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BUILDING A MODEL CATEGORY OUT OF MULTIPLIER IDEAL SHEAVES

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ABSTRACT. We will construct a Quillen model structure out of the multiplier ideal sheaves on a smooth quasi-projective variety using earlier works of Isaksen and Barnea and Schlank. We also show that fibrant objects of this model category are made of kawamata log terminal pairs in birational geometry.

Contents

1. Introduction

In this note, we begin a homotopical study of multiplier ideal sheaves. Our guiding principle is not new: To understand a multiplier ideal sheaf, we try to understand a category built of multiplier ideal sheaves. Hence the homotopy theory we are aiming at is categorical; more precisely, we use Quillen's framework of (closed) model categories ([Quillen, 1967]). We will construct a model structure using the multiplier ideal sheaves on a smooth quasi-projective variety using earlier works of Isaksen [Isaksen, 2004] and Barnea and Schlank [Barnea and Schlank, 2016]. Interestingly, fibrant objects of this model category are made of kawamata log terminal pairs in birational geometry.

Now let us recall the multiplier ideal sheaves. Multiplier ideal sheaves can be created from effective divisors, non-zero ideal sheaves, and linear systems. Because of the motivation discussed below we explain the multiplier ideal sheaves of effective divisors.

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Let X be a smooth projective variety defined over the set of complex numbers. Let D be an effective $\mathbb{Q}$-divisor on X, i.e., a formal finite linear combination of reduced and irreducible codimension one subvarieties of X with non-negative rational coefficients. Let $f : Y \to X$ be a projective birational morphism from a smooth variety Y such that the support of the exceptional locus $\mathrm{Ex}(f)$ of f on Y is a divisor on Y and the components of the pullback f^*D of D and $\mathrm{Ex}(f)$ intersect transversally. Then the multiplier ideal sheaf $\mathscr{I}(X, D)$ of the pair (X, D) is defined by

$$\mathscr{I}(X, D) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor f^*D \rfloor). \tag{1}$$

It is an ideal sheaf on X and does not depend on the choice of f. For more than the last two decades, many applications of multiplier ideal sheaves have been found in algebraic geometry. We refer to the book [Lazarsfeld, 2004] for a comprehensive introduction to this subject and some applications. For a later development we refer to [Ein an Mustata, 2006].

One important property of multiplier ideal sheaves is the vanishing of higher cohomology groups

$$H^i(X, \mathscr{I}(X, D) \otimes \mathcal{O}_X(K_X + L)) = 0 \quad \text{for } i > 0 \tag{2}$$

where L is an ample divisor on X such that $L - D$ is an ample $\mathbb{Q}$-divisor and K_X is the canonical divisor on X. A ($\mathbb{Q}$-)divisor on X is ample if the numbers produced by the intersections with all the subvarieties including X itself are positive. The vanishing (2) was first proved by Nadel in the analytic setting. When $D = 0$, hence $\mathscr{I}(X, D) = \mathcal{O}_X$, it is the classical Kodaira vanishing theorem. The vanishing property (2) played a key role in providing a partial answer to a conjecture raised by Takao Fujita.

In [Fujita, 1987], Fujita raised two conjectures on adjoint line bundles on smooth projective varieties. One of them says that if X is a smooth projective variety defined over the set of complex numbers and L is an ample divisor on X, the adjoint line bundle associated with $K_X + mL$ is globally generated for all integer $m > \dim X$. The adjoint line bundle is globally generated if for every point $p \in X$, there is a section $s \in H^0(X, \mathcal{O}_X(K_X + mL))$ such that $s(p) \neq 0$, or equivalently there is a divisor $D \in |K_X + mL|$ such that $p \notin D$. Thus if the conjecture is true, the associated rational map $\phi_{K_X+mL} : X \to \mathbb{P}^N$ is in fact a morphism defined on every point of X. The conjecture is known for small dimensions ([Reider, 1988], [Ein and Lazarsfeld, 1993], [Kawamata, 1997], [Ye and Zhu, 2015]). One of the motivations for this project is to understand the conjecture in terms of homotopy theory.

As we mentioned in the beginning, the homotopy theory that we want to build is a model category. To build such a category, we will define a category $\mathcal{M}(X)$ for every smooth quasi-projective variety X that may be considered as the category of all multiplier ideal sheaves (1) on X with the resolutions f associated with them. However the category $\mathcal{M}(X)$ has a rather simple structure. It is a preorder, i.e. the hom-sets have at most one morphism. It does not have enough morphisms for the lifting axiom or the factorization axiom of model categories. To remedy this, we will consider the pro-category, introduced by Grothendieck in [Grothendieck, 1960] and developed in [Artin, Grothendieck, and

Verdier, 1972], of $\mathcal{M}(X)$ and apply the results of Isaksen [Isaksen, 2004] and Barnea and Schlank [Barnea and Schlank, 2016].

Based on the earlier works [Grossman, 1975] and [Edwards and Hastings, 1976], Isaksen introduced strict model structures for the pro-categories of model categories in [Isaksen, 2004]. Recently, Barnea and Schlank generalized his result by introducing weak fibration categories in [Barnea and Schlank, 2016]. A weak fibration structure on a category with finite limits consists of two sets of morphisms. One is a set of weak equivalences and the other is a set of fibrations. In [Barnea and Schlank, 2016] the authors show that the pro-category of a weak fibration category has three sets of morphisms that satisfy all the axioms (Definition 7.34) of the model category once they have the two out of three property. The weak equivalences and the fibrations in a weak fibration category generate the weak equivalences and the fibrations in the model structure of the associated pro-category respectively. We will show that $\mathcal{M}(X)$ is a weak fibration category.

Our view on multiplier ideal sheaves is that they are tools that help us to understand birational morphisms. In other words, what is really important is birational morphisms. With this in mind, we will define a weak fibration structure on $\mathcal{M}(X)$ as in the following paragraph. However, because we want our construction to have wider applications, we treat weighted ideal sheaves instead of effective $\mathbb{Q}$-divisors. A weighed ideal sheaf $\mathcal{I}^c$ for a non-negative rational c can be thought of taking c-th power of $\mathcal{I}$. The multiplier ideal sheaf of an effective $\mathbb{Q}$-divisor D is the multiplier ideal sheaf of the associated weighted invertible ideal sheaf.

We denote by $\mathcal{B}$ the category of normal quasi-projective varieties and projective birational morphisms. The category $\mathcal{M}$ is built on $\mathcal{B}$ with an additional structure. Its objects $(Y, \underline{\mathcal{I}}^c)$ consist of a normal quasi-projective variety Y and a reduced finite set of weighted ideal sheaves $\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ on Y (Definition 4.6, Definition 4.7). The hom-set $\mathcal{M}((Y, \underline{\mathcal{I}}^c), (Z, \underline{\mathcal{J}}^d))$ consists of morphisms $f : Y \to Z$ in $\mathcal{B}$ such that $\underline{\mathcal{I}}^c \leq f^*\underline{\mathcal{J}}^d$ where $f^*\underline{\mathcal{J}}^d$ is the pullback of $\underline{\mathcal{J}}^d$ on Y (Definition 4.6, Definition 4.14). Then, given a smooth quasi-projective variety X, $\mathcal{M}(X)$ is defined to be the comma category of $\mathcal{M}$ over $(X, \mathcal{O}_X^0)$.

$$\mathcal{M}(X) = (\mathcal{M} \downarrow (X, \mathcal{O}_X^0)) \tag{3}$$

In Section 5, we will define the multiplier ideal sheaf (Definition 5.14)

$$\mathscr{I}(f) \subseteq \mathcal{O}_X \tag{4}$$

of an object f of $\mathcal{M}(X)$ generalizing (1).

In Section 6, we will show that the category $\mathcal{M}(X)$ has finite limits and a weak fibration structure $(\mathcal{W}(X), \mathcal{F}(X))$ where $\mathcal{W}(X)$ is the set of all morphisms in $\mathcal{M}(X)$ whose underlying morphisms are isomorphisms in $\mathcal{B}$ and $\mathcal{F}(X)$ consists of all morphisms in $\mathcal{M}(X)$ whose domains and codomains produce the same multiplier ideal sheaves.

In Section 7, we will prove the main theorem, Theorem 1.1, after reviewing some well-known facts for pro-categories and model categories. Let $\mathrm{Pro}(X)$ be the pro-category (Definition 7.3) of the weak fibration category $\mathcal{M}(X)$

$$\mathrm{Pro}(X) = \mathrm{Pro}(\mathcal{M}(X)) \tag{5}$$

and define three sets

$$\mathcal{W}_{\mathrm{Pro}}(X) = \mathrm{Lw}^{\cong}(\mathcal{W}(X)) \tag{6}$$
$$\mathcal{F}_{\mathrm{Pro}}(X) = \mathrm{Sp}^{\cong}(\mathcal{F}(X)) \tag{7}$$
$$\mathcal{C}_{\mathrm{Pro}}(X) = {}^{\boxslash}(\mathcal{W}(X) \cap \mathcal{F}(X)) \tag{8}$$

of morphisms in $\mathrm{Pro}(X)$. Given a non-empty set $\mathcal{S}$ of morphisms in $\mathcal{M}(X)$, $\mathrm{Lw}^{\cong}(\mathcal{S})$ is the set of all morphisms isomorphic to the morphisms associated with natural transformations whose components are in $\mathcal{S}$. $\mathrm{Sp}^{\cong}(\mathcal{S})$ is similar to $\mathrm{Lw}^{\cong}(\mathcal{S})$, but its elements have an additional property that enables us to control their behavior inductively (Definition 7.42). ${}^{\boxslash}\mathcal{S}$ is the set of all morphisms in $\mathrm{Pro}(X)$ satisfying the left lifting property with respect to every morphism in $\mathcal{S}$. Given two morphisms f and g in $\mathrm{Pro}(X)$, f satisfies the left lifting property with respect to g if every commutative diagram
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$
in $\mathrm{Pro}(X)$ of solid arrows has a lifting of dotted arrow (Definition 7.31).

In Section 7.50, we will show that $\mathcal{W}_{\mathrm{Pro}}(X)$ has the two out of three property by slightly modifying the argument in [Isaksen, 2004]. Then by Theorem 4.18 in [Barnea and Schlank, 2016] (Theorem 7.46), we have two functorial weak factorization systems (Definition 7.32)

$$(\mathcal{W}_{\mathrm{Pro}}(X) \cap \mathcal{C}_{\mathrm{Pro}}(X), \mathcal{F}_{\mathrm{Pro}}(X)) \tag{9}$$

and

$$(\mathcal{C}_{\mathrm{Pro}}(X), \mathcal{W}_{\mathrm{Pro}}(X) \cap \mathcal{F}_{\mathrm{Pro}}(X)) \tag{10}$$

on the pro-category $\mathrm{Pro}(X)$. Hence $(\mathcal{W}_{\mathrm{Pro}}(X), \mathcal{C}_{\mathrm{Pro}}(X), \mathcal{F}_{\mathrm{Pro}}(X))$ provides a functorial model structure (Definition 7.34) on $\mathrm{Pro}(X)$. The following is our main theorem.

1.1. Theorem. *Let X be a smooth quasi-projective variety. Then*

1. $\mathrm{Pro}(X)$ *is a preorder.*
2. $\mathrm{Pro}(X)$ *has small limits and small colimits.*
3. $(\mathcal{W}_{\mathrm{Pro}}(X), \mathcal{C}_{\mathrm{Pro}}(X), \mathcal{F}_{\mathrm{Pro}}(X))$ *is a functorial model structure on* $\mathrm{Pro}(X)$.
4. $\mathcal{W}_{\mathrm{Pro}}(X) \cap \mathcal{C}_{\mathrm{Pro}}(X) = {}^{\boxslash}\mathcal{F}(X)$.
5. $\mathcal{W}_{\mathrm{Pro}}(X) \cap \mathcal{F}_{\mathrm{Pro}}(X) = \mathrm{Sp}^{\cong}(\mathcal{W}(X) \cap \mathcal{F}(X))$.

There is a different model structure on $\mathrm{Pro}(X)$ with the same set of weak equivalences, hence with the same homotopy category. See Proposition 3.1 in [Droz and Zakharevich, 2015]. However, these two are quite different in that every morphism is a cofibration in the Droz-Zakharevich model structure on $\mathrm{Pro}(X)$. It seems that cofibrant objects and cofibrations in general are rather special in our model structure. For example, no object of $\mathcal{M}(X)$ is cofibrant in $\mathrm{Pro}(X)$. See Proposition 8.2.

Contrary to the cofibrant objects, the fibrant objects have an explicit description in terms of kawamata log terminal pairs in birational geometry. Given an object $f : (Y, \underline{\mathcal{J}}^c) \to (X, \mathcal{O}_X^0)$ of $\mathcal{M}(X)$, we will say that f is klt (kawamata log terminal) if

$$\mathscr{I}(f) = \mathcal{O}_X \tag{11}$$

holds. If $\underline{\mathcal{J}}^c = \{\mathcal{J}_1^{c_1}, \dots, \mathcal{J}_m^{c_m}\}$, then f is klt iff $f : (Y, \mathcal{J}_i^{c_i}) \to (X, \mathcal{O}_X)$ is klt for every $i = 1, \dots, m$. See Lemma 8.5. We call an object $F : \mathcal{I} \to \mathcal{M}(X)$ of $\mathrm{Pro}(X)$ klt if every component of F is klt, i.e. $F(i)$ is klt for every $i \in \mathcal{I}$. It is a natural generalization of the original klt pairs to our setting. See the discussion in the beginning of Section 8.3. We will prove the following result in Section 8.

1.2. Theorem. *Let X be a smooth quasi-projective variety. Then for every object F of* $\mathrm{Pro}(X)$*, the following are equivalent.*

1. *F is a fibrant object in* $\mathrm{Pro}(X)$*.*
2. *F is klt.*

Later in Section 9, we will reformulate the conjecture of Fujita using a variant (Theorem 9.5) of Theorem 1.2 discussed in Section 9.1.

Before we close the introduction, we make two remarks. First, there is a variant of the model structure in Theorem 1.1. Altering the multiplier ideal sheaves of objects in $\mathcal{M}(X)$ slightly, we get another weak fibration structure $(\mathcal{W}^{\mathrm{lc}}(X), \mathcal{F}^{\mathrm{lc}}(X))$ on $\mathcal{M}(X)$, hence another model structure

$$(\mathcal{W}^{\mathrm{lc}}_{\mathrm{Pro}}(X), \mathcal{C}^{\mathrm{lc}}_{\mathrm{Pro}}(X), \mathcal{F}^{\mathrm{lc}}_{\mathrm{Pro}}(X)) \tag{12}$$

on $\mathrm{Pro}(X)$. The set $\mathcal{W}^{\mathrm{lc}}(X)$ of weak equivalences is $\mathcal{W}(X)$. But $\mathcal{F}^{\mathrm{lc}}(X)$ is different from $\mathcal{F}(X)$.

Let $f : (Y, \underline{\mathcal{J}}^c) \to (X, \mathcal{O}_X^0)$ be an object of $\mathcal{M}(X)$. Let $\underline{\mathcal{J}}^c = \{\mathcal{J}_1^{c_1}, \dots, \mathcal{J}_m^{c_m}\}$. For any positive rational number ϵ, we denote by $\underline{\mathcal{J}}^{\epsilon \cdot c}$ the finite set $\{\mathcal{J}_1^{\epsilon \cdot c_1}, \dots, \mathcal{J}_m^{\epsilon \cdot c_m}\}$. The new multiplier ideal sheaf $\mathscr{I}^{\mathrm{lc}}(f)$ of f is

$$\mathscr{I}^{\mathrm{lc}}(f) = \mathscr{I}((Y, \underline{\mathcal{J}}^{(1-\epsilon)\cdot c}) \xrightarrow{f} (X, \mathcal{O}_X^0)) \tag{13}$$

where ϵ is a positive rational number such that

$$\mathscr{I}((Y, \underline{\mathcal{J}}^{(1-\eta)\cdot c}) \xrightarrow{f} (X, \mathcal{O}_X^0)) = \mathscr{I}((Y, \underline{\mathcal{J}}^{(1-\epsilon)\cdot c}) \xrightarrow{f} (X, \mathcal{O}_X^0)) \tag{14}$$

holds for every $0 < \eta < \epsilon$.

There are other singularities important in birational geometry. One of them is log canonical singularities. Kawamata log terminal singularities are log canonical singularities and the set of log canonical singularities forms a largest class where the discrepancies make sense.

We can define log canonical objects in $\mathrm{Pro}(X)$ just as we define kawamata log terminal objects in $\mathrm{Pro}(X)$. Then with a little change in the proof of Theorem 1.2, it can be shown that the fibrant objects in the new model structure (12) are determined by log canonical objects in the same way that the fibrant objects in the model category of Theorem 1.1 are determined by kawamata log terminal objects by Theorem 1.2.

Finally, the results in this paper can be extended from a smooth quasi projective variety X to a pair (X, B) of a normal quasi-projective variety X with an effective $\mathbb{Q}$-divisor B on X such that $K_X + B$ is $\mathbb{Q}$-Cartier, i.e., some integral multiplier of $K_X + B$ is Cartier.

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2. Conventions

2.1. Categories. Throughout this note, we will work with a fixed Grothendieck universe $\mathcal{U}$ (cf. Chapter I.6 in [Mac Lan, 1998]).

A **small set** is an element of $\mathcal{U}$. We will use the term **set** to include small sets, subsets of $\mathcal{U}$, and other sets such as $\{\mathcal{U}\}$.

A category $\mathcal{C}$ is **locally small** if for every object a, b of $\mathcal{C}$, the hom-set $\mathcal{C}(a, b)$ is a small set.

In this note, we will assume that every category is locally small. It enables us to contain the hom-sets of pro-categories within $\mathcal{U}$: the pro-category of a locally small category is again locally small. However, we do not assume that the set of objects of a category is small.

A category is **small** if both the set of objects and the set of arrows are small sets.

Since the set of integers is a small set and the power set of a small set is again a small set by axioms of the universe $\mathcal{U}$, both the set of quasi-projective varieties defined over the field of complex numbers and the set of all morphisms between them are small sets.

A **preorder** is a category in which, given any two objects a and b, there is at most one morphism $a \to b$. In any preorder P, we define a binary relation $\geq$ on the set of the objects of P with

$$a \geq b \text{ iff there is a morphism } a \to b \text{ in } P. \tag{15}$$

Note that following [Barnea and Schlank, 2016], we are using $\geq$ instead of $\leq$.

