\Sigma}Gp\mathbb{E}_\sharp$, is included in $Ab\mathbb{E}$ and stable under finite limits in $Ab\mathbb{E}$; accordingly this core is an additive category.*

Proof. Since (f,s) is in Σ, the terminal map $K[f] \to 1$ is in Σ, but $K[f]$ being an internal group the terminal map $K[f] \to 1$ is Σ-special by Corollary 2.9, and $K[f]$ is a Σ-special object. Conversely when $K[f]$ is Σ-special, the following rightward pullback in $\mathbb{E}$, as in diagram (1) above:

$$\begin{tikzcd}
K[f] \arrow[r, tail, "k_1"] \arrow[d, shift right] & X \arrow[l, bend right=60, "l"'] \arrow[d, shift right, "f"'] \\
1 \arrow[r, tail] \arrow[u, shift right] & Y \arrow[u, shift right, "s"'] \arrow[l, bend left=60]
\end{tikzcd}$$

makes Σ-special the split epimorphism (f,s) which therefore belongs to Σ. The second point is Theorem 5.10. ∎

We recalled above that the category *Qnd* of quandles is a Mal'tsev category relatively to the two classes $\Sigma' \subset \Sigma$ of respectively the acupincturing and puncturing split epimorphisms. Accordingly the category $Gp(Qnd)$ is both $\bar{\Sigma}$-penessentially affine and $\bar{\Sigma}'$-penessentially affine. Recall from Corollary 5.26 in [8] that the inclusion $Gp(AQd) \rightarrowtail Gp(Qnd)$ is actually an isomorphism. The core $\bar{\Sigma}'Gp(Qnd)_\sharp = \bar{\Sigma}'Gp(AQd)_\sharp$ is the additive category $Ab(LAQd)$ of the internal abelian groups in the latin autonomous quandles. Let us also recall from [8] the following observations. Denote by $ZautGp$ the category whose objects are the pairs $((G,\cdot),g)$ of a group and an automorphism g such that for all x the element $h(x) = g(x^{-1}) \cdot x$ is in the center of G (which makes h an endomorphism of groups), and whose morphisms are the group homomorphisms commuting with these automorphisms. The forgetful functor $U : ZautGp \to Gp$ being conservative and the category Gp being protomodular, the category $ZautGp$ is protomodular as well and consequently a Mal'tsev category.
There is a functor $Al : ZautGp \to Gp(Qnd) = Gp(AQd)$ defined by $Al((G,\cdot),g) = (X, \triangleright_g)$ with $x \triangleright_g y = g(x \cdot y^{-1}) \cdot y = g(x) \cdot g(y)^{-1} \cdot y = g(x) \cdot h(y)$. *The functor Al is an isomorphism of categories.*
We shall denote by the same symbols the classes of split epimorphisms in $ZautGp$ induced by the classes $\bar{\Sigma}' \subset \bar{\Sigma}$ in $Gp(Qnd)$. A split epimorphism (f,s) in $ZautGp$ is in $\bar{\Sigma}$ (resp. $\bar{\Sigma}'$) if and only if the restriction of h to $K[f]$ is surjective (resp. bijective). So, we can assert that the pointed protomodular category $ZautGp$ is both $\bar{\Sigma}'$-penessentially affine and $\bar{\Sigma}$-penessentially affine. The core $\bar{\Sigma}'ZautGp_\sharp$ is the additive full subcategory $LAutAb$ of

$ZautGp$ whose objects are the pairs $((A,+),g)$ of an abelian group and an automorphism g such that $Id_A - g$ is an isomorphism.

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