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A SIMPLICIAL DESCRIPTION OF THE HOMOTOPY CATEGORY OF SIMPLICIAL GROUPOIDS

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ABSTRACT. In this paper we use Quillen's model structure given by Dwyer-Kan for the category of simplicial groupoids (with discrete object of objects) to describe in this category, in the simplicial language, the fundamental homotopy theoretical constructions of path and cylinder objects. We then characterize the associated left and right homotopy relations in terms of simplicial identities and give a simplicial description of the homotopy category of simplicial groupoids. Finally, we show loop and suspension functors in the pointed case.

1. Introduction

1.1. SUMMARY. A well-known and quite powerful context in which an abstract homotopy theory can be developed is supplied by a category with a closed model structure in the sense of Quillen [16]. The category **Simp(Gp)** of simplicial groups is a remarkable example of what a closed model category is, and the homotopy theory in **Simp(Gp)** developed by Kan [12] occurs as the homotopy theory associated to this closed model structure. According to the terminology used by Quillen, we have that the homotopy theory in **Simp(Gp)** is equivalent to the homotopy theory in the category of reduced simplicial sets and this last is equivalent to the homotopy theory in the category of pointed connected topological spaces.

If Y is an object of a closed model category $\mathcal{C}$, a path object for Y is a factorization of the diagonal morphism

$$Y \xrightarrow{\sigma} Y^I \xrightarrow{(\partial_0,\partial_1)} Y \times Y \ ,$$

where (∂_0, ∂_1) is a fibration and σ is a weak equivalence.

If $f, g \in Hom_{\underline{\mathbf{C}}}(X, Y)$, a right homotopy from f to g is defined as a morphism $h : X \to Y^I$ such that $\partial_0 h = g$ and $\partial_1 h = f$. The morphism f is said to be right homotopic to g if such a right homotopy exists. When Y is fibrant "is right homotopic to" is an equivalence relation on $Hom_{\mathcal{C}}(X, Y)$. The notions of cylinder object and left homotopy are defined in a dual manner. Moreover, if X is cofibrant and Y is fibrant, then the left and right homotopy relations on $Hom_{\mathcal{C}}(X, Y)$ coincide. If $[X, Y]$ denotes the set of equivalence classes, the category $\pi\mathcal{C}_{cf}$, whose objects are the objects of $\mathcal{C}$ that are both

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fibrant and cofibrant, and $Hom_{\pi\mathcal{C}_{cf}}(X,Y) = [X,Y]$, with composition induced from that of $\mathcal{C}$, is equivalent to the homotopy category of $\mathcal{C}$, $Ho(\mathcal{C})$, which is defined to be the localization of $\mathcal{C}$, [5], with respect to the class of weak equivalences.

Moreover, if $\mathcal{C}$ is a closed simplicial model category (see [16]), X is cofibrant and Y is fibrant then the left and the right homotopy relations on $Hom_{\mathcal{C}}(X,Y)$ coincide with the simplicial homotopy relation. This is the case in $\mathbf{Simp(Gp)}$ when $G_\bullet$ is a free simplicial group in which case, for any simplicial group $H_\bullet$, $Hom_{Ho(\mathbf{Simp(Gp)})}(G_\bullet, H_\bullet) = [G_\bullet, H_\bullet]$, the set of simplicial homotopy classes of simplicial morphisms from $G_\bullet$ to $H_\bullet$.

In [3] Dwyer-Kan demonstrated that $\mathbf{Simp(Gpd)}_*$, the category of simplicial groupoids (with discrete object of objects), admits a closed model structure and the associated homotopy theory was then shown to be equivalent to the (unpointed) homotopy theory in the category $\mathbf{Simp(Set)}$ of simplicial sets and therefore to that one in the category of topological spaces. This was done by extending the well-known adjoint situation, [14],

$$G : \mathbf{Simp(set)_{red}} \leftrightarrow \mathbf{Simp(Gp)} : \overline{W}$$

to a pair of adjoint functors

$$G : \mathbf{Simp(Set)} \leftrightarrow \mathbf{Simp(Gpd)}_* : \overline{W} \ ,$$

which induces the equivalence of homotopy theories. In particular, their homotopy categories are equivalent and there is a 1-1 correspondence of homotopy classes of maps. Moreover, analogously to the case of simplicial groups, in the model category $\mathbf{Simp(Gpd)}_*$ every object is fibrant and the cofibrant objects are the free simplicial groupoids and their retracts.