A **poset** is a preorder such that $a \geq b$ and $a \leq b$ imply $a = b$.

2.2. Varieties. In this note, the varieties are reduced and irreducible, of dimension greater than one, and defined over the field $\mathbb{C}$ of complex numbers.

3. Review on Basic Notions in Birational Geometry

Here we collect basic notions in birational geometry and well-known properties of birational morphisms. We refer to [Kollár and Mori, 1998] for a comprehensive introduction on this subject.

3.1. Definition. *Let X be a variety. A* ***prime divisor*** *on X is a reduced and irreducible subvariety of X with codimension 1. A* $\mathbb{Q}$***-divisor*** *on X is a formal finite linear combination*

$$D = \sum_i a_i D_i \tag{16}$$

of distinct prime divisors with $a_i \in \mathbb{Q}$. A $\mathbb{Q}$-divisor D on X is $\mathbb{Q}$-***Cartier*** *if mD is a Cartier divisor for some $m \in \mathbb{N}$.*

3.2. Definition. *Let X be a variety. The* ***support*** *of a divisor $\sum_i d_i D_i$ on X is the subset $\cup_i D_i$ of X. A divisor $\sum_i d_i D_i$ has* ***simple normal crossings*** *if every D_i is smooth and they intersects transversally. A $\mathbb{Q}$-divisor $\sum_i a_i D_i$ has a* ***simple normal crossing support*** *if $\sum_i D_i$ has simple normal crossings.*

3.3. Definition. *Let X be a variety. Let $D = \sum_i a_i D_i$ be a $\mathbb{Q}$-divisor on X. We define*

$$\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i \tag{17}$$

where $\lfloor a_i \rfloor$ is the round-down of a_i for every i, and

$$\lceil D \rceil = \sum_i \lceil a_i \rceil D_i \tag{18}$$

where $\lceil a_i \rceil$ is the round-up of a_i for every i.

3.4. Definition. *Let X be a variety. Let $P = \sum_i a_i D_i$ and $N = \sum_i b_i D_i$ be two $\mathbb{Q}$-divisors on X. We write*

$$P \geq N \tag{19}$$

if $a_i \geq b_i$ holds for every i. A $\mathbb{Q}$-divisor D is called ***effective*** *if*

$$D \geq 0 \tag{20}$$

holds where 0 is the zero divisor on X. In other words, a $\mathbb{Q}$-divisor $\sum_i a_i D_i$ on X is effective iff $a_i \geq 0$ for every i.

3.5. Definition. *Let $f : Y \to X$ be a projective birational morphism between quasi-projective varieties. A $\mathbb{Q}$-divisor $\sum_i a_i D_i$ on Y is called f-**exceptional** if*

$$\dim f(D_i) < \dim X - 1 \tag{21}$$

holds for every i.

3.6. Definition. *Let $f : Y \to X$ be a projective birational morphism between quasi-projective varieties. We denote by* $\mathrm{Ex}(f)$ *the exceptional locus of f.*

$$\mathrm{Ex}(f) = \{y \in Y \mid \dim f^{-1}(f(y)) \geq 1\} \tag{22}$$

We remark that if X is normal, f is not an isomorphism near a point y of Y iff $y \in \mathrm{Ex}(f)$ holds. See (3.20) and the following remark in [Mumford 95].

3.7. Definition. *Let $f : Y \to X$ be a projective birational morphism between smooth varieties. We denote by $K_{Y/X}$ the relative canonical divisor of f.*

$$K_{Y/X} = K_Y - f^*K_X \tag{23}$$

We remark that $K_{Y/X}$ is defined naturally as an effective divisor on Y supported on $\mathrm{Ex}(f)$*. In particular, $K_{Y/X}$ is an f-exceptional divisor on Y.*

Below we collect some properties of birational morphisms.

3.8. Lemma. *Let $f, g : X \to Y$ be morphisms between varieties.*

1. *If there exists a dominant morphism $h : W \to X$ such that $fh = gh$ then $f = g$ holds.*
2. *If f and g are birational and there exists a birational morphism $h : Y \to Z$ such that $hf = hg$ then $f = g$ holds.*

Proof. We will prove (1). The proof of (2) is similar. So we will omit it.

Since $fh = gh$ holds and h is dominant, there exists a Zariski open subset U of X such that $f|_U = g|_U$. Then $f = g$ holds because Y is separated over $\mathbb{C}$ (Exercise 4.2 on Chapter II in [Hartshorne, 1977]). ∎

3.9. Lemma. *Given two projective birational morphisms $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ between quasi-projective varieties, there exists an unique irreducible component of $Y_1 \times_X Y_2$ dominating Y_1 and Y_2.*

Proof. Because f_1 and f_2 are birational, there are open subsets U of Y_1 and V of Y_2 such that $f_1(U) = f_2(V)$ holds and $f_1|_U$ and $f_2|_V$ are isomorphisms. Consider the following pullback diagram.

$$\begin{tikzcd} Y_1 \times_X Y_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"] & X \end{tikzcd} \tag{24}$$

By the universal property of the pullback, we have

$$U \times_{f_1(U)} V \cong \pi_1^{-1}(U) \cap \pi_2^{-1}(V). \tag{25}$$

So $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ is reduced and irreducible. Let W be the Zariski closure of $\pi_1^{-1}(U) \cap \pi_2^{-1}(V)$ in $Y_1 \times_X Y_2$. Then W is an irreducible component of $Y_1 \times_X Y_2$ dominating Y_1 and Y_2.

Let W' be an irreducible component of $Y_1 \times_X Y_2$ dominating Y_1 and Y_2. Since U is dense in Y_1, $W' \cap \pi_1^{-1}(U)$ is dense in W'. Similarly $W' \cap \pi_2^{-1}(V)$ is dense in W'. Then $W' \cap W$ is dense in W'. Hence $W = W'$. Thus W is the unique irreducible component of $Y_1 \times_X Y_2$ dominating Y_1 and Y_2. ∎

3.10. Lemma. *Given two projective birational morphisms $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ between quasi-projective varieties, there exist projective birational morphisms $g_1 : Z \to Y_1$ and $g_2 : Z \to Y_2$ from a smooth quasi-projective variety Z such that $f_1 \cdot g_1 = f_2 \cdot g_2$.*

$$\begin{array}{ccc} Z & \xrightarrow{g_2} & Y_2 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{f_1} & X \end{array} \tag{26}$$

Proof. There exists an irreducible component W of $Y_1 \times_X Y_2$ dominating Y_1 and Y_2 by Lemma 3.9. Then we may obtain Z by resolving the singularities of W. ∎

3.11. Remark. The diagram (26) is called a **common resolution** of f_1 and f_2.

3.12. Notation. Let X be a variety and Y be a subscheme of X. Given a Cartier divisor D on X, we denote the sheaf $\mathcal{O}_Y \otimes \mathcal{O}_X(D)$ by $\mathcal{O}_Y(D)$ for simplicity.

$$\mathcal{O}_Y(D) = \mathcal{O}_Y \otimes \mathcal{O}_X(D) \tag{27}$$

3.13. Lemma. [Lemma 1.3.2 in [Kawamata, Matsuda, and Matsuki, 1987]] *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y onto a variety X. If P is an effective f-exceptional divisor on Y then*

$$f_* \mathcal{O}_P(P) = 0 \tag{28}$$

holds.

3.14. Lemma. *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y onto a variety X. Let P and N be effective divisors on Y without common component. If P is a f-exceptional divisor on Y then*

$$f_* \mathcal{O}_Y(-N) = f_* \mathcal{O}_Y(P - N) \tag{29}$$

holds.

Proof. Consider the following short exact sequence.

$$0 \to \mathcal{O}_Y(-N) \to \mathcal{O}_Y(P - N) \to \mathcal{O}_P(P - N) \to 0 \tag{30}$$

Since P is a f-exceptional divisor, we have $f_* \mathcal{O}_P(P) = 0$ by Lemma 3.13. Since P and N have no common component, $\mathcal{O}_P(P - N)$ is a subsheaf of $\mathcal{O}_P(P)$. Therefore, we have $f_* \mathcal{O}_P(P - N) = 0$, hence the equality (29). ∎

3.15. Lemma. *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y onto a normal variety X. If N is an effective divisor on Y then*

$$f_*\mathcal{O}_Y(-N) \subseteq \mathcal{O}_X \tag{31}$$

holds.

Proof. Since X is normal, we have $f_*\mathcal{O}_Y = \mathcal{O}_X$, hence $f_*\mathcal{O}_Y(-N) \subseteq \mathcal{O}_X$. ∎

3.16. Lemma. *Let $f : Y \to X$ be a proper birational morphism from a smooth variety Y onto a normal variety X. If P is an effective f-exceptional divisor on Y then*

$$f_*\mathcal{O}_Y(P) = \mathcal{O}_X \tag{32}$$

holds.

Proof. Consider the following short exact sequence.

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(P) \to \mathcal{O}_P(P) \to 0 \tag{33}$$

Since X is normal, $f_*\mathcal{O}_Y = \mathcal{O}_X$ holds. We have $f_*\mathcal{O}_P(P) = 0$ by Lemma 3.13. Therefore $f_*\mathcal{O}_Y(P) = \mathcal{O}_X$ holds. ∎

4. The Category $\mathcal{M}(X)$

Here we introduce categories $\mathcal{B}$, $\mathcal{M}$, and $\mathcal{M}(X)$. We also study their finite limits and finite colimits.

4.1. Category $\mathcal{B}$. Even though we are mainly interested in the category $\mathcal{M}$ and its comma categories $\mathcal{M}(X)$, the category $\mathcal{B}$ is of the fundamental importance. Our view on multiplier ideal sheaves is that they aid us to understand the category $\mathcal{B}$ by capturing a part of information on birational morphisms, which will be encoded as fibrations in our weak fibration structure, Definition 6.9, on $\mathcal{M}(X)$.

4.2. Definition. *We denote by $\mathcal{B}$ the category of normal quasi-projective varieties and projective birational morphisms between them.*

4.3. Remark. Proper morphisms between quasi-projective varieties are projective. See Chapter II.5 in [Grothendieck, 1961].

In model categories, one often needs to construct pullbacks and pushouts. The category $\mathcal{B}$ does not have pushouts in general, but it has pullbacks.

4.4. Lemma. *The category $\mathcal{B}$ has pullbacks.*

PROOF. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{B}$. By Lemma 3.9, there exists an unique irreducible component W' of $X \times_Z Y$ dominating X and Y. Let W be the normalization of W'. Then we have the following commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{p_2} & Y \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z \end{array} \tag{34}$$

where p_1 and p_2 are the canonical projections.

Now by the universal property of the fiber product, a pair of birational morphisms $q_1 : N \to X$ and $q_2 : N \to Y$ from a variety N satisfying $f \cdot q_1 = g \cdot q_2$ induces a morphism $q : N \to X \times_Z Y$ such that $q_1 = p_1 \cdot q$ and $q_2 = p_2 \cdot q$ hold. Since W' is the unique component of $X \times_Z Y$ dominating X and Y, q factors through W'. If N is normal, q factors through W by the universal property of the normalization $W \to W'$. Therefore, the diagram (34) is the pullback of (f, g) in $\mathcal{B}$. ∎

4.5. CATEGORY $\mathcal{M}$. Here we define a category $\mathcal{M}$ and show that $\mathcal{M}$ has finite limits. An object of $\mathcal{M}$ is a pair of a normal quasi-projective variety and a finite set of weighted ideal sheaves on it. One can associate with a $\mathbb{Q}$-divisor an weighted invertible ideal sheaf. In this sense, weighted ideal sheaves are generalizations of $\mathbb{Q}$-divisors.

4.6. DEFINITION. *Let X be a variety.*

1. *A weighted ideal sheaf on X is a pair $(\mathcal{I}, c)$ of a nonzero ideal sheaf $\mathcal{I}$ on X and a non-negative rational number $c \in \mathbb{Q}_{\geq 0}$. For simplicity, we write*
$$\mathcal{I}^c \tag{35}$$
instead of $(\mathcal{I}, c)$. $\mathcal{I}^c$ should be thought of taking c-th power of $\mathcal{I}$.
2. *We define a binary relation $\leq$ on the set of weighted ideal sheaves on X by*
$$\mathcal{I}^c \leq \mathcal{J}^d \text{ iff } \mathcal{I} \subseteq \mathcal{J} \text{ and } c \geq d. \tag{36}$$
3. *We define a binary relation $\leq$ on the set of finite sets of weighted ideal sheaves on X by*
$$\{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\} \leq \{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\} \text{ iff } \begin{cases} \text{for every } \mathcal{J}_j \text{ there exists } \mathcal{I}_i \\ \text{such that } \mathcal{I}_i^{c_i} \leq \mathcal{J}_j^{d_j}. \end{cases} \tag{37}$$

We realize that (36) is a rather crude way to compare weighted ideal sheaves. For example $(\mathcal{I}^2)^{\frac{1}{2}}$ and $\mathcal{I}$ are not comparable with this relation even though they will produce the same multiplier ideal sheaf. But (36) is a partial ordering and it seems suitable for our purpose. See Lemma 4.21.

4.7. Definition. *Let X be a variety.*

1. *A finite set $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ of weighted ideal sheaves on X is called* ***reduced*** *if for every $1 \le i \neq j \le m$,*
$$\mathcal{I}_i^{c_i} \not\leq \mathcal{I}_j^{c_j} \tag{38}$$
holds.

2. *Let $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ be a finite set of weighted ideal sheaves on X. The* ***reduction***
$$\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}} \tag{39}$$
of $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ is the reduced subset of $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ such that for every $\mathcal{I}_i^{c_i}$ in $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ there exists $\mathcal{I}_j^{c_j}$ in $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}}$ satisfying
$$\mathcal{I}_j^{c_j} \le \mathcal{I}_i^{c_i}. \tag{40}$$

4.8. Remark. If $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ is a finite set of weighted ideal sheaves on a variety then $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}}$ is the subset consisting of the minimal elements in $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ with respect to the relation (36).

4.9. Remark. If $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ is a finite set of weighted ideal sheaves on a variety then
$$\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\} \text{ is reduced iff } \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}} = \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\} \tag{41}$$
holds.

4.10. Lemma. *For every finite set $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ of weighted ideal sheaves on a variety*
$$\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\} \le \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}} \tag{42}$$
and
$$\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\} \ge \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}} \tag{43}$$
hold.

Proof. The inequalities hold by the definition. ■

4.11. Remark. The relation (37) is preserved under reductions. In fact, if we have two finite sets $\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ and $\{\mathcal{J}_1^{d_1}, \ldots, \mathcal{J}_n^{d_n}\}$ of weighted ideal sheaves on a variety then
$$\{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\} \le \{\mathcal{J}_1^{d_1}, \ldots, \mathcal{J}_n^{d_n}\} \text{ iff } \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}} \le \{\mathcal{J}_1^{d_1}, \ldots, \mathcal{J}_n^{d_n}\}_{\mathrm{red}} \tag{44}$$
holds by Lemma 4.10.

4.12. Remark. Lemma 4.10 shows that the relation (37) is not antisymmetric in general. However, if $\{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$ and $\{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}$ are finite sets of weighted ideal sheaves on a variety then

$$\left.\begin{array}{l}\{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\} \le \{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\} \\ \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\} \ge \{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}\end{array}\right\} \text{ iff } \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}_{\mathrm{red}} = \{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}_{\mathrm{red}} \tag{45}$$

holds by Remark 4.11. So, the relation is antisymmetric among the reduced finite sets of weighted ideal sheaves.

4.13. Notation. Given a birational morphism $f : X \to Y$ and an ideal sheaf $\mathcal{I}$ on Y, we will use both of

$$f^*\mathcal{I} \text{ and } \mathcal{I} \cdot \mathcal{O}_X \tag{46}$$

to denote the ideal sheaf on X generated by $\mathcal{I}$. The notation $f^*\mathcal{I}$ is used to denote the pullback of $\mathcal{I}$ as an $\mathcal{O}_X$-module in some literature. However, we never use $f^*\mathcal{I}$ to denote the $\mathcal{O}_X$-module in this note.

4.14. Definition. *Let $f : X \to Y$ be a projective birational morphism between quasi-projective varieties.*

1. *Let $\mathcal{J}^d$ be a weighted ideal sheaf on Y. We define a weighted ideal sheaf $f^*(\mathcal{J}^d)$ on X by*
$$f^*(\mathcal{J}^d) = (f^*\mathcal{J})^d \tag{47}$$
where $f^*\mathcal{J}$ is the ideal sheaf (46) on X generated by $\mathcal{J}$. For simplicity we omit the parenthesis and write
$$f^*\mathcal{J}^d. \tag{48}$$
2. *Let $\{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}$ be a finite set of weighted ideal sheaves on Y. We define a finite set $f^*\{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}$ of weighted ideal sheaves on X by*
$$f^*\{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\} = \{f^*\mathcal{J}_1^{d_1}, \dots, f^*\mathcal{J}_n^{d_n}\}. \tag{49}$$

4.15. Remark. The relation (37) is preserved under projective birational morphisms, but the reducedness of finite sets of weighted ideal sheaves is not.

4.16. Notation. Let $\{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$ be a finite set of weighted ideal sheaves on a variety. We will often write $\underline{\mathcal{I}}^c$ instead of $\{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$.

$$\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\} \tag{50}$$

4.17. Definition. *We define a category $\mathcal{M}$ as follows.*

1. *The set of objects consists of all the pairs $(X, \underline{\mathcal{I}}^c)$ of a normal quasi-projective variety X and a* ***reduced*** *finite set $\underline{\mathcal{I}}^c$ of weighted ideal sheaves on X.*
2. *The hom-set $\mathcal{M}((X, \underline{\mathcal{I}}^c), (Y, \underline{\mathcal{J}}^d))$ consists of all projective birational morphisms $f : X \to Y$ satisfying*
$$\underline{\mathcal{I}}^c \le f^*\underline{\mathcal{J}}^d. \tag{51}$$

4.18. Remark. $\mathcal{M}$ is a small category. See Section 2.1.

4.19. Notation. If $\underline{\mathcal{I}}^c$ contains only one weighted ideal sheaf $\mathcal{I}_1^{c_1}$, we write $(X, \mathcal{I}_1^{c_1})$ instead of $(X, \underline{\mathcal{I}}^c)$.

$$(X, \mathcal{I}_1^{c_1}) = (X, \{\mathcal{I}_1^{c_1}\}) \tag{52}$$

4.20. Remark. If $(Y, \underline{\mathcal{I}}^c)$ and $(Z, \underline{\mathcal{J}}^d)$ are objects of $\mathcal{M}$ and $f : Y \to Z$ is a projective birational morphism then (51) holds iff

$$\underline{\mathcal{I}}^c \leq (f^* \underline{\mathcal{J}}^d)_{\mathrm{red}} \tag{53}$$

holds by Lemma 4.10.

The next lemma shows that isomorphisms in $\mathcal{M}$ have the expected characteristic.

4.21. Lemma. *Let $f : (X, \underline{\mathcal{I}}^c) \to (Y, \underline{\mathcal{J}}^d)$ be a morphism in $\mathcal{M}$. Then the following three properties are equivalent.*

1. *f is an isomorphism in $\mathcal{M}$.*
2. *There is a morphism $g : (Y, \underline{\mathcal{J}}^d) \to (X, \underline{\mathcal{I}}^c)$ in $\mathcal{M}$ such that*

$$gf = 1_X \tag{54}$$

   holds in $\mathcal{B}$.

3. *f is an isomorphism as a morphism in $\mathcal{B}$ and $\underline{\mathcal{I}}^c = f^* \underline{\mathcal{J}}^d$.*

Proof. We will show $(2 \Rightarrow 3)$. $(1 \Rightarrow 2)$ and $(3 \Rightarrow 1)$ are clear.

First, because f is birational, $fgf = f$ implies that

$$fg = 1_Y \tag{55}$$

holds in $\mathcal{B}$ by Lemma 3.8.(1).

Let $\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$ and $\underline{\mathcal{J}}^d = \{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}$. Given $\mathcal{I}_i$, there is $\mathcal{J}_j$ and $\mathcal{I}_k$ such that $\mathcal{J}_j^{d_j} \leq g^* \mathcal{I}_i^{c_i}$ and $\mathcal{I}_k^{c_k} \leq f^* \mathcal{J}_j^{d_j}$. Hence $\mathcal{I}_k^{c_k} \leq \mathcal{I}_i^{c_i}$ by (54). Because $\underline{\mathcal{I}}^c$ is reduced, $i = k$. Then $\mathcal{I}_i^{c_i} = f^* \mathcal{J}_j^{d_j}$, and

$$\mathcal{J}_j^{d_j} = g^* \mathcal{I}_i^{c_i} \tag{56}$$

because of (55). Now (54) implies that $\underline{\mathcal{I}}^c$ and $g^* \underline{\mathcal{I}}^c$ has the same cardinality and (56) implies that $g^* \underline{\mathcal{I}}^c$ is a subset of $\underline{\mathcal{J}}^d$.

Thus by symmetry, $\underline{\mathcal{J}}^d$ and $\underline{\mathcal{I}}^c$ have the same cardinality and $\underline{\mathcal{I}}^c = f^* \underline{\mathcal{J}}^d$ holds. ∎

4.22. Limits and Colimits in $\mathcal{M}$. Here we consider pullbacks and pushouts in $\mathcal{M}$.

4.23. Definition. *Let* $\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}\}$ *and* $\underline{\mathcal{J}}^d = \{\mathcal{J}_1^{d_1}, \ldots, \mathcal{J}_n^{d_n}\}$ *be two sets of weighted ideal sheaves on a variety. We define a reduced finite set* $\underline{\mathcal{I}}^c * \underline{\mathcal{J}}^d$ *by*

$$\underline{\mathcal{I}}^c * \underline{\mathcal{J}}^d = \{\mathcal{I}_1^{c_1}, \ldots, \mathcal{I}_m^{c_m}, \mathcal{J}_1^{d_1}, \ldots, \mathcal{J}_n^{d_n}\}_{\mathrm{red}}. \tag{57}$$

4.24. Lemma. *The category* $\mathcal{M}$ *has pullbacks.*

Proof. Let $f : (X, \underline{\mathcal{I}}^c) \to (Z, \underline{\mathcal{K}}^e)$ and $g : (Y, \underline{\mathcal{J}}^d) \to (Z, \underline{\mathcal{K}}^e)$ be morphisms in $\mathcal{M}$. Let W be the normalization of the unique irreducible component of $X \times_Z Y$ dominating X and Y in (34). Let

$$\underline{\mathcal{H}}^b = p_1^*\underline{\mathcal{I}}^c * p_2^*\underline{\mathcal{J}}^d \tag{58}$$

where p_1 and p_2 are the canonical projections in (34). Then the diagram

$$\begin{array}{ccc} (W, \underline{\mathcal{H}}^b) & \xrightarrow{p_2} & (Y, \underline{\mathcal{J}}^d) \\ \Big\downarrow{\scriptstyle p_1} & & \Big\downarrow{\scriptstyle g} \\ (X, \underline{\mathcal{I}}^c) & \xrightarrow{f} & (Z, \underline{\mathcal{K}}^e) \end{array} \tag{59}$$

is the pullback of (f, g) in $\mathcal{M}$. ∎

One may want to make the category $\mathcal{M}$ simpler by working with a single weighted ideal sheaf instead of a finite set. But it seems unsuitable to form a category with pullbacks. Let us consider the following example.

4.25. Example. Let $\mathcal{T}$ be the full subcategory of $\mathcal{M}$ consisting of objects

$$(X, \mathcal{I}_1^{c_1}) = (X, \{\mathcal{I}_1^{c_1}\}). \tag{60}$$

We will show that $\mathcal{T}$ does not have pullbacks.

Let X be a smooth quasi-projective surface and p be a point of X. Let C_1 and C_2 be two distinct smooth irreducible curves on X containing p. Let

$$D_1 = 2C_1 + C_2, \tag{61}$$

$$D_2 = C_1 + 2C_2, \tag{62}$$

and

$$D = 2C_1 + 2C_2. \tag{63}$$

Let $\pi : \widetilde{X} \to X$ be the blowing up at p with the exceptional divisor E. Let $\widetilde{C}_i$ be the proper transformation of C_i in $\widetilde{X}$ for $i = 1, 2$.

Consider two morphisms in $\mathcal{T}$

$$f : (X, \mathcal{O}_X(-D_1)^1) \to (X, \mathcal{O}_X^0) \tag{64}$$

and
$$g : (X, \mathcal{O}_X(-D_2)^1) \to (X, \mathcal{O}_X^0) \tag{65}$$
where f and g are induced by 1_X. Suppose that we have the following pullback of (f, g) in $\mathcal{T}$.
$$\begin{tikzcd} (Y, \mathcal{J}^c) \arrow[r, "p_2"] \arrow[d, "p_1"] & (X, \mathcal{O}_X(-D_2)^1) \arrow[d, "g"] \\ (X, \mathcal{O}_X(-D_1)^1) \arrow[r, "f"] & (X, \mathcal{O}_X^0) \end{tikzcd} \tag{66}$$
Since we also have the following commutative diagram in $\mathcal{T}$
$$\begin{tikzcd} (X, \mathcal{O}_X(-D)^1) \arrow[r, "q_2"] \arrow[d, "q_1"] & (X, \mathcal{O}_X(-D_2)^1) \arrow[d, "g"] \\ (X, \mathcal{O}_X(-D_1)^1) \arrow[r, "f"] & (X, \mathcal{O}_X^0) \end{tikzcd} \tag{67}$$
where q_1 and q_2 are induced by 1_X, the diagram (67) must factor through the diagram (66). So we have the following commutative diagram in $\mathcal{T}$.
$$\begin{tikzcd} (X, \mathcal{O}_X(-D)^1) \arrow[r, "q_2"] \arrow[d, "q_1"] \arrow[dr, "h"] & (X, \mathcal{O}_X(-D_2)^1) \\ (X, \mathcal{O}_X(-D_1)^1) & (Y, \mathcal{J}^c) \arrow[l, "p_1"] \arrow[u, "p_2"] \end{tikzcd} \tag{68}$$
Then $c = 1$ and
$$\mathcal{O}_X(-D) \subseteq h^*\mathcal{J} \subseteq \mathcal{O}_X(-D_1) \cap \mathcal{O}_X(-D_2) = \mathcal{O}_X(-D). \tag{69}$$
Hence $\mathcal{O}_X(-D) = h^*\mathcal{J}$. Since $p_1 \cdot h = 1_X$ in $\mathcal{B}$, $h \cdot p_1 = 1_Y$ also holds in $\mathcal{B}$ by Lemma 3.8.(1). Then h is an isomorphism in $\mathcal{T}$ by Lemma 4.21. Hence the diagram (67) also provides the pullback of (f, g) in $\mathcal{T}$.

On the other hand, we have another commutative diagram in $\mathcal{T}$
$$\begin{tikzcd} (\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-2\widetilde{C}_1 - 2\widetilde{C}_2 - 3E)^1) \arrow[r, "r_2"] \arrow[d, "r_1"] & (X, \mathcal{O}_X(-D_2)^1) \arrow[d, "g"] \\ (X, \mathcal{O}_X(-D_1)^1) \arrow[r, "f"] & (X, \mathcal{O}_X^0) \end{tikzcd} \tag{70}$$
where r_1 and r_2 are induced by π. Hence the diagram (70) must factor through the diagram (67).
$$\begin{tikzcd} (\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-2\widetilde{C}_1 - 2\widetilde{C}_2 - 3E)^1) \arrow[r, "r_2"] \arrow[d, "r_1"] \arrow[dr, "k"] & (X, \mathcal{O}_X(-D_2)^1) \\ (X, \mathcal{O}_X(-D_1)^1) & (X, \mathcal{O}_X(-D)^1) \arrow[l, "q_1"] \arrow[u, "q_2"] \end{tikzcd} \tag{71}$$

However such a morphism k in $\mathcal{T}$ does not exist because

$$\mathcal{O}_{\widetilde{X}}(-2\widetilde{C}_1 - 2\widetilde{C}_2 - 3E) \not\subseteq \mathcal{O}_{\widetilde{X}}(-\pi^* D). \tag{72}$$

Hence $\mathcal{T}$ does not have pullbacks.

The category $\mathcal{M}$ does not have pushouts in general. But the following special case will be enough for us.

4.26. Lemma. *Let $f : (W, \underline{\mathcal{H}}^b) \to (X, \underline{\mathcal{I}}^c)$ and $g : (W, \underline{\mathcal{H}}^b) \to (Y, \underline{\mathcal{J}}^d)$ be morphisms in $\mathcal{M}$. If f is an isomorphism as a morphism in $\mathcal{B}$ then the pushout of (f, g) exists in $\mathcal{M}$.*

Proof. We may assume that $W = X$ and f is induced by the identity 1_X on X. Let $\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$ and let $\{\widetilde{\mathcal{I}}_1^{c_1}, \dots, \widetilde{\mathcal{I}}_m^{c_m}\}$ be the finite set of the ideal sheaves on Y such that $\widetilde{\mathcal{I}}_i$ is the intersection of all ideal sheaves $\widehat{\mathcal{I}}$ on Y satisfying $\mathcal{I}_i \subseteq g^*\widehat{\mathcal{I}}$. Let $\underline{\mathcal{J}}^d = \{\mathcal{J}_1^{d_1}, \dots, \mathcal{J}_n^{d_n}\}$. For every $1 \le i \le m$ and $1 \le j \le n$, let $\mathcal{K}_{ij} = \widetilde{\mathcal{I}}_i + \mathcal{J}_j$ and $e_{ij} = \min\{c_i, d_j\}$. Let $\underline{\mathcal{K}}^e = \{\mathcal{K}_{11}^{e_{11}}, \dots, \mathcal{K}_{mn}^{e_{mn}}\}_{\mathrm{red}}$. Then $(Y, \underline{\mathcal{K}}^e)$ is the pushout of (f, g) in $\mathcal{M}$. ∎

4.27. Category $\mathcal{M}(X)$. Here we introduce our main object of study, the category $\mathcal{M}(X)$. It is a small category with finite limits. As we will see in Section 7.21, these two properties will ensure that the pro-category of $\mathcal{M}(X)$ has small colimits and small limits.

4.28. Definition. *Let X be a normal quasi-projective variety. We define a category $\mathcal{M}(X)$ by the comma category of $\mathcal{M}$ over $(X, \mathcal{O}_X^0)$.*

$$\mathcal{M}(X) = (\mathcal{M} \downarrow (X, \mathcal{O}_X^0)) \tag{73}$$

So an object of $\mathcal{M}(X)$ is a morphism $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ in $\mathcal{M}$, and a morphism in $\mathcal{M}(X)$ is a commutative diagram

$$\begin{tikzcd}
(Y, \underline{\mathcal{I}}^c) \arrow[rr, "h"] \arrow[dr, "f"'] & & (Z, \underline{\mathcal{J}}^d) \arrow[dl, "g"] \\
& (X, \mathcal{O}_X^0) &
\end{tikzcd} \tag{74}$$

in $\mathcal{M}$. We will often denote the morphism (74) with h.

The proof of the following lemma is a simple diagram chasing. So we omit it.

4.29. Lemma. *Let $\mathcal{K}$ be a category with finite limits. Let $f : a \to z$ and $g : b \to z$ be two morphisms in $\mathcal{K}$. Assume that the following diagram is the pullback of (f, g) in $\mathcal{K}$.*

$$\begin{tikzcd}
c \arrow[r, "p"] \arrow[d, "q"] & a \arrow[d, "f"] \\
b \arrow[r, "g"] & z
\end{tikzcd} \tag{75}$$

Let $h = f \cdot p$. Then $h : c \to z$ together with the morphisms $p : h \to f$ and $q : h \to g$ in the comma category $(\mathcal{K} \downarrow z)$ is the product of f and g in $(\mathcal{K} \downarrow z)$.

4.30. LEMMA. *Let X be a normal quasi-projective variety.*

1. *$\mathcal{M}(X)$ is a preorder, i.e. for every f and g in $\mathcal{M}(X)$, $\mathcal{M}(X)(f,g)$ has at most one element.*

2. *$\mathcal{M}(X)$ has finite limits.*

PROOF. (1) It follows from Lemma 3.8.(2).

(2) First, $\mathcal{M}(X)$ has the canonical terminal object $1_{(X,\mathcal{O}_X^0)} : (X,\mathcal{O}_X^0) \to (X,\mathcal{O}_X^0)$. Second, products in $\mathcal{M}(X)$ are pullbacks in $\mathcal{M}$ by Lemma 4.29, which exist by Lemma 4.24. Finally, $\mathcal{M}(X)$ has equalizers because of (1) for a trivial reason. ∎

We will use the following lemma to calculate finite limits in $\mathcal{M}(X)$.

4.31. LEMMA. *Let $F : \mathcal{D} \to \mathcal{K}$ be a functor. If $\mathcal{K}$ is a preorder then the following properties hold.*

1. *The limit $\lim F$ exists in $\mathcal{K}$ iff the product $\prod_{d\in\mathcal{D}} F(d)$ exists in $\mathcal{K}$ iff the product $\prod_{d\in\mathcal{D}_{\max}} F(d)$ exists in $\mathcal{K}$ where $\mathcal{D}_{\max}$ is the subset of $\mathrm{ob}\mathcal{D}$ consisting of the maximal elements in $\mathrm{ob}\mathcal{D}$ with respect to $d \geq d'$ which is defined by the existence of a morphism $d \to d'$ in $\mathcal{D}$.*

2. *If the limit of F exists in $\mathcal{K}$ then*
$$\lim F = \prod_{d\in\mathcal{D}} F(d) \tag{76}$$
$$= \prod_{d\in\mathcal{D}_{\max}} F(d) \tag{77}$$
hold.

PROOF. Since $\mathcal{K}$ is a preorder, every diagram in $\mathcal{K}$ commutes. Hence the results hold. ∎

4.32. DEFINITION. *By an abuse of notation, we use the same letter U to denote the forgetful functors*
$$U : \mathcal{M} \to \mathcal{B} \tag{78}$$
and
$$U : \mathcal{M}(X) \to \mathcal{B}. \tag{79}$$

Finally, just like Lemma 4.26, we have pushouts in $\mathcal{M}(X)$ too if one of the morphisms is an isomorphism in $\mathcal{B}$.

4.33. LEMMA. *Let $f : \alpha \to \beta$ and $g : \alpha \to \gamma$ be morphisms in $\mathcal{M}(X)$. If $U(f)$ is an isomorphism then the pushout*
$$\begin{array}{ccc} \alpha & \xrightarrow{g} & \gamma \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} \\ \beta & \xrightarrow{v} & \delta \end{array} \tag{80}$$
of (f,g) exists in $\mathcal{M}(X)$ and $U(u)$ is an isomorphism.

PROOF. It follows from Lemma 4.26 and the general fact that the projection $(\mathcal{K} \downarrow z) \to \mathcal{K}$ from a comma category of a category $\mathcal{K}$ over $z \in \mathcal{K}$ creates pushouts. See Exercise V.1.1 in [Mac Lan, 1998]. ∎

5. The Multiplier Ideal Sheaf: A Generalization

5.1. REVIEW ON MULTIPLIER IDEAL SHEAVES. Here we recall some basic facts on multiplier ideal sheaves on smooth varieties.

5.2. DEFINITION. *Let X be a smooth quasi-projective variety. Let $\mathcal{I}$ be a non-zero ideal sheaf on X and $c \in \mathbb{Q}_{\geq 0}$ be a non-negative rational number. A* ***log resolution*** *of $(X, \mathcal{I}^c)$ is a projective birational morphism $f : Y \to X$ from a smooth quasi-projective variety Y such that*

1. *$\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$ for some effective Cartier divisor D on Y,*
2. *$\mathrm{Ex}(f)$ is a divisor on Y (See Definition 3.6), and*
3. *$\mathrm{Ex}(f) + D$ has a simple normal crossing support (See Definition 3.2).*

5.3. REMARK. A log resolution of $(X, \mathcal{I}^c)$ always exists by the theorem of Hironaka (cf. Theorem 0.2 in [Kollár and Mori, 1998], Theorem 3.26 in [Kollar, 2007]).

5.4. DEFINITION. *Let X be a smooth quasi-projective variety. Let $\mathcal{I}$ be a non-zero ideal sheaf on X and $c \in \mathbb{Q}_{\geq 0}$ be a non-negative rational number. Let $f : Y \to X$ be a log resolution of $(X, \mathcal{I}^c)$. The multiplier ideal sheaf $\mathscr{I}(X, \mathcal{I}^c)$ of $(X, \mathcal{I}^c)$ is defined by*

$$\mathscr{I}(X, \mathcal{I}^c) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor cD \rfloor) \tag{81}$$

(See Definition 3.3 and Definition 3.7).