The first aim of the present paper is to give explicit constructions of path space and cylinder object in the closed model category of simplicial groupoids and then, although $\mathbf{Simp(Gpd)}_*$ is not a closed simplicial model category, we characterize the associated right and left homotopy relations in terms of simplicial identities, which correspond by the functor $\overline{W}$ to the simplicial homotopy identities in simplicial sets. Thus, $\mathbf{Simp(Gpd)}_*$ behaves as it was a closed simplicial model category, that is, there is a notion of simplicial homotopy when the source is cofibrant ant the target fibrant and it coincides with the axiomatic left homotopy and right homotopy that Quillen develops via cylinder or path objects. The surprise of this fact can be clarified if one looks at the objects of $\mathbf{Simp(Gpd)}_*$ inside of the category of all simplicial groupoids (i.e., with objects not necessarily discrete) and then one restricts the notion of simplicial homotopy here to those simplicial groupoids. Recall that there are true Quillen model structures on the category of all simplicial groupoids, one created by the Moerdijk's model structure on bisimplicial sets [13] and the dimensionwise nerve $N : \mathbf{Simp(Gpd)} \to \mathbf{BiSsets}$, and the other by Joyal-Tierney in [10], obtained as a particular case (when $\mathcal{E} = Sets$) of the model structure on the category $\mathbf{Gpd(S(\mathcal{E}))}$ of simplicial groupoids in any Grothendieck topos $\mathcal{E}$. Both model structures should yield to equivalent homotopy theories (they have same weak equivalences and cofibrations ordered by inclusion). Besides, as it is deduced from ([10] Theorem 10), the inclusion $I : \mathbf{Simp(Gpd)}_* \to \mathbf{Simp(Gpd)}$ induces an equivalence of homotopy categories and there is a 1-1 correspondence of homotopy classes of maps $[G, H] \cong [I(G), I(H)]$ for any objects $G, H \in \mathbf{Simp(Gpd)}_*$.

The fact that the axiomatic right and left homotopy in $\mathbf{Simp(Gpd)}_*$ can be expressed in the simplicial language gives a more manageable version of the set of homotopy classes of maps between two simplicial groupoids and thus we are able of giving, as in simplicial groups, a simplicial description of the homotopy category of simplicial groupoids. Also we think that, in this way and in order to study non-connected n-types , it is possible to give truncated versions of the fundamental homotopy constructions and the homotopy relation, following the same approach as in simplicial groups [1], [7] and in simplicial sets [4].

Finally, we show pointed versions of the homotopy constructions of path and cylinder objects that allow us to give the corresponding constructions of loop and suspension functors.

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1.2. Notation and terminology. We will freely use the results of [14], [16] and [3]. We denote by $\mathbf{Simp(Gp)}$ the category of simplicial groups and by $\mathbf{Simp(Gpd)}_*$ the category of simplicial groupoids with discrete object of objects. An object $X \in \mathbf{Simp(Gpd)}_*$ can be represented by means of a diagram

$$\begin{tikzcd}
\cdots & X_n \arrow[r, shift left, "d_n"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X_{n-1} \arrow[l, bend right=60, "s_0"'] \arrow[l, bend right=80, "s_{n-1}"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & \cdots & X_2 \arrow[r, shift left=2, "d_2"] \arrow[r] \arrow[r, shift right=2, "d_0"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X_1 \arrow[l, bend right=60, "s_0"'] \arrow[l, bend right=80, "s_1"'] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & X_0 \arrow[l, bend right=60, "s_0"'] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
\cdots & O \arrow[r, equal] & O & \cdots & O \arrow[r, equal] & O \arrow[r, equal] & O
\end{tikzcd}$$

where each $\begin{tikzcd} X_n \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend right=50, "I"'] \end{tikzcd}$ is a groupoid and d_i and s_j are groupoid morphisms satisfying the usual simplicial identities (see [14]). Note that if $O = *$ is a one-point set, then X can be seen as a simplicial group and so $\mathbf{Simp(Gp)}$ is a full subcategory of $\mathbf{Simp(Gpd)}_*$. For any two objects $p, q \in O$ in X, $X(p, q)$ will denote the simplicial set of morphisms in X from p to q. If $p = q$, $X(p) = X(p, p)$ is a simplicial group called the simplicial group of automorphisms in p. Both categories $\mathbf{Simp(Gp)}$ and $\mathbf{Simp(Gpd)}_*$ are closed model categories in the sense of Quillen [16]. Below we recall the structure in $\mathbf{Simp(Gpd)}_*$ [