5.5. REMARK. In Definition 5.4 we write

$$K_{Y/X} - \lfloor cD \rfloor = P - N \tag{82}$$

where P and N are effective divisors without common component. Then

$$\mathscr{I}(X, \mathcal{I}^c) = f_* \mathcal{O}_Y(-N) \tag{83}$$

holds by (29), so $\mathscr{I}(X, \mathcal{I}^c)$ is an ideal sheaf on X by (31).

It is well-known that the multiplier ideal sheaf (81) does not depend on the choice of the log resolution f. The key point is the following result.

5.6. Lemma. [Lemma 9.2.19 in [Lazarsfeld, 2004]] *Let Y be a smooth quasi-projective variety. Let $\mu : Z \to Y$ be a projective birational morphism from a smooth variety Z. Let D be an effective $\mathbb{Q}$-divisor on Y with a simple normal crossing support. Then*

$$K_{Z/Y} - \lfloor \mu^* D \rfloor + \mu^* \lfloor D \rfloor \geq 0 \tag{84}$$

and

$$\mathcal{O}_Y(\lfloor D \rfloor) = \mu_* \mathcal{O}_Z(K_{Z/Y} - \lfloor \mu^* D \rfloor) \tag{85}$$

hold.

5.7. Remark. The assertion (84) is proved during the proof of Lemma 9.2.19 in [Lazarsfeld, 2004].

5.8. A Generalization. Here we define the multiplier ideal sheaf of an object of the category $\mathcal{M}(X)$. However the reducedness of $\underline{\mathcal{I}}^c$ in Definition 4.17 is not required to define it. So here we will work with a larger category $\mathcal{N}(X)$ defined below instead of $\mathcal{M}(X)$.

5.9. Definition. *We define a category $\mathcal{N}$ as follows.*

1. *The set of objects consists of all the pairs $(X, \underline{\mathcal{I}}^c)$ of a normal quasi-projective variety X and a (**not necessarily reduced**) finite set $\underline{\mathcal{I}}^c$ of weighted ideal sheaves on X.*
2. *The hom-set $\mathcal{N}((X, \underline{\mathcal{I}}^c), (Y, \underline{\mathcal{J}}^d))$ consists of all projective birational morphisms $f : X \to Y$ satisfying*
$$\underline{\mathcal{I}}^c \leq f^* \underline{\mathcal{J}}^d. \tag{86}$$

5.10. Definition. *Let X be a normal quasi-projective variety. We define a category $\mathcal{N}(X)$ by the comma category of $\mathcal{N}$ over $(X, \mathcal{O}_X^0)$.*

$$\mathcal{N}(X) = (\mathcal{N} \downarrow (X, \mathcal{O}_X^0)) \tag{87}$$

5.11. Remark. Clearly $\mathcal{M}$ and $\mathcal{M}(X)$ are full subcategories of $\mathcal{N}$ and $\mathcal{N}(X)$ respectively.

5.12. Definition. *Let X be a smooth quasi-projective variety. Let $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ be an object of $\mathcal{N}(X)$. Let $\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$. A projective birational morphism $\mu : Z \to Y$ from a smooth quasi-projective variety Z is called a **log resolution of** f if*

1. *for every $\mathcal{I}_i$, there exists an effective Cartier divisor D_i in Z such that $\mathcal{I}_i \cdot \mathcal{O}_Z = \mathcal{O}_Z(-D_i)$,*
2. $\mathrm{Ex}(f \cdot \mu)$ *is a divisor on Z, and*
3. $\mathrm{Ex}(f \cdot \mu) + \sum_{i=1}^m D_i$ *has a simple normal crossing support.*

5.13. Remark. Again a log resolution of f exists by the theorem of Hironaka.

5.14. Definition. *Let X be a smooth quasi-projective variety. Let $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ be an object of $\mathcal{N}(X)$. Let $\underline{\mathcal{I}}^c = \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}$. Let $\mu : Z \to Y$ be a log resolution of f. Let $D_i \subset Z$ be an effective Cartier divisor in Z such that $\mathcal{I}_i \cdot \mathcal{O}_Z = \mathcal{O}_Z(-D_i)$. We write $c_i D_i = \sum_{k=1}^{l} a_i^k E_k$ where E_k's are prime divisors and $a_i^k \in \mathbb{Q}_{\geq 0}$. Then we define*

$$D_{f,\mu} = \sum_{k=1}^{l} \max\{a_1^k, \dots, a_m^k\} E_k \tag{88}$$

and

$$\mathscr{I}_\mu(f) = (f \cdot \mu)_* \mathcal{O}_Z(K_{Z/X} - \lfloor D_{f,\mu} \rfloor). \tag{89}$$

Again the right hand side of (89) is an ideal sheaf on X. Just like the usual multiplier ideal sheaf (81), the following Lemma 5.17 will show that $\mathscr{I}_\mu(f)$ does not depend on the choice of μ. So we will write $\mathscr{I}(f)$ instead of $\mathscr{I}_\mu(f)$.

5.15. Remark. If $m = 1$ and $f = 1_X$ then $\mathscr{I}(f)$ is the usual multiplier ideal sheaf $\mathscr{I}(X, \mathcal{I}_1^{c_1})$ in (81).

5.16. Remark. This relative view on multiplier ideal sheaves is not new. See for example [Ein and Lazarsfeld, 1993]. I learned an explicit treatment from [Ein, 1995].

The proof of the following lemma has little difference with that of the usual multiplier ideal sheaves. We provide the proof for the convenience of the reader.

5.17. Lemma. [cf. Theorem 9.2.18 in [Lazarsfeld, 2004]] *The ideal sheaf $\mathscr{I}_\mu(f)$ in (89) does not depend on the choice of the log resolution μ of f.*

Proof. Let $\mu' : Z' \to Y$ be another log resolution of f. By taking a common resolution (Lemma 3.10) of μ and μ' we may assume that μ' factors through μ. Let $\mu' = \mu \cdot \rho$ for some $\rho : Z' \to Z$. For simplicity, we denote $D_{f,\mu}$ and $D_{f,\mu'}$ with D_μ and $D_{\mu'}$ respectively.

Since $D_{\mu'} \leq \rho^* D_\mu$ holds by definition, the following calculation and (84) show that the $\mathbb{Q}$-divisor (90) is effective.

$$K_{Z'/X} - \lfloor D_{\mu'} \rfloor - \rho^*(K_{Z/X} - \lfloor D_\mu \rfloor) \tag{90}$$

$$= K_{Z'/Z} - \lfloor D_{\mu'} \rfloor + \rho^* \lfloor D_\mu \rfloor \tag{91}$$

$$\geq K_{Z'/Z} - \lfloor \rho^* D_\mu \rfloor + \rho^* \lfloor D_\mu \rfloor \tag{92}$$

The divisor (91) is ρ-exceptional. By Lemma 3.16

$$\rho_* \mathcal{O}_{Z'}(K_{Z'/X} - \lfloor D_{\mu'} \rfloor - \rho^*(K_{Z/X} - \lfloor D_\mu \rfloor)) = \mathcal{O}_Z \tag{93}$$

holds. Therefore the projection formula (Exercise 5.1 on Chapter II in [Hartshorne, 1977]) implies that

$$\rho_* \mathcal{O}_Z(K_{Z'/X} - \lfloor D_{\mu'} \rfloor) = \mathcal{O}_Y(K_{Z/X} - \lfloor D_\mu \rfloor) \tag{94}$$

holds, hence $\mathscr{I}_{\mu'}(f) = \mathscr{I}_\mu(f)$. ∎

5.18. Lemma. *Let X be a smooth quasi-projective variety. Let $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ be an object of $\mathcal{N}(X)$. Let $\mu : Z \to Y$ be a projective birational morphism from a smooth variety Z. Then*

$$\mathscr{I}(f) = \mathscr{I}((Z, f^*\underline{\mathcal{I}}^c) \xrightarrow{f \cdot \mu} (X, \mathcal{O}_X^0)) \tag{95}$$

holds.

Proof. It follows from Lemma 5.17. ∎

5.19. Lemma. *Let X be a smooth quasi-projective variety. Let f and g be objects of $\mathcal{N}(X)$. If there is a morphism $h : f \to g$ in $\mathcal{N}(X)$ then*

$$\mathscr{I}(f) \subseteq \mathscr{I}(g) \tag{96}$$

holds.

Proof. We write $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ and $g : (Z, \underline{\mathcal{J}}^d) \to (X, \mathcal{O}_X^0)$. Let $\mu : W \to Y$ be a log resolution of $(Y, \underline{\mathcal{I}}^c * h^*\underline{\mathcal{J}}^d) \xrightarrow{f} (X, \mathcal{O}_X^0)$. Then $D_{f,\mu} \geq D_{g,h\cdot\mu}$ holds by definition. So the inclusion (96) holds by Lemma 5.17. ∎

5.20. Lemma. *Let X be a smooth quasi-projective variety. Let $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ be an object of $\mathcal{N}(X)$. Then*

$$\mathscr{I}(f) = \mathscr{I}((Y, (\underline{\mathcal{I}}^c)_{\mathrm{red}}) \xrightarrow{f} (X, \mathcal{O}_X^0)) \tag{97}$$

holds.

Proof. It follows from Lemma 4.10 and Lemma 5.19. ∎

6. Weak Fibration Structure on $\mathcal{M}(X)$

In [Quillen, 1967] Quillen introduced model categories as a general setting in which one can do homotopy theory. A modern definition of a (closed) model structure on a category $\mathcal{M}$ with finite limits and finite colimits consists of three sets $\mathcal{W}$, $\mathcal{C}$, $\mathcal{F}$ of morphisms in $\mathcal{M}$ such that

1. $\mathcal{W}$ satisfies the two out of three property (Definition 6.2) and
2. $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems (Definition 7.32).

In [Barnea and Schlank, 2016], the authors introduce weak fibration categories and show that the pro-category of a weak fibration category has three sets $\mathcal{W}$, $\mathcal{C}$, $\mathcal{F}$ of morphisms such that $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorization systems. They call a weak fibration category **pro-admissible** if $\mathcal{W}$ satisfies the two out of three property.

Here we will recall the definition of the weak fibration structure and show that $\mathcal{M}(X)$ has such a structure.

The dual notions are weak cofibration categories and their ind-categories. Our choice of fibration over cofibration was to take advantage of the existence of pullbacks in the category $\mathcal{B}$.

6.1. Weak Fibration Structures. If one wants to treat a certain set of morphisms in a category as isomorphisms then one can form its homotopy category by inverting the elements in the set. The essence of such a set is captured by the following property.

6.2. Definition. [Definition 14.1.4 in [May and Ponto, 2012]] *Let $\mathcal{K}$ be a category. Let $\mathcal{W}$ be a nonempty set of morphisms in $\mathcal{K}$. We say that $\mathcal{W}$ satisfies the* ***two out of three property*** *if the following three conditions hold: For every $g, h \in \mathrm{Mor}(\mathcal{K})$ with $\mathrm{dom}(h) = \mathrm{cod}(g)$,*

(M) $g \in \mathcal{W}$ and $h \in \mathcal{W}$ imply $hg \in \mathcal{W}$.

(L) $hg \in \mathcal{W}$ and $h \in \mathcal{W}$ imply $g \in \mathcal{W}$.

(R) $hg \in \mathcal{W}$ and $g \in \mathcal{W}$ imply $h \in \mathcal{W}$.

6.3. Definition. *Let $\mathcal{K}$ be a category. Let $\mathcal{L}$ and $\mathcal{R}$ be two sets of morphisms in $\mathcal{K}$. Then we write*

$$\mathrm{Mor}(\mathcal{K}) = \mathcal{R} \circ \mathcal{L} \tag{98}$$

if every morphism $a \to c$ in $\mathcal{K}$ can be factored as $a \xrightarrow{f} b \xrightarrow{g} c$ where $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

6.4. Definition. [Definition 1.2 in [Barnea and Schlank, 2016]] *A* ***weak fibration structure*** *on a category $\mathcal{K}$ with finite limits consists of two sets $\mathcal{W}$ and $\mathcal{F}$ of morphisms in $\mathcal{K}$ satisfying the following four properties.*

1. *$\mathcal{W}$ satisfies the two out of three property and contains all the isomorphisms.*
2. *$\mathcal{F}$ is closed under composition and contains all the isomorphisms.*
3. *$\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are closed under base change.*
4. *$\mathrm{Mor}(\mathcal{M}) = \mathcal{F} \circ \mathcal{W}$ holds.*

For simplicity, we denote $\mathcal{W} \cap \mathcal{F}$ with $\mathcal{F}_t$. We call an element of $\mathcal{W}$ a ***weak equivalence*** *and call an element of $\mathcal{F}$ a* ***fibration****. An element of $\mathcal{F}_t$ is called a* ***trivial fibration****. An object of $\mathcal{K}$ is called* ***fibrant*** *if the unique morphism to the terminal object is a fibration.*

6.5. Definition. *A category with finite limits and a weak fibration structure is called a* ***weak fibration category****.*

6.6. Example. For every model category $(\mathcal{M}; \mathcal{W}, \mathcal{C}, \mathcal{F})$, $(\mathcal{M}; \mathcal{W}, \mathcal{F})$ is a weak fibration category and $(\mathcal{M}; \mathcal{W}, \mathcal{C})$ is a weak cofibration category.

6.7. Example. Let $(\mathcal{K}; \mathcal{W}, \mathcal{F})$ be a weak fibration category. Let $\mathcal{K}_f$ be the full subcategory of the fibrant objects. Then $(\mathcal{K}_f; \mathcal{W} \cap \mathcal{K}_f, \mathcal{F} \cap \mathcal{F}_f)$ is a category of fibrant objects in the sense of K. Brown ([Brown, 1973]).

6.8. Weak Fibration Structure on $\mathcal{M}(X)$. Here we will construct a weak fibration structure on $\mathcal{M}(X)$.

6.9. Definition. *Let X be a smooth quasi-projective variety. We defined two sets $\mathcal{W}(X)$ and $\mathcal{F}(X)$ of morphisms in $\mathcal{M}(X)$ as follows: Let $h : f \to g$ be a morphism in $\mathcal{M}(X)$.*

$$\begin{tikzcd} (Y, \underline{\mathcal{I}}^c) \arrow[rr, "h"] \arrow[dr, "f"'] & & (Z, \underline{\mathcal{J}}^d) \arrow[dl, "g"] \\ & (X, \mathcal{O}_X^0) & \end{tikzcd} \tag{99}$$

1. *h belongs to $\mathcal{W}(X)$ iff $U(h)$ is an isomorphism where U is the forgetful functor (79).*

2. *h belongs to $\mathcal{F}(X)$ iff $\mathscr{I}(f) = \mathscr{I}(g)$.*

For simplicity, we denote $\mathcal{W}(X) \cap \mathcal{F}(X)$ with $\mathcal{F}_t(X)$.

6.10. Remark. In Definition 6.9, the condition imposed on the weak equivalences is perhaps too strong and the set $\mathcal{W}(X)$ is too small. Of course the choice should be made by the application in mind. Because here we want to build a general theory, we take the set of weak equivalences as above.

6.11. Remark. If we consider $\mathscr{I}$ as a functor from $\mathcal{M}(X)$ to the category of ideal sheaves on X (Lemma 5.19), then the condition (2) says that $h \in \mathcal{F}(X)$ iff $\mathscr{I}(h)$ is an identity.

6.12. Lemma. *If X is a smooth quasi-projective variety then the following two properties hold.*

1. *$\mathcal{W}(X)$ is stable under base change.*

2. *All the cobase changes of elements of $\mathcal{W}(X)$ exist and belong to $\mathcal{W}(X)$.*

Proof. (1) The projection $\mathcal{M}(X) \to \mathcal{M}$ preserves pullbacks. Hence the forgetful functor $U : \mathcal{M}(X) \to \mathcal{B}$ also preserves pullbacks. On the other hand, a morphism f in $\mathcal{M}(X)$ is in $\mathcal{W}(X)$ iff $U(f)$ is an isomorphism in $\mathcal{B}$, and all the isomorphisms in $\mathcal{B}$ are preserved by base change. Hence (1) holds.

(2) holds by Lemma 4.33. ■

6.13. Definition. *Let X be a normal quasi-projective variety. Let f and g be two objects of $\mathcal{M}(X)$. We denote the product of f and g with*

$$f * g. \tag{100}$$

Recall that the product $f * g$ can be obtained from the pullback of (f, g) in $\mathcal{M}$. See Lemma 4.29.

6.14. DEFINITION. *Let Y be a smooth variety. Let D_1 and D_2 be $\mathbb{Q}$-divisors on Y. We define a $\mathbb{Q}$-divisor $D_1 * D_2$ on Y with*

$$D_1 * D_2 = \sum_l \max\{a_1^l, a_2^l\} E_l \tag{101}$$

where $D_1 = \sum_l a_1^l E_l$, $D_2 = \sum_l a_2^l E_l$, and E_l's are prime divisors on Y.

6.15. LEMMA. *Let X be a smooth quasi-projective variety. Let f and g be two objects of $\mathcal{M}(X)$. Let $p : f * g \to f$ and $q : f * g \to g$ be the canonical projections. If μ is a log resolution of $f * g$ then $p \cdot \mu$ and $q \cdot \mu$ are log resolutions of f and g respectively and*

$$D_{f*g,\mu} = D_{f,p\cdot\mu} * D_{g,q\cdot\mu} \tag{102}$$

holds.

PROOF. It easily follows from definitions. ■

6.16. PROPOSITION. *If X is a smooth quasi-projective variety then $\mathcal{M}(X)$ is a category with finite limits and $(\mathcal{W}(X), \mathcal{F}(X))$ is a weak fibration structure for $\mathcal{M}(X)$.*

PROOF. $\mathcal{M}(X)$ has finite limits by Lemma 4.30.(2).

Lemma 4.21 implies that $\mathrm{Iso}(\mathcal{M}(X)) \subseteq \mathcal{W}(X) \cap \mathcal{F}(X)$. The rest of (1) and (2) in Definition 6.4 follows from the definitions.

Let $h : f \to g$ be a morphism in $\mathcal{M}(X)$.

$$\begin{tikzcd} (Y, \underline{\mathcal{I}}^c) \arrow[rr, "h"] \arrow[dr, "f"'] & & (Z, \underline{\mathcal{J}}^d) \arrow[dl, "g"] \\ & (X, \mathcal{O}_X^0) & \end{tikzcd} \tag{103}$$

As a morphism in $\mathcal{M}$, $h : (Y, \underline{\mathcal{I}}^c) \to (Z, \underline{\mathcal{J}}^d)$ can be factored as

$$(Y, \underline{\mathcal{I}}^c) \xrightarrow{h_1} (Y, (f^*\underline{\mathcal{J}}^d)_{\mathrm{red}}) \xrightarrow{h_2} (Z, \underline{\mathcal{J}}^d) \tag{104}$$

where $U(h_1) = 1_Y$ and $U(h_2) = U(h)$. The morphism h_1 is in $\mathcal{M}$ by (42). The morphism h_2 is in $\mathcal{M}$ by (43). Let $f' : (Y, (f^*\underline{\mathcal{J}}^d)_{\mathrm{red}}) \to (X, \mathcal{O}_X^0)$ be the object of $\mathcal{M}(X)$ satisfying $U(f') = U(f)$. Then we have morphisms $h_1 : f \to f'$ and $h_2 : f' \to g$ in $\mathcal{M}(X)$. By the definition $h_1 \in \mathcal{W}(X)$ holds. By (95) and (97) $h_2 \in \mathcal{F}(X)$ holds. Therefore, (4) holds.

$\mathcal{W}(X)$ is stable under base change by Lemma 6.12.(1). So to verify the property (3) we only need to show that $\mathcal{F}(X)$ is stable under base change. Consider the following pullback diagram of (g, h) in $\mathcal{M}(X)$.

$$\begin{tikzcd} f_4 \arrow[r, "p_1"] \arrow[d, "p_2"] & f_2 \arrow[d, "g"] \\ f_3 \arrow[r, "h"] & f_1 \end{tikzcd} \tag{105}$$

where $f_i : (Y_i, \underline{\mathcal{I}}_i^{c_i}) \to (X, \mathcal{O}_X^0)$ for $i = 1, 2, 3, 4$ and $g \in \mathcal{F}(X)$. By (77)

$$f_4 = f_2 * f_3 \tag{106}$$

holds. Then by Lemma 4.29 we have $\underline{\mathcal{I}}_4^{c_4} = p_1^* \underline{\mathcal{I}}_2^{c_2} * p_2^* \underline{\mathcal{I}}_3^{c_3}$. Let $\mu : Y \to Y_4$ be a log resolution of

$$f : (Y_4, \underline{\mathcal{I}}_4^{c_4} * (g \cdot p_1)^* \underline{\mathcal{I}}_1^{c_1}) \to (X, \mathcal{O}_X^0) \tag{107}$$

where $U(f) = U(f_4)$. μ is also a log resolution of f_4, and induces log resolutions of f_1, f_2, and f_3. By (102)

$$D_{(f_4, \mu)} = D_{(f_2, p_1 \cdot \mu)} * D_{(f_3, p_2 \cdot \mu)} \tag{108}$$

holds. Then the following Lemma 6.17 implies that $\mathcal{F}(X)$ is closed under base change. ■

6.17. Lemma. *Let $f : Y \to X$ be a projective birational morphism between smooth quasi-projective varieties. Let D_1, D_2, and D_3 be effective $\mathbb{Q}$-divisors on Y satisfying $D_2 \geq D_1$ and $D_3 \geq D_1$. Let $D_4 = D_2 * D_3$. Then*

$$f_* \mathcal{O}_Y(K_{Y/X} - \lfloor D_1 \rfloor) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor D_2 \rfloor) \tag{109}$$

implies

$$f_* \mathcal{O}_Y(K_{Y/X} - \lfloor D_3 \rfloor) = f_* \mathcal{O}_Y(K_{Y/X} - \lfloor D_4 \rfloor). \tag{110}$$

Proof. Let A_1 and A_2 be effective $\mathbb{Q}$-divisors such that $D_3 = A_1 + D_1$ and $D_4 = A_2 + D_2$. By definition we have $A_1 \geq A_2$. We want to use the induction on the number of the irreducible components of A_1.

Let E be an irreducible component of A_1. Let c_1 and c_2 be the coefficients of E in A_1 and A_2 respectively. Then we can write $A_1 = c_1 E + B_1$ and $A_2 = c_2 E + B_2$ for some effective $\mathbb{Q}$-divisors B_1 and B_2 such that $\mathrm{ord}_E(B_1 + B_2) = 0$. From $A_1 \geq A_2$, we have $c_1 \geq c_2$. Consider the following inequalities.

$$\begin{array}{ccccc} D_4 & \geq & B_2 + D_2 & \geq & D_2 \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ D_3 & \geq & B_1 + D_1 & \geq & D_1 \end{array} \tag{111}$$

Since $D_4 = D_3 * D_2$, we have $D_4 = D_3 * (D_2 + B_2)$. Since $\mathrm{ord}_E(B_1 + D_1) = \mathrm{ord}_E(D_1) \leq \mathrm{ord}_E(D_2)$, we have $\mathrm{ord}_E((B_1 + D_1) * D_2) = \mathrm{ord}_E D_2 = \mathrm{ord}_E(B_2 + D_2)$. So, $(B_1 + D_1) * D_2 = (B_2 + D_2)$. Therefore, by the induction hypothesis, we may assume that $B_1 = B_2 = 0$ and

$$D_3 = c_1 E + D_1 \text{ and } D_4 = c_2 E + D_2 \tag{112}$$

hold. In particular, we may assume that

$$c_2 + a_2 = \max\{c_1 + a_1, a_2\} \tag{113}$$

where $a_1 = \mathrm{ord}_E D_1$ and $a_2 = \mathrm{ord}_E D_2$.

If $c_2 + a_2 = a_2$ holds in (113) then $D_4 = D_2$. Therefore,

$$f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_1 \rfloor) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_4 \rfloor) \tag{114}$$

by (109), hence

$$f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_3 \rfloor) = f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_4 \rfloor).$$

So from now on, we assume that

$$c_2 + a_2 = c_1 + a_1 \tag{115}$$

holds.

We write, as in (82),

$$K_{Y/X} - \lfloor D_i \rfloor = P_i - N_i \tag{116}$$

where P_i and N_i are effective divisors on Y without common component for $i = 1, 2, 3, 4$. We have

$$f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_i \rfloor) = f_*\mathcal{O}_Y(-N_i) \tag{117}$$

by (29). Hence

$$f_*\mathcal{O}_Y(-N_1) = f_*\mathcal{O}_Y(-N_2) \tag{118}$$

holds by (109). Let F be a prime divisor on Y. Then

$$\mathrm{ord}_F N_1 = \mathrm{ord}_F N_3 \text{ if } F \neq E \tag{119}$$

$$\mathrm{ord}_F N_1 \leq \mathrm{ord}_F N_3 \text{ if } F = E \tag{120}$$

and

$$\mathrm{ord}_F N_2 = \mathrm{ord}_F N_4 \text{ if } F \neq E \tag{121}$$

$$\mathrm{ord}_F N_2 \leq \mathrm{ord}_F N_4 \text{ if } F = E \tag{122}$$

hold by (112). Let $b = \mathrm{ord}_E K_{Y/X}$. Then because of (115), we have

$$\lceil b - (c_1 + a_1) \rceil = \lceil b - (c_2 + a_2) \rceil. \tag{123}$$

We denote this common integer with d. Then

$$\mathrm{ord}_E(P_3 - N_3) = d = \mathrm{ord}_E(P_4 - N_4) \tag{124}$$

holds.

Assume that $d \geq 0$. Since P_3 (resp. P_4) and N_3 (resp. N_4) have no common component,

$$\mathrm{ord}_E N_3 = 0 = \mathrm{ord}_E N_4 \tag{125}$$

by (124). So

$$\mathrm{ord}_E N_1 = 0 = \mathrm{ord}_E N_2 \tag{126}$$

by (119) and (121). Then

$$N_1 = N_3 \text{ and } N_2 = N_4 \tag{127}$$

by (120) and (122). Therefore,

$$\begin{aligned} f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_3 \rfloor) =& f_*\mathcal{O}_Y(-N_3) && (128)\\ =& f_*\mathcal{O}_Y(-N_1) && (129)\\ =& f_*\mathcal{O}_Y(-N_2) && (130)\\ =& f_*\mathcal{O}_Y(-N_4) && (131)\\ =& f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_4 \rfloor) && (132) \end{aligned}$$

by (117), (118), and (127).

Assume that $d < 0$. We have

$$\mathcal{O}_Y(-N_3) = \mathcal{O}_Y(-N_1) \cap \mathcal{O}_Y(dE) \tag{133}$$

and

$$\mathcal{O}_Y(-N_4) = \mathcal{O}_Y(-N_2) \cap \mathcal{O}_Y(dE) \tag{134}$$

by (119) - (122). Then

$$\begin{aligned} f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_3 \rfloor) =& f_*\mathcal{O}_Y(-N_3) && (135)\\ =& f_*(\mathcal{O}_Y(-N_1) \cap \mathcal{O}_Y(dE)) && (136)\\ =& f_*(\mathcal{O}_Y(-N_1)) \cap f_*(\mathcal{O}_Y(dE)) && (137)\\ =& f_*(\mathcal{O}_Y(-N_2)) \cap f_*(\mathcal{O}_Y(dE)) && (138)\\ =& f_*(\mathcal{O}_Y(-N_2) \cap \mathcal{O}_Y(dE)) && (139)\\ =& f_*\mathcal{O}_Y(-N_4) && (140)\\ =& f_*\mathcal{O}_Y(K_{Y/X} - \lfloor D_4 \rfloor) && (141) \end{aligned}$$

by (117), (118), (133), and (134). ■

7. Model Structure on Pro(X)

Here we will prove Theorem 1.1 after recollecting some well known notions and facts. We review pro-categories in Section 7.1, and recall their basic properties including the uniform approximation theorem in Section 7.12 and completeness and cocompleteness of pro-categories in Section 7.21. Next we recall model categories in Section 7.28 and review a result of Barnea and Schlank [Barnea and Schlank, 2016] in Section 7.38. The proof of Theorem 1.1 will be given at the end of this section after we establish in Section 7.50 the two out of three property of $\mathcal{W}_{\mathrm{Pro}}(X)$ by slightly modifying the original proof of Isaksen in [Isaksen, 2004].

7.1. Pro-Categories. Here and the following three subsections, we will recall some properties of pro-categories. Our main reference is the appendix of [Artin and Mazur, 1969]. One can also consult the original [Artin, Grothendieck, and Verdier, 1972] or [Kashiwara and Schapira, 2006].

7.2. Definition. *A non-empty category $\mathcal{I}$ is called **cofiltering** if*

1. *for every $i, j \in \mathcal{I}$, there are an object $k \in \mathcal{I}$ and morphisms $k \to i$ and $k \to j$ in $\mathcal{I}$.*
2. *for every $i, j \in \mathcal{I}$ and $f, g : i \rightrightarrows j$, there is a morphism $h : k \to i$ in $\mathcal{I}$ such that $fh = gh$.*

In [Artin and Mazur, 1969], the authors define a pro-object as a contravariant functor from a small filtering category. In this note, a pro-object is a covariant functor from a small cofiltering categories.

7.3. Definition. *Given a category $\mathcal{K}$, we define a category* $\operatorname{Pro}(\mathcal{K})$ *as follows.*

1. *Objects are functors $X : \mathcal{I} \to \mathcal{K}$ from a small cofiltering category $\mathcal{I}$.*
2. *Given two objects $X : \mathcal{I} \to \mathcal{K}$ and $Y : \mathcal{J} \to \mathcal{K}$, the hom-set* $\operatorname{Pro}(\mathcal{K})(X, Y)$ *is*

$$\operatorname{Pro}(\mathcal{K})(X, Y) = \lim_{\mathcal{J}} \operatorname*{colim}_{\mathcal{I}^{\mathrm{op}}} \mathcal{K}(X_i, Y_j). \tag{142}$$

We call $\operatorname{Pro}(\mathcal{K})$ *the **pro-category** of $\mathcal{K}$. Objects of* $\operatorname{Pro}(\mathcal{K})$ *are called **pro-projects** of $\mathcal{K}$.*

7.4. Remark. $\operatorname{Pro}(\mathcal{K})$ is again a (locally small) category.

7.5. Remark. $\operatorname{Pro}(\mathcal{K})$ depends on the choice of the universe $\mathcal{U}$ that we made at the beginning of Section 2.1.

7.6. Remark. To give a morphism $f : X \to Y$ in $\operatorname{Pro}(\mathcal{K})$ is to specify

$$f_j \in \operatorname*{colim}_{\mathcal{I}} \mathcal{K}(X_i, Y_j) \tag{143}$$

for every $j \in \mathcal{J}$ satisfying certain compatibility conditions.

7.7. Remark. A category $\mathcal{K}$ can be considered as a full subcategory of its pro-category $\operatorname{Pro}(\mathcal{K})$ by considering an object $X \in \mathcal{K}$ as a functor from an one object category without non-identity morphisms.

7.8. Definition. *Let $X : \mathcal{I} \to \mathcal{K}$ be a pro-object of a category $\mathcal{K}$. For every $i \in \mathcal{I}$, we call X_i the i-th **component** of X.*

7.9. Definition. *Let $f : X \to Y$ be a morphism in* $\operatorname{Pro}(\mathcal{K})$ *given by a compatible set of elements $\{f_j\}_{j \in \mathcal{J}}$ where $f_j \in \operatorname{colim}_{\mathcal{I}^{\mathrm{op}}} \mathcal{K}(X_i, Y_j)$. We call f_j the j-th **coordinate** of f. We will say a morphism $f_{ij} : X_i \to Y_j$ in $\mathcal{K}$ **represents** f if the image of f_{ij} in* $\operatorname{colim}_{\mathcal{I}^{\mathrm{op}}} \mathcal{K}(X_i, Y_j)$ *is f_j, the j-th coordinate of f.*

7.10. Pro-Category of Preorder. Here we will show that if $\mathcal{K}$ is a preorder then the pro-category of $\mathcal{K}$ is also a preorder.

7.11. Lemma. *Let $\mathcal{K}$ be a preorder.*

1. *Every diagram in $\mathcal{K}$ commutes.*
2. *For every pro-object $X : \mathcal{I} \to \mathcal{K}$ and $Y \in \mathcal{K}$, $\operatorname{colim}_{\mathcal{I}^{op}} \mathcal{K}(X_i, Y)$ has at most one element.*
3. *Let $X : \mathcal{I} \to \mathcal{K}$ and $Y : \mathcal{J} \to \mathcal{K}$ be objects of $\operatorname{Pro}(\mathcal{K})$. Then to give a morphism $f : X \to Y$ in $\operatorname{Pro}(\mathcal{K})$ is equivalent to specify, for every $j \in \mathcal{J}$, an object $i \in \mathcal{I}$ and a morphism $f_{ij} : X_i \to Y_j$ (without worrying about the compatibilities).*
4. *$\operatorname{Pro}(\mathcal{K})$ is a preorder.*
5. *A morphism $f : X \to Y$ in $\operatorname{Pro}(\mathcal{K})$ is an isomorphism iff there exists a morphism $g : Y \to X$.*

Proof. (1) follows from $\mathcal{K}$ being a preorder.

(2) Assume that there exist i_1, i_2 such that $\mathcal{K}(X_{i_1}, Y)$ and $\mathcal{K}(X_{i_2}, Y)$ are non-empty. Let $f_1 \in \mathcal{K}(X_{i_1}, Y)$ and $f_2 \in \mathcal{K}(X_{i_2}, Y)$. Since $\mathcal{I}$ is cofiltering, there exist an object $i \in \mathcal{I}$ with morphisms $u_1 : i \to i_1$ and $u_2 : i \to i_2$ in $\mathcal{I}$. Since every diagram commutes in $\mathcal{K}$ by (1), following diagram commutes.

$$\begin{array}{ccc} X_i & \xrightarrow{X_{u_1}} & X_{i_1} \\ \downarrow{\scriptstyle X_{u_2}} & & \downarrow{\scriptstyle f_1} \\ X_{i_2} & \xrightarrow{f_2} & Y \end{array} \tag{144}$$

Therefore, $\operatorname{colim}_{\mathcal{I}^{op}} \mathcal{K}(X_i, Y)$ has at most one element.

(3) Assume that for every $j \in \mathcal{J}$ we have an object $i \in \mathcal{I}$ and a morphism $f_{ij} : X_i \to Y_j$ in $\mathcal{K}$. We want to show that they are compatible.

Let us choose two of them: $f_1 : X_{i_1} \to Y_{j_1}$ and $f_2 : X_{i_2} \to Y_{j_2}$. Let $v : j_1 \to j_2$ be a morphism in $\mathcal{J}$. Then, f_2 and $Y_v \cdot f_1$ represent the same element in $\operatorname{colim}_{\mathcal{I}^{op}} \mathcal{K}(X_i, Y_{j_2})$ by (2). Hence f_{ij}'s are compatible.

(4) follows from (2).

(5) follows from (4). ∎

7.12. Uniform Approximation. Now we want to recall from [Artin and Mazur, 1969] the uniform approximation of loopless finite diagrams in pro-categories. Here the **loopless** means that the beginning and the end of a chain of morphisms in the diagram are always distinct. Later in Section 7.50, it is used to prove the two out of three property for the weak equivalences in the pro-category of $\mathcal{M}(X)$. The uniform approximation theorem is also useful in dealing with finite limits and finite colimits (Lemma 7.22).

From the definition (142) one might think that morphisms in $\mathrm{Pro}(\mathcal{K})$ are difficult to handle. But, up to isomorphism, one can replace them with natural transformations between functors with the **same** index categories. More precisely, the uniform approximation theorem in [Artin and Mazur, 1969] for diagrams with finite loopless index categories can be stated elegantly as follows ([Meyer, 1980], See Theorem 3.3 in [Isaksen, 2002] for a generalization). Given a finite loopless category $\mathcal{D}$, there is an essentially surjective functor

$$\mathrm{Pro}(\mathcal{K}^{\mathcal{D}}) \to \mathrm{Pro}(\mathcal{K})^{\mathcal{D}}. \tag{145}$$

In particular, the calculation of pullbacks and pushouts can be done level-wise up to isomorphism. However, we will need a bit more explicit statement, Lemma 7.20, to prove the two out of three property for the set of weak equivalences in $\mathrm{Pro}(X)$.

First we recall the cofinal functor.

7.13. Definition. [Definition 2.8 in [Barnea and Schlank, 2016]] *A functor $P : \mathcal{I} \to \mathcal{J}$ is called **cofinal** if for every $j \in \mathcal{J}$, the comma category $(P \downarrow j)$ is non-empty and connected, i.e., the following two conditions hold.*

1. *There exist $i \in \mathcal{I}$ and a morphism $P(i) \to j$ in $\mathcal{J}$.*
2. *For every $i_1, i_2 \in \mathcal{I}$ and morphisms $f : P(i_1) \to j$ and $g : P(i_2) \to j$, there exist $i \in \mathcal{I}$ and morphisms $u : i \to i_1$ and $v : i \to i_2$ such that $f \cdot P(u) = g \cdot P(v)$.*

7.14. Remark. In Appendix (1.5) in [Artin and Mazur, 1969], the authors give a definition of cofinal functor slightly different from the above. However if $\mathcal{I}$ is cofiltering then these two definitions coincide.

7.15. Remark. The compositions of cofinal functors are cofinal.

7.16. Remark. If $P : \mathcal{I} \to \mathcal{J}$ is a cofinal functor and $\mathcal{I}$ is cofiltering then $\mathcal{J}$ is also cofiltering.

7.17. Lemma. [Appendix (1.8) and (2.5) in [Artin and Mazur, 1969]] *Let $P : \mathcal{I} \to \mathcal{J}$ be a cofinal functor between small cofiltering categories. Let $X : \mathcal{J} \to \mathcal{K}$ be an object of* $\mathrm{Pro}(\mathcal{K})$. *Then the canonical morphism*

$$X \to P^* X \tag{146}$$

in $\mathrm{Pro}(\mathcal{K})$ *is an isomorphism.*

A natural transformation between functors $X, Y : \mathcal{I} \to \mathcal{K}$ from a small cofiltering category $\mathcal{I}$ naturally produces a morphism in $\mathrm{Pro}(\mathcal{K})$. The next lemma shows that every morphism in $\mathrm{Pro}(\mathcal{K})$ can be transformed into such a morphism by re-indexing the domain categories. But first, we recall two definitions from [Isaksen, 2004].

7.18. Definition. *Let $\mathcal{K}$ be a category. A* ***level presentation*** *of a morphism $f : X \to Y$ in* $\mathrm{Pro}(\mathcal{K})$ *consists of*

1. *a small cofiltering category $\mathcal{I}$,*
2. *two objects $\widetilde{X}, \widetilde{Y} : \mathcal{I} \to \mathcal{K}$ of* $\mathrm{Pro}(\mathcal{K})$*,*
3. *a natural transformation $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, and*
4. *isomorphisms $X \to \widetilde{X}$ and $Y \to \widetilde{Y}$ in* $\mathrm{Pro}(\mathcal{K})$

such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"] & Y \arrow[d, "\cong"] \\ \widetilde{X} \arrow[r, "\widetilde{f}"] & \widetilde{Y} \end{tikzcd} \tag{147}$$

in $\mathrm{Pro}(\mathcal{K})$ *commutes. For simplicity, we will denote the level presentation of f by the morphism $\widetilde{f}$.*

The previous notion generalizes to diagrams in $\mathrm{Pro}(\mathcal{K})$.

7.19. Definition. *Let $\mathcal{K}$ be a category. Let $X : \mathcal{D} \to \mathrm{Pro}(\mathcal{K})$ be a functor from a category $\mathcal{D}$. A* ***level presentation*** *of X in* $\mathrm{Pro}(\mathcal{K})$ *consists of*

1. *a small cofiltering category $\mathcal{I}$,*
2. *a functor $\widetilde{X} : \mathcal{D} \times \mathcal{I} \to \mathcal{K}$, and*
3. *an isomorphism $\tau_a : X_a \to \widetilde{X}_a$ in* $\mathrm{Pro}(\mathcal{K})$ *for every object a of $\mathcal{D}$*

such that for every morphism $\phi : a \to b$ in $\mathcal{D}$, the diagram

$$\begin{tikzcd} X_a \arrow[r, "X_\phi"] \arrow[d, "\cong"', "\tau_a"] & X_b \arrow[d, "\cong"', "\tau_b"] \\ \widetilde{X}_a \arrow[r, "\widetilde{X}_\phi"] & \widetilde{X}_b \end{tikzcd} \tag{148}$$

in $\mathrm{Pro}(\mathcal{K})$ *commutes.*

The level presentation is a flexible notion. But we will need a bit more stronger statement as in the following lemma.

7.20. LEMMA. [Appendix (3.3) in [Artin and Mazur, 1969] "uniform approximation"] *Let $\mathcal{K}$ be a category. Let $\mathcal{D}$ be a finite loopless category. Then for every functor $X : \mathcal{D} \to \mathrm{Pro}(\mathcal{K})$, there exist*

1. *a small cofiltering category $\mathcal{I}$,*
2. *a functor $\widetilde{X} : \mathcal{D} \times \mathcal{I} \to \mathcal{K}$, and*
3. *a cofinal functor $P_a : \mathcal{I} \to \mathrm{dom}(X_a)$ for every object a of $\mathcal{D}$*

such that

(4) for every object a of $\mathcal{D}$

$$\widetilde{X}_a = P_a^* X_a \tag{149}$$

holds,

(5) for every morphism $\phi : a \to b$ in $\mathcal{D}$ the diagram

$$\begin{tikzcd} X_a \arrow[r, "X_\phi"] \arrow[d, "\cong"] & X_b \arrow[d, "\cong"] \\ \widetilde{X}_a \arrow[r, "\widetilde{X}_\phi"] & \widetilde{X}_b \end{tikzcd} \tag{150}$$

in $\mathrm{Pro}(\mathcal{K})$ commutes where the vertical morphisms are the canonical isomorphisms in Lemma 7.17, and

(6) for every morphism ϕ in $\mathcal{D}$ and every object i of $\mathcal{I}$ $\widetilde{X}_{\phi,i}$ represents X_ϕ.

In particular, every finite loopless diagram in $\mathrm{Pro}(\mathcal{K})$ has a level presentation.

7.21. LIMITS AND COLIMITS IN PRO-CATEGORIES. Here we recall from [Artin and Mazur, 1969] that if a category is **small** and has finite limits then its pro-category has small limits and small colimits.

First, we discuss finite limits in pro-categories. Let $\mathcal{K}$ be a category with finite (co)limits and $\mathcal{I}$ be a small cofiltering category. The Proposition (4.1) in [Artin and Mazur, 1969] says that the functor

$$\mathcal{K}^{\mathcal{I}} \to \mathrm{Pro}(\mathcal{K}) \tag{151}$$

associating $X : \mathcal{I} \to \mathcal{K}$ with the corresponding pro-object in $\mathrm{Pro}(\mathcal{K})$ commutes with finite (co)limits. In fact the proof of the proposition shows how to compute (co)limits of finite diagrams in $\mathrm{Pro}(\mathcal{K})$ when their level presentations exist.

7.22. Lemma. [Proposition (4.1) in [Artin and Mazur, 1969]] *Let $\mathcal{K}$ be a category. Let $X : \mathcal{D} \to \mathrm{Pro}(\mathcal{K})$ be a finite diagram in $\mathrm{Pro}(\mathcal{K})$ and $\widetilde{X} : \mathcal{D} \times \mathcal{I} \to \mathcal{K}$ be a level presentation of X in Definition 7.19.*

1. *If the colimit $\mathrm{colim}_d \widetilde{X}_{d,i}$ exists for every $i \in \mathcal{I}$ then the object $\{\mathrm{colim}_d \widetilde{X}_{d,i}\}_{i\in\mathcal{I}}$ of $\mathrm{Pro}(\mathcal{K})$ is the colimit of X.*
2. *If the limit $\lim_d \widetilde{X}_{d,i}$ exists for every $i \in \mathcal{I}$ then the object $\{\lim_d \widetilde{X}_{d,i}\}_{i\in\mathcal{I}}$ of $\mathrm{Pro}(\mathcal{K})$ is the limit of X.*

Finite limits exist if finite products and equalizers exist by Corollary 1 on Chapter 5.2 in [Mac Lan, 1998]. So the following lemma follows from Lemma 7.20 and Lemma 7.22.(2).

7.23. Lemma. [Appendix (4.2) in [Artin and Mazur, 1969]] *If a category $\mathcal{K}$ has finite limits then so does $\mathrm{Pro}(\mathcal{K})$.*

If a category is small and has finite limits then its pro-category has small colimits.

7.24. Lemma. [Appendix (4.3) in [Artin and Mazur, 1969]; cf. Proposition 11.1 in [Isaksen, 2001]] *If $\mathcal{K}$ is a small category with finite limits then $\mathrm{Pro}(\mathcal{K})$ has small colimits.*

7.25. Remark. Lemma 7.24 also follows from the well-known special adjoint functor theorem (cf. Corollary on Chapter V.8 in [Mac Lan, 1998]) and the following Lemma 7.27. Again the smallness of $\mathcal{K}$ is essential in verifying the assumptions in the special adjoint functor theorem.

By Theorem 2 on Chapter 5.2 in [Mac Lan, 1998] and the dual of Theorem 1 on Chapter 9.1 in [Mac Lan, 1998], a (small) limit can be decomposed into a (small) cofiltering limit and finite limits.

7.26. Lemma. [Appendix (4.4) in [Artin and Mazur, 1969]; cf. Theorem 4.1 in [Isaksen, 2002]] *For any category $\mathcal{K}$, its pro-category $\mathrm{Pro}(\mathcal{K})$ has small cofiltering limits.*

7.27. Lemma. *If $\mathcal{K}$ is a category with finite limits then $\mathrm{Pro}(\mathcal{K})$ has small limits.*

Proof. By Theorem 2 on Chapter 5.2 in [Mac Lan, 1998] and the dual of Theorem 1 on Chapter 9.1 in [Mac Lan, 1998], $\mathrm{Pro}(\mathcal{K})$ has small limits if it has small cofiltering limits and finite limits. Hence $\mathrm{Pro}(\mathcal{K})$ has small limits by Lemma 7.23 and Lemma 7.26. ∎

Lemma 7.27 can also be proved as in the proof of Proposition 11.1 in [Isaksen, 2001]. There only the finite limits are used during the proof of the completeness of the pro-category.

7.28. Model Structures. Here we recall model categories. For our purpose, the content of Chapter 14 in [May and Ponto, 2012] is sufficient. For a comprehensive reference we refer to [Hirschhorn, 2003] or [Hovey, 1999], or the original [Quillen, 1967].

7.29. DEFINITION. *Let $\mathcal{M}$ be a category. Let $\mathcal{S}$ be a set of morphisms in $\mathcal{M}$.*

1. *Let f and g be two morphisms in $\mathcal{M}$. If there is a commutative diagram*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \tag{152}$$

   *where the compositions of the horizontal morphisms are identities then we say that f is a **retract** of g.*

2. *We denote by $\mathrm{R}(\mathcal{S})$ the set of morphisms in $\mathcal{M}$ that are retracts of morphisms in $\mathcal{S}$. We call it the **retract closure** of $\mathcal{S}$.*

3. *We say that $\mathcal{S}$ is **closed under retracts** if $\mathrm{R}(\mathcal{S}) = \mathcal{S}$.*

7.30. REMARK. We note that $\mathcal{S} \subseteq \mathrm{R}(\mathcal{S})$ and $\mathrm{R}(\mathrm{R}(\mathcal{S})) = \mathrm{R}(\mathcal{S})$ hold.

7.31. DEFINITION. *Let $\mathcal{M}$ be a category. Let $\mathcal{L}$ and $\mathcal{R}$ be two sets of morphisms in $\mathcal{M}$.*

1. *Let f and g be morphisms in $\mathcal{M}$. We write $f \boxslash g$ iff for every commutative square*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd} \tag{153}$$

   *of solid arrows there is a lifting of the dotted arrow making the whole diagram commutes. If $f \boxslash g$ holds, we say that f has the **left lifting property** with respect to g and g has the **right lifting property** with respect to f.*

2. *We write $\mathcal{L} \boxslash \mathcal{R}$ iff for every $f \in \mathcal{L}$ and $g \in \mathcal{R}$ $f \boxslash g$ holds.*

3. *We define two sets*

$$\mathcal{L}^{\boxslash} = \{g \in \mathrm{Mor}(\mathcal{M}) \mid f \boxslash g \text{ for every } f \in \mathcal{L}\} \tag{154}$$

   and

$$^{\boxslash}\mathcal{R} = \{f \in \mathrm{Mor}(\mathcal{M}) \mid f \boxslash g \text{ for every } g \in \mathcal{R}\}. \tag{155}$$

   of morphisms in $\mathcal{M}$.

7.32. DEFINITION. [Definition 14.1.11 in [May and Ponto, 2012]] *Let $\mathcal{M}$ be a category. A **weak factorization system** on $\mathcal{M}$ is a pair $(\mathcal{L}, \mathcal{R})$ of two sets of morphisms in $\mathcal{M}$ such that the following three conditions hold.*

1. $\mathrm{R}(\mathcal{L}) = \mathcal{L}$ *and* $\mathrm{R}(\mathcal{R}) = \mathcal{R}$.

2. $\mathcal{L} \boxslash \mathcal{R}$.

3. $\mathrm{Mor}(\mathcal{M}) = \mathcal{R} \circ \mathcal{L}$ *(Definition 6.3).*

*A weak factorization systems is called **functorial** if the factorization in (3) is functorial.*

7.33. Lemma. [Proposition 14.1.13 in [May and Ponto, 2012]] *Let* $\mathcal{M}$ *be a category. Let* $(\mathcal{L}, \mathcal{R})$ *be a pair of sets of morphisms in* $\mathcal{M}$ *such that* $\mathrm{Mor}(\mathcal{M}) = \mathcal{R} \circ \mathcal{L}$*. Then (1) and (2) in Definition 7.32 hold iff* $\mathcal{L}^{\boxslash} = \mathcal{R}$ *and* $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ *hold.*

7.34. Definition. *Let* $\mathcal{M}$ *be a category. A* ***(functorial) model structure*** *on* $\mathcal{M}$ *consists of three sets* $\mathcal{W}$, $\mathcal{C}$, *and* $\mathcal{F}$ *of morphisms in* $\mathcal{M}$ *satisfying the following two conditions.*

1. $\mathcal{W}$ *satisfies the two out of three property (Definition 6.2).*

2. $(\mathcal{W} \cap \mathcal{C}, \mathcal{F})$ *and* $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ *are (functorial) weak factorization systems on* $\mathcal{M}$.

We call an element of $\mathcal{W}$ *a* ***weak equivalence****, an element of* $\mathcal{C}$ *a cofibration, and an element of* $\mathcal{F}$ *a* ***fibration****. An element of* $\mathcal{W} \cap \mathcal{C}$ *is called a* ***trivial cofibration****. An element of* $\mathcal{W} \cap \mathcal{F}$ *is called a* ***trivial fibration****. An object of* $\mathcal{M}$ *is called* ***cofibrant*** *if the unique morphism from the initial object is a cofibration. An object of* $\mathcal{M}$ *is called* ***fibrant*** *if the unique morphism to the terminal object is a fibration.*

7.35. Remark. It is well known that the condition (2) implies $\mathcal{C}$ and $\mathcal{F}$ are closed under composition and $\mathrm{Iso}(\mathcal{M}) = \mathcal{W} \cap \mathcal{C} \cap \mathcal{F}$ holds. In particular, $\mathcal{C}$ and $\mathcal{F}$ form subcategories of $\mathcal{M}$. In general, (2) does not imply that $\mathcal{W}$ is closed under composition. But the condition (1) does.

7.36. Definition. *A category* $\mathcal{M}$ *is called a* ***(functorial) model category*** *if the following two conditions hold.*

1. $\mathcal{M}$ *has small colimits and small limits.*

2. $\mathcal{M}$ *has a (functorial) model structure.*

7.37. Remark. Quillen's original definition of the (closed) model category does not assume the functoriality and assumes only the existence of finite limits and finite colimits.

7.38. Barnea-Schlank Model Structure. Here we will state a result of Barnea and Schlank in [Barnea and Schlank, 2016] on the existence of a pair of weak factorization systems on $\mathrm{Pro}(\mathcal{K})$.

First, we need some definitions. Recall from (15) that a morphism $a \to b$ in a poset T is denoted by $a \geq b$.

7.39. Definition. *A poset* T *is called* ***cofinite*** *if the subset* $\{b \leq a \mid b \in T\}$ *of* T *is finite for every* $a \in T$.

7.40. Definition. [Definition 2.11 in [Barnea and Schlank, 2016]] *Let* $\mathcal{K}$ *be a category with finite limits. Let* $\mathcal{S}$ *be a non-empty set of morphisms in* $\mathcal{K}$.

1. *We denote by* $\mathrm{Lw}(\mathcal{S})$ *the set of natural transformations* $f : X \to Y$ *between functors* $X, Y : \mathcal{I} \to \mathcal{K}$ *from a small category* $\mathcal{I}$ *such that* f_i *is in* $\mathcal{S}$ *for every* $i \in \mathcal{I}$.

2. *We denote by* $\mathrm{Sp}(\mathcal{S})$ *the set of natural transformations* $f : X \to Y$ *between functors* $X, Y : \mathcal{I} \to \mathcal{K}$ *from a small cofinite poset* $\mathcal{I}$ *such that the natural map*

$$X_t \to Y_t \underset{\lim_{s<t} Y_s}{\times} \lim_{s<t} X_s \tag{156}$$

is in $\mathcal{S}$ *for every* $t \in \mathcal{I}$.

7.41. Remark. The second conditions (2) implies that if $t \in \mathcal{I}$ is a minimal object of $\mathcal{I}$, that is there is no non-trivial morphism from t, then f_t is in $\mathcal{S}$.

7.42. Definition. [Definition 2.12.(4) and (5) in [Barnea and Schlank, 2016]] *Let* $\mathcal{K}$ *be a category with finite limits. Let* $\mathcal{S}$ *be a non-empty set of morphisms in* $\mathcal{K}$.

1. *We denote by* $\mathrm{Lw}^{\cong}(\mathcal{S})$ *the set of all morphisms* f *in* $\mathrm{Pro}(\mathcal{K})$ *that has a level presentation (Definition 7.18)* $\widetilde{f}$ *in* $\mathrm{Lw}(\mathcal{S})$.
2. *We denote by* $\mathrm{Sp}^{\cong}(\mathcal{S})$ *the set of all morphisms* f *in* $\mathrm{Pro}(\mathcal{K})$ *that has a level presentation* $\widetilde{f}$ *in* $\mathrm{Sp}(\mathcal{S})$.

7.43. Remark. Note that in Definition 7.40, $\mathcal{I}$ is not required to be cofiltering. But the index category of the level presentation $\widetilde{f}$ in Definition 7.42 is cofiltering.

Finally, we define a model structure on $\mathrm{Pro}(\mathcal{K})$ when $\mathcal{K}$ is a weak fibration category.

7.44. Definition. *Let* $\mathcal{K}$ *be a category with finite limits. Given a weak fibration structure* $(\mathcal{W}, \mathcal{F})$ *on* $\mathcal{K}$, *we define three sets* $\mathcal{W}_{\mathrm{Pro}}$, $\mathcal{C}_{\mathrm{Pro}}$, *and* $\mathcal{F}_{\mathrm{Pro}}$ *of morphisms in* $\mathrm{Pro}(\mathcal{K})$ *as follows.*

$$\mathcal{W}_{\mathrm{Pro}} = \mathrm{Lw}^{\cong}(\mathcal{W}) \tag{157}$$

$$\mathcal{C}_{\mathrm{Pro}} = {}^{\boxslash}(\mathcal{W} \cap \mathcal{F}) \tag{158}$$

$$\mathcal{F}_{\mathrm{Pro}} = \mathrm{R}(\mathrm{Sp}^{\cong}(\mathcal{F})) \tag{159}$$

7.45. Definition. *Let* $\mathcal{K}$ *be a category with finite limits. Then a weak fibration structure* $(\mathcal{W}, \mathcal{F})$ *on* $\mathcal{K}$ *is called* ***pro-admissible*** *if* $\mathcal{W}_{\mathrm{Pro}}$ *has the two out of three property.*

The following theorem of Barnea and Schlank is one of the key ingredients in the construction of a model structure on $\mathrm{Pro}(X)$.

7.46. Theorem. [Theorem 4.18 in [Barnea and Schlank, 2016], Proposition 3.18 in [Barnea and Schlank, 2016]] *Let* $\mathcal{K}$ *be a small category with finite limits. Let* $(\mathcal{W}, \mathcal{F})$ *be a pro-admissible weak fibration structure on* $\mathcal{K}$. *Then the following properties hold.*

1. *The following are functorial weak factorization systems in* $\mathrm{Pro}(\mathcal{K})$.

$$(\mathcal{C}_{\mathrm{Pro}}, \mathcal{W}_{\mathrm{Pro}} \cap \mathcal{F}_{\mathrm{Pro}}) \qquad (\mathcal{W}_{\mathrm{Pro}} \cap \mathcal{C}_{\mathrm{Pro}}, \mathcal{F}_{\mathrm{Pro}}) \tag{160}$$

2. $\mathcal{W}_{\mathrm{Pro}} \cap \mathcal{C}_{\mathrm{Pro}} = {}^{\boxslash}\mathcal{F}$.
3. $\mathcal{W}_{\mathrm{Pro}} \cap \mathcal{F}_{\mathrm{Pro}} = \mathrm{R}(\mathrm{Sp}^{\cong}(\mathcal{W} \cap \mathcal{F}))$.

7.47. Remark. In Theorem 4.18 in [Barnea and Schlank, 2016], the above theorem is stated without the functoriality condition in (1) under a weaker assumption that $\mathcal{K}$ is homotopically small (Definition 4.12 in [Barnea and Schlank, 2016]). The proof of Theorem 4.18 in [Barnea and Schlank, 2016] relies on Proposition 3.17 in [Barnea and Schlank, 2016]. However, as explained in Remark 2.19 in [Barnea and Schlank, 2014], if we use Proposition 3.18 in [Barnea and Schlank, 2016] instead and assume that $\mathcal{K}$ is small then we have the functoriality as stated above.

The follows lemma shows that we do not need to take retract closures in (159) and Theorem 7.46.(3).

7.48. Lemma. *Let $\mathcal{K}$ be a category with finite limits. Let $\mathcal{S}$ be a set of morphisms in $\mathcal{K}$. If $\mathcal{K}$ is a preorder then*

$$\mathrm{R}(\mathrm{Sp}^{\cong}(\mathcal{S})) = \mathrm{Sp}^{\cong}(\mathcal{S}) \tag{161}$$

holds.

Proof. It follows from Lemma 7.11.(5). ■

7.49. Lemma. [Corollary 2.20 in [Barnea and Schlank, 2016]] *Let $\mathcal{K}$ be a category with finite limits. Let $\mathcal{S}$ be a set of morphisms in $\mathcal{K}$ such that $\mathcal{S}$ contains the set of all isomorphisms and is closed under composition. If $\mathcal{S}$ is closed under base change then*

$$\mathrm{Sp}^{\cong}(\mathcal{S}) \subseteq \mathrm{Lw}^{\cong}(\mathcal{S}) \tag{162}$$

holds.

7.50. Two out of Three Property for $\mathrm{Pro}(X)$. Recall that $\mathrm{Pro}(X)$ is the category $\mathrm{Pro}(\mathcal{M}(X))$ of pro-objects in $\mathcal{M}(X)$ defined in (5) and $\mathcal{W}_{\mathrm{Pro}}(X)$ is the set $\mathrm{Lw}^{\cong}(\mathcal{W}(X))$ of weak equivalences in $\mathrm{Pro}(\mathcal{M}(X))$ defined in (6). Here we prove the two out of three property (Definition 6.2) for $\mathcal{W}_{\mathrm{Pro}}(X)$. It is essentially done in [Isaksen, 2004]. All we need is to modify the proof slightly and use Lemma 4.33.

In [Isaksen, 2004], the author assumes that the base category is a proper model category. What is relevant to the proof of the two out of three property is the properness. In our case, a much stronger condition holds: all elements in $\mathcal{W}(X)$ are stable under base change and cobase change.

7.51. Lemma. *If X is a smooth quasi-projective variety then $\mathrm{Lw}^{\cong}(\mathcal{W}(X))$ satisfies the two out of three property.*

Proof. Since $\mathcal{W}(X)$ has the two out of three property, the result follows from Lemma 6.12 and the following Lemma 7.52. ■

7.52. Lemma. [Lemma 3.5 and Lemma 3.6 in [Isaksen, 2004]] *Let $\mathcal{K}$ be a category. Let $\mathcal{W}$ be a non-empty set of morphisms in $\mathcal{K}$. If $\mathcal{W}$ satisfies the following three properties then $\mathrm{Lw}^{\cong}(\mathcal{W})$ has the two out of three property.*

1. *$\mathcal{W}$ has the two out of three property.*
2. *All the base changes of elements of $\mathcal{W}$ exist and belong to $\mathcal{W}$.*
3. *All the cobase changes of elements of $\mathcal{W}$ exist and belong to $\mathcal{W}$.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms in $\mathrm{Pro}(\mathcal{K})$.

(M) Assume that $f, g \in \mathrm{Lw}^{\cong}(\mathcal{W})$. Then by definition, gf is isomorphic in $\mathrm{Pro}(\mathcal{K})$ to the following diagram

$$\widetilde{X} \xrightarrow{\widetilde{f}} \widetilde{Y}_1 \xleftarrow{\lambda} \widetilde{Y}_2 \xrightarrow{\widetilde{g}} \widetilde{Z} \tag{163}$$

where $\widetilde{f}$ and $\widetilde{g}$ in $\mathrm{Lw}(\mathcal{W})$ are level presentations of f and g respectively, and λ is an isomorphism in $\mathrm{Pro}(\mathcal{K})$. Using Lemma 7.20, we we may replace λ with its level presentation and still assume that

$$\widetilde{f}, \widetilde{g} \in \mathrm{Lw}(\mathcal{W}) \tag{164}$$

holds. In particular, every pro-object in diagram (163) has the same index category.

Now let

$$\begin{tikzcd} W \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2", "\cong"'] & \widetilde{Y}_2 \arrow[d, "\lambda", "\cong"'] \\ X \arrow[r, "\widetilde{f}"] & \widetilde{Y}_1 \end{tikzcd} \tag{165}$$

be the level-wise pullback diagram provided by Lemma 7.22.(2). The morphism pr_2 is an isomorphism in $\mathrm{Pro}(\mathcal{K})$ because it is a base change of an isomorphism λ. Since $\widetilde{f}$ belongs to $\mathrm{Lw}(\mathcal{W})$,

$$\mathrm{pr}_1 \in \mathrm{Lw}(\mathcal{W}) \tag{166}$$

holds by the assumption (2). Then, since $\mathcal{W}$ is closed under composition, $\widetilde{g} \cdot \mathrm{pr}_1$ belongs to $\mathrm{Lw}(\mathcal{W})$. Hence gf belongs to $\mathrm{Lw}^{\cong}(\mathcal{W})$.

(L) Assume that $gf, g \in \mathrm{Lw}^{\cong}(\mathcal{W})$. Then by definition, there is the following commutative diagram in $\mathrm{Pro}(\mathcal{K})$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\lambda", "\cong"'] & Y \arrow[d, "\rho", "\cong"'] \\ \widetilde{X} \arrow[r, "\rho f \lambda^{-1}"] \arrow[d, "\widetilde{gf}"] & \widetilde{Y} \arrow[d, "\widetilde{g}"] \\ \widetilde{Z}_1 \arrow[r, "\tau", "\cong"'] & \widetilde{Z}_2 \end{tikzcd} \tag{167}$$

where $\widetilde{gf}$ and $\widetilde{g}$ in $\mathrm{Lw}(\mathcal{W})$ are level presentations of gf and g respectively. By applying Lemma 7.20 to the lower square in diagram (167), we may replace τ and $\rho \cdot f \cdot \lambda^{-1}$ with their level presentation and still assume that

$$\widetilde{gf}, \widetilde{g} \in \mathrm{Lw}(\mathcal{W}) \tag{168}$$

holds. In particular, the pro-objects in the lower square have the same index category.

Now consider the pullback of the lower right corner of diagram (167) which we also calculate level-wise by Lemma 7.22.(2).

$$\begin{tikzcd}
\widetilde{X} \arrow[rr, "\rho f \lambda^{-1}"] \arrow[dr, "h"] \arrow[dd, "\widetilde{gf}"'] & & \widetilde{Y} \arrow[dd, "\widetilde{g}"] \\
& \widetilde{Z}_1 \times_{\widetilde{Z}_2} \widetilde{Y} \arrow[ur, "\mathrm{pr}_1", "\cong"'] \arrow[dl, "\mathrm{pr}_2"] & \\
\widetilde{Z}_1 \arrow[rr, "\tau", "\cong"'] & & \widetilde{Z}_2
\end{tikzcd} \tag{169}$$

Since τ is an isomorphism in $\mathrm{Pro}(\mathcal{K})$, pr_1 is also an isomorphism in $\mathrm{Pro}(\mathcal{K})$. Since $\widetilde{g}$ is in $\mathrm{Lw}(\mathcal{W})$,

$$\mathrm{pr}_2 \in \mathrm{Lw}(\mathcal{W}) \tag{170}$$

holds by the assumption (2). Then by the two out of three property for $\mathcal{W}$, h also belongs to $\mathrm{Lw}(\mathcal{W})$. Hence f has a level presentation h in $\mathrm{Lw}(\mathcal{W})$ and belongs to $\mathrm{Lw}^{\cong}(\mathcal{W})$.

(R) The proof of (R) is dual to (L). So we omit it. ∎

7.53. Proof of Theorem 1.1. Let X be a smooth quasi-projective variety. First, $\mathrm{Pro}(X)$ is a preorder by Lemma 7.11.(4). The pro-category $\mathrm{Pro}(X)$ has small limits and small colimits by Lemma 7.24 and Lemma 7.27.

The category $\mathcal{M}(X)$ is a weak fibration category by Proposition 6.16 with the weak fibration structure $(\mathcal{W}(X), \mathcal{F}(X))$. Furthermore the weak fibration category

$$(\mathcal{M}(X); \mathcal{W}(X), \mathcal{F}(X)) \tag{171}$$

is pro-admissible by Lemma 7.51. Then, the remaining properties (3), (4), and (5) follow from Theorem 7.46 and Lemma 7.48

8. Cofibrant Objects and Fibrant Objects

Here we make some remarks on fibrant objects and cofibrant objects in $\mathrm{Pro}(X)$. We will see that the cofibrant objects should have many components (Definition 7.8). And the fibrant objects are determined by klt pairs in birational geometry.

8.1. Cofibrant Objects. Let $f : (Y, \underline{\mathcal{J}}^c) \to (X, \mathcal{O}_X^0)$ and $g : (Y, \underline{\mathcal{J}}^{c+\epsilon}) \to (X, \mathcal{O}_X^0)$ be objects of $\mathcal{M}(X)$ satisfying $U(f) = U(g)$ where $\underline{\mathcal{J}}^c = \{\mathcal{J}_1^{c_1}, \dots, \mathcal{J}_m^{c_m}\}$ and $\underline{\mathcal{J}}^{c+\epsilon} = \{\mathcal{J}_1^{c_1+\epsilon}, \dots, \mathcal{J}_k^{c_m+\epsilon}\}$ for some positive rational ϵ. Then for a sufficiently small ϵ, the morphism

$$h : g \to f \tag{172}$$

in $\mathcal{M}(X)$ satisfying $U(h) = 1_Y$ belongs to $\mathcal{F}_t(X)$, hence to $\mathcal{F}_{\mathrm{Pro}}(X) \cap \mathcal{W}_{\mathrm{Pro}}(X)$. But the following lifting problem can not be solved in $\mathcal{M}(X)$, hence in $\mathrm{Pro}(X)$.

$$\begin{tikzcd} & g \arrow[d, "h"] \\ f \arrow[ur, dotted] \arrow[r, "1"] & f \end{tikzcd}$$

Therefore we observe the following property.

8.2. Proposition. *Let X be a smooth quasi-projective variety. Then there is no object in $\mathcal{M}(X)$ cofibrant in* $\mathrm{Pro}(X)$.

On the other hand, let $F : 2 \to \mathcal{M}(X)$ be a functor such that

1. $F(0) = g$ and $F(1) = f$ and
2. $F(0 \to 1) = h$

hold where 2 is the category with two objects 0 and 1 and one non-identity morphism $0 \to 1$. Then the following diagram in $\mathrm{Pro}(X)$ has a lifting h_0

$$\begin{tikzcd} & g \arrow[d, "h"] \\ F \arrow[ur, dotted, "h_0"] \arrow[r, "h_1"] & f \end{tikzcd}$$

where $h_0 \in \mathrm{Pro}(X)(F, g)$ and $h_1 \in \mathrm{Pro}(X)(F, g)$ are induced by 1_g and 1_f respectively.

So considering every possible f, it seems reasonable to expect that the cofibrant objects in $\mathrm{Pro}(X)$ have many components (Definition 7.8).

8.3. Fibrant Objects. There is an interpretation of fibrant objects using a concept in birational geometry. In birational geometry, certain singularities play an important role. One of them is the klt singularities.

Recall that a pair (X, D) of a smooth quasi-projective variety and an effective $\mathbb{Q}$-divisor D on X is **klt (kawamata log terminal)** if there exists a log resolution $\mu : Y \to X$ of (X, D) such that

$$K_{Y/X} - \lfloor \mu^* D \rfloor \geq 0. \tag{173}$$

From the decomposition of $K_{Y/X} - \lfloor \mu^* D \rfloor$ in (82), we know that the inequality (173) holds iff $N = 0$. Thus (173) is equivalent to

$$\mathscr{I}(X, D) = \mathcal{O}_X \tag{174}$$

by (29). Let us generalize it to $\mathcal{M}(X)$.

8.4. Definition. *Let X be a smooth quasi-projective variety.*

1. *Let $f : (Y, \underline{\mathcal{I}}^c) \to (X, \mathcal{O}_X^0)$ be an object of $\mathcal{M}(X)$. We say that f is **klt** if*

$$\mathscr{I}(f) = \mathcal{O}_X \tag{175}$$

holds.

2. *Let $F : \mathcal{I} \to \mathcal{M}(X)$ be an object of $\mathrm{Pro}(X)$. We say that F is **klt** if every component F_i, $i \in \mathcal{I}$, of F is klt.*

First, we will show that fibrant objects in $\mathcal{M}(X)$ are made up of the klt pairs. Recall that an object of $\mathcal{M}(X)$ is fibrant if the unique map to the terminal object belongs to $\mathcal{F}(X)$.

8.5. Lemma. *Let X be a smooth quasi-projective variety. Then for every object*

$$f : (Y, \{\mathcal{I}_1^{c_1}, \dots, \mathcal{I}_m^{c_m}\}) \to (X, \mathcal{O}_X^0) \tag{176}$$

of $\mathcal{M}(X)$, the following are equivalent.

1. *f is fibrant in $\mathcal{M}(X)$.*
2. *f is klt.*
3. *For every $i = 1, \dots, m$, the object*

$$f_i : (Y, \mathcal{I}_i^{c_i}) \to (X, \mathcal{O}_X^0) \tag{177}$$

of $\mathcal{M}(X)$ is klt where $U(f_i) = U(f)$.

Proof. (1)⇔(2) f is fibrant in $\mathcal{M}(X)$ iff $\mathscr{I}(f) = \mathscr{I}(1_{(X,\mathcal{O}_X^0)})$. But $\mathscr{I}(1_{(X,\mathcal{O}_X^0)}) = \mathcal{O}_X$.

(2)⇔(3) Let $\mu : Z \to Y$ be a log resolution of f. Let D_i be a Cartier divisor in Z such that $\mathcal{I}_i \cdot \mathcal{O}_Z = \mathcal{O}_Z(-D_i)$. We write $c_i D_i = \sum_{k=1}^{l} a_i^k E_k$ where E_k's are prime divisors and $a_i^k \in \mathbb{Q}_{\geq 0}$. Then by definition (88),

$$D_{f,\mu} = \sum_{k=1}^{l} \max\{a_1^k, \dots, a_m^k\} E_k. \tag{178}$$

Now $\mathscr{I}(f) = \mathcal{O}_X$ iff $K_{Z/Y} - \lfloor D_{f,\mu} \rfloor$ is effective iff $K_{Z/Y} - \lfloor c_i D_i \rfloor$ is effective for all i. ∎

The following two lemmas are general results on small pro-admissible weak fibration categories that are preorders. Recall that given a small pro-admissible weak fibration category $(\mathcal{K}; \mathcal{W}, \mathcal{F})$, we have the model structure

$$(\mathcal{W}_{\mathrm{Pro}}, \mathcal{C}_{\mathrm{Pro}}, \mathcal{F}_{\mathrm{Pro}}) \tag{179}$$

on $\mathrm{Pro}(\mathcal{K})$ by Theorem 7.46.

The first lemma shows that, in contrast to cofibrant objects in $\mathrm{Pro}(X)$, an object f of $\mathcal{M}(X)$ is fibrant in $\mathrm{Pro}(X)$ iff f is fibrant in $\mathcal{M}(X)$.

8.6. Lemma. *Let $(\mathcal{K}; \mathcal{W}, \mathcal{F})$ be a small pro-admissible weak fibration category. Then the following properties hold.*

1. $\mathcal{F} \subseteq \mathcal{F}_{\mathrm{Pro}}$.
2. *If $\mathcal{K}$ is a preorder then $\mathcal{F} = \mathcal{F}_{\mathrm{Pro}} \cap \mathrm{Mor}(\mathcal{K})$ holds.*

Proof. Assume that f is a fibration in $\mathcal{K}$, i.e., $f \in \mathcal{F}$. Clearly, $\mathcal{F} \subseteq ({}^{\boxslash}\mathcal{F})^{\boxslash}$. Since $({}^{\boxslash}\mathcal{F})^{\boxslash} = (\mathcal{W}_{\mathrm{Pro}} \cap \mathcal{C}_{\mathrm{Pro}})^{\boxslash} = \mathcal{F}_{\mathrm{Pro}}$ by Theorem 7.46.(2) and Lemma 7.33, f is a fibration in $\mathrm{Pro}(\mathcal{K})$.

Let f be a morphism in $\mathcal{K}$. Assume that f is a fibration in $\mathrm{Pro}(\mathcal{K})$. Then $f \in \mathrm{Sp}^{\cong}(\mathcal{F})$, hence $f \in \mathrm{Lw}^{\cong}(\mathcal{F})$ by Lemma 7.49. Let $\widetilde{f}$ in $\mathrm{Lw}(\mathcal{F})$ be a level presentation of f as in (147) and $\mathcal{I}$ be the index category of $\widetilde{X}$ and $\widetilde{Y}$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cong\downarrow\alpha & & \cong\downarrow\beta \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \end{array} \tag{180}$$

We will show that f is isomorphic to one of the components of $\widetilde{f}$.

First, for every $i \in \mathcal{I}$, we have the following commutative diagram in $\mathcal{K}$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow\alpha_i & & \downarrow\beta_i \\ \widetilde{X}_i & \xrightarrow{\widetilde{f}_i} & \widetilde{Y}_i \end{array} \tag{181}$$

where α_i and β_i are morphisms representing α and β respectively. Since $\widetilde{f} \in \mathrm{Lw}(\mathcal{F})$, $\widetilde{f}_i \in \mathcal{F}$.

On the other hand, since $\mathcal{I}$ is cofiltering, there is $i \in \mathcal{I}$ such that the following diagram in $\mathcal{K}$ commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ (\alpha^{-1})_i\uparrow & & (\beta^{-1})_i\uparrow \\ \widetilde{X}_i & \xrightarrow{\widetilde{f}_i} & \widetilde{Y}_i \end{array} \tag{182}$$

where $(\alpha^{-1})_i$ and $(\beta^{-1})_i$ are morphisms representing α^{-1} and β^{-1} respectively.

Then since $\mathcal{K}$ is a preorder, α_i and β_i are isomorphisms in $\mathcal{K}$. Hence $\widetilde{f_i} \in \mathcal{F}$ implies $f \in \mathcal{F}$. ∎

The second lemma shows that an object $F : \mathcal{I} \to \mathcal{M}(X)$ of $\mathrm{Pro}(X)$ is fibrant iff every component F_i, $i \in \mathcal{I}$, of F is fibrant.

8.7. Lemma. *Let $(\mathcal{K}; \mathcal{W}, \mathcal{F})$ be a small pro-admissible weak fibration category. We assume that $\mathcal{K}$ is a preorder. Let $F : \mathcal{I} \to \mathcal{K}$ be an object of* $\mathrm{Pro}(\mathcal{K})$.

1. *Assume that $\mathcal{F}$ has the following property: for every $g, h \in \mathcal{F}$ with* $\mathrm{dom}(h) = \mathrm{cod}(g)$, *$hg \in \mathcal{F}$ implies $h \in \mathcal{F}$. Then, F_i is a fibrant object in* $\mathrm{Pro}(\mathcal{K})$ *for every $i \in \mathcal{I}$ if F is a fibrant object in* $\mathrm{Pro}(\mathcal{K})$.

2. *If F_i is a fibrant object in* $\mathrm{Pro}(\mathcal{K})$ *for every $i \in \mathcal{I}$ then F is a fibrant object in* $\mathrm{Pro}(\mathcal{K})$.

Proof. (1) First we make observations on terminal objects of $\mathrm{Pro}(\mathcal{K})$. Let $*$ be a terminal object of $\mathcal{K}$. Then $*$ is also a terminal object in $\mathrm{Pro}(\mathcal{K})$. Furthermore if T is a terminal object in $\mathrm{Pro}(\mathcal{K})$ then every component of T is isomorphic to $*$. This is because a morphism in $\mathcal{K}$ that has a left or right inverse is already an isomorphism, which is a consequence of $\mathcal{K}$ being preorder.

Since F is fibrant, the unique morphism $F \to *$ belongs to $\mathrm{Lw}^{\cong}(\mathcal{F})$ by Lemma 7.49. So there is a level presentation $G \to T$ in $\mathrm{Lw}(\mathcal{F})$ of $F \to *$ where $G, T : \mathcal{J} \to \mathcal{K}$. In particular, every component of G is fibrant in $\mathcal{K}$ because every component of T is isomorphic to $*$.

Since there is a (iso)morphism $f : G \to F$, for every $i \in \mathcal{I}$, there exist $j \in \mathcal{J}$ and a morphism $f_{ji} : G_j \to F_i$ representing f. Then since G_j is fibrant, F_i is also fibrant by our assumption.

(2) Let $f : A \to B$ be a trivial cofibration. Let $g : A \to F$ be a morphism in $\mathrm{Pro}(\mathcal{K})$. We want to find a lifting h of the following diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & F \\ B \arrow[ur, dotted, "h"'] \end{tikzcd} \tag{183}$$

Since F_i is fibrant in $\mathrm{Pro}(\mathcal{K})$ for every $i \in \mathcal{I}$, there is a lifting h_i of the following diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & F \arrow[r] & F_i \\ B \arrow[urr, dotted, "h_i"'] \end{tikzcd} \tag{184}$$

where the second horizontal morphism is represented by the identity on F_i. Then $\{h_i\}_{i \in \mathcal{I}}$ form a morphism $h : B \to F$ by Lemma 7.11.(3). Since $\mathrm{Pro}(\mathcal{K})$ is a preorder by Lemma 7.11.(4), $hf = g$ holds. ∎

8.8. Remark. If $\mathcal{K}$ is a preorder, limits of fibrant objects are again fibrant in $\mathrm{Pro}(\mathcal{K})$. And if $F : \mathcal{I} \to \mathcal{K}$ is a pro-object then F is the limit of F_i's considered as objects of $\mathrm{Pro}(\mathcal{K})$. In this way one can also show that if F_i's are fibrant then F is fibrant. However, to show these we again need Lemma 7.11.(3) and (4).

8.9. Proof of Theorem 1.2. Note that $\mathcal{F}(X)$ satisfies the hypothesis in Lemma 8.7.(1). So the theorem follows from Lemma 8.5, Lemma 8.6, and Lemma 8.7.

9. A Reformulation of the Conjecture of Fujita

In this final section, we will introduce a variant of the model category constructed in Section 7, and use it to give a model theoretic reformulation of the conjecture of Fujita.

Throughout this section, we fix a smooth projective variety X and an ample divisor L on X.

9.1. Model Category $\mathrm{Pro}(U, L)_q$. In this subsection, we fix an open subvariety U of X and a positive rational number $q \in \mathbb{Q}_{>0}$. Here we will construct a model category $\mathrm{Pro}(U, L)_q$. The proof is similar to that of the model category $\mathrm{Pro}(X)$. So we will omit it.

9.2. Definition. *We denote by*

$$\mathcal{M}(U, L)_q \tag{185}$$

the full subcategory of $\mathcal{M}(U)$ *consisting of objects*

$$\alpha : (Y, \{\mathcal{O}_Y(-f^*(D_1|_U))^{c_1}, \dots, \mathcal{O}_Y(-f^*(D_m|_U))^{c_m}\}) \to (U, \mathcal{O}_U^0) \tag{186}$$

where f *is the underlying morphism of* α *and* D_i *is an effective divisor on* X *such that*

$$D_i \in |d_i L| \tag{187}$$

for some $d_i \in \mathbb{Z}_{\geq 0}$ *satisfying*

$$c_i d_i < q \tag{188}$$

for $i = 1, \dots, m$.

9.3. Definition. *We denote* $\mathrm{Mor}(\mathcal{M}(U, L)_q) \cap \mathcal{W}(U)$ *and* $\mathrm{Mor}(\mathcal{M}(U, L)_q) \cap \mathcal{F}(U)$ *by* $\mathcal{W}(U, L)_q$ *and* $\mathcal{F}(U, L)_q$ *respectively.*

$$\mathcal{W}(U, L)_q = \mathrm{Mor}(\mathcal{M}(U, L)_q) \cap \mathcal{W}(U) \tag{189}$$

$$\mathcal{F}(U, L)_q = \mathrm{Mor}(\mathcal{M}(U, L)_q) \cap \mathcal{F}(U) \tag{190}$$

Then one can show that

$$(\mathcal{M}(U, L)_q; \mathcal{W}(U, L)_q, \mathcal{F}(U, L)_q) \tag{191}$$

is a pro-admissible weak fibration category. Let us denote the pro-category of $\mathcal{M}(U,L)_q$ by

$$\mathrm{Pro}(U,L)_q \tag{192}$$

and denote the three sets of morphisms in $\mathrm{Pro}(U,L)_q$ produced by Definition 7.44 by

$$(\mathcal{W}_{\mathrm{Pro}}(U,L)_q, \mathcal{C}_{\mathrm{Pro}}(U,L)_q, \mathcal{F}_{\mathrm{Pro}}(U,L)_q). \tag{193}$$

Now, as before, we have the following result by Theorem 7.46.

9.4. THEOREM. *Let X be a smooth projective variety. Let L be an ample divisor on X. Let U be an open subvariety of X. Let q be a positive rational number. Then*

1. $\mathrm{Pro}(U,L)_q$ *is a preorder.*
2. $\mathrm{Pro}(U,L)_q$ *has small limits and small colimits.*
3. $(\mathcal{W}_{\mathrm{Pro}}(U,L)_q, \mathcal{C}_{\mathrm{Pro}}(U,L)_q, \mathcal{F}_{\mathrm{Pro}}(U,L)_q)$ *is a functorial model structure on* $\mathrm{Pro}(U,L)_q$.
4. $\mathcal{W}_{\mathrm{Pro}}(U,L)_q \cap \mathcal{C}_{\mathrm{Pro}}(U,L)_q = {}^{\boxslash}\mathcal{F}(U,L)_q$.
5. $\mathcal{W}_{\mathrm{Pro}}(U,L)_q \cap \mathcal{F}_{\mathrm{Pro}}(U,L)_q = \mathrm{Sp}^{\cong}(\mathcal{W}(U,L)_q \cap \mathcal{F}(U,L)_q)$.

We also have the following Theorem 9.5 analogous to Theorem 1.2. The proof of Theorem 1.2 is based on Lemma 8.5 and the two general lemmas, Lemma 8.6 and Lemma 8.7. So the same proof works for the following theorem and we will omit it.

Note that $\mathrm{Pro}(U,L)_q$ is a subcategory of $\mathrm{Pro}(U)$ because $\mathcal{M}(U,L)_q$ is a subcategory of $\mathcal{M}(U)$. So it makes sense to say that an object of $\mathrm{Pro}(U,L)_q$ is klt.

9.5. THEOREM. *Let X be a smooth projective variety. Let L be an ample divisor on X. Let U be an open subvariety of X. Let q be a positive rational number. Then for every object F of* $\mathrm{Pro}(U,L)_q$, *the following are equivalent.*

1. *F is a fibrant object of* $\mathrm{Pro}(U,L)_q$.
2. *F is klt.*

Finally, the following lemma connects the klt objects in $\mathcal{M}(U,L)_q$ and the multiplier ideal sheaves of $\mathbb{Q}$-divisors in (1).

9.6. LEMMA. *For every object*

$$\alpha : (Y, \{\mathcal{O}_Y(-f^*(D_1|_U))^{c_1}, \ldots, \mathcal{O}_Y(-f^*(D_m|_U))^{c_m}\}) \to (U, \mathcal{O}_U^0) \tag{194}$$

of $\mathcal{M}(U,L)_q$, *the following are equivalent.*

1. *α is klt.*

2. For every $i = 1, \dots, m$, the object

$$\alpha_i : (Y, \mathcal{O}_Y(-f^*(D_i|_U))^{c_i}) \to (U, \mathcal{O}_U^0) \tag{195}$$

of $\mathcal{M}(U, L)_q$ is klt.

3. For every $= 1, \dots, m$, the multiplier ideal sheaf

$$\mathscr{I}(X, \mathcal{O}_X(-D_i)^{c_i}) \tag{196}$$

of the weighted ideal sheaf $\mathcal{O}_X(-D_i)^{c_i}$ defined in (81) satisfies

$$\mathscr{I}(X, \mathcal{O}_X(-D_i)^{c_i})|_U = \mathcal{O}_U. \tag{197}$$

4. For every $= 1, \dots, m$, the multiplier ideal sheaf

$$\mathscr{I}(X, c_i D_i) \tag{198}$$

of the $\mathbb{Q}$-divisor $c_i D_i$ defined in (1) satisfies

$$\mathscr{I}(X, c_i D_i)|_U = \mathcal{O}_U. \tag{199}$$

Proof. (1) and (2) are equivalent by Lemma 8.5.

(2) and (3) are equivalent by Lemma 5.17.

(3) and (4) are equivalent by definition. ∎

9.7. A Reformulation. First, we explain the well-known connection ([Ein, 1997], [Lazarsfeld, 2004]) between the conjecture of Fujita and multiplier ideal sheaves.

As we mentioned in the introduction multiplier ideal sheaves have a strong vanishing property. Let m be a positive integer. Let D be an effective $\mathbb{Q}$-divisor on X. If D is $\mathbb{Q}$-linearly equivalent to cL ($D \overset{\mathbb{Q}}{\sim} cL$) for some rational $c < m$, i.e., there is an integer k such that kD and kcL are divisors (with integral coefficients) and kD is linearly equivalent to kcL, then

$$H^i(X, \mathscr{I}(X, D) \otimes \mathcal{O}_X(K_X + mL)) = 0, \quad \text{for } i > 0. \tag{200}$$

So in terms of multiplier ideal sheaves, the conjecture is reduced to proving the following statement. If $m \geq \dim X + 1$ then for every point p of X there exists an effective $\mathbb{Q}$-divisor D on X such that

1. D is $\mathbb{Q}$-linearly equivalent to cL for some $c < m$,
2. $\mathscr{I}(X, D) \subseteq m_p$ where m_p is the maximal ideal sheaf of X at p, and
3. $\mathscr{I}(X, D)|_U = \mathcal{O}_U$ where U is a punctured neighborhood of X at p.

If such a D exists, we have the following exact sequence by (200)

$$H^0(X, \mathcal{O}_X(K_X + mL)) \to H^0(Z, \mathcal{O}_Z(K_X + mL)) \to 0 \tag{201}$$

where Z is the subscheme of X defined by $\mathscr{I}(X, D)$. Since $\{p\} \subseteq Z$ is a connected component of Z by (2) and (3), we can find a section in $H^0(Z, \mathcal{O}_Z(K_X + mL))$ non-vanishing at p. Then the exactness of the sequence (201) guarantees that $K_X + mL$ has a section non-vanishing at p.

Using a general property of ample divisors, it is easy to show that such a D indeed exists for a sufficiently large but inexplicit m. In fact, there is an explicit bound for all dimensions. In [Angehrn and Siu, 1995], the authors prove that if $m \geq \frac{(\dim X+1)(\dim X)}{2} + 1$ then such a D exists. This bound was improved by Helmke in [Helmke, 1997]. However to create a $\mathbb{Q}$-divisor D satisfying (2) and (3), we in general need to create intermediate effective $\mathbb{Q}$-divisors

$$D_1, \ldots, D_k = D \tag{202}$$

on X such that the sequence of the dimensions at p of the schemes Z_i

$$Z_i = Z_{\mathscr{I}(X, D_i)} \tag{203}$$

associated with the multiplier ideal sheaves $\mathscr{I}(X, D_i)$ converges to zero. If it is inevitable to work with several $\mathbb{Q}$-divisors, it may be worthwhile to work simultaneously with the set of all the $\mathbb{Q}$-divisors more consciously.

Consider the set S of all effective $\mathbb{Q}$-divisors D on X, $\mathbb{Q}$-linearly equivalent to qL for some $q \in \mathbb{Q}_{\geq 0}$.

$$S = \coprod_{q \in \mathbb{Q}_{\geq 0}} \{D \mid D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ satisfying } D \overset{\mathbb{Q}}{\sim} qL\} \tag{204}$$

Given a positive rational $q \in \mathbb{Q}_{>0}$, we let

$$S_q = \{D \in S \mid D \overset{\mathbb{Q}}{\sim} cL \text{ for some } 0 \leq c < q\}. \tag{205}$$

Then the existence of a $\mathbb{Q}$-divisor satisfying the above three conditions is equivalent to the following statement.

$$\{D \in S_m \mid (2) \text{ and } (3) \text{ hold for } D\} \neq \emptyset \tag{206}$$

Given an inclusion $i : V \to W$ of two open subvarieties of X we have the induced functor

$$i^* : \mathrm{Pro}(W, L)_m \to \mathrm{Pro}(V, L)_m \tag{207}$$

by restriction. Then the condition (206) is equivalent to the existence of an open subvariety W of X containing p and a non-fibrant object of $\mathrm{Pro}(W, L)_m$ that is mapped to a fibrant object of $\mathrm{Pro}(W - \{p\}, L)_m$ by Theorem 9.5 and Lemma 9.6. So we can reformulate the conditions (1), (2), and (3) as follows.

If $m \geq \dim X + 1$ then for every point p of X there exist a open subvariety W of X containing p and a fibrant object of $\mathrm{Pro}(W - \{p\}, L)_m$ having a non-fibrant lifting in $\mathrm{Pro}(W, L)_m$.

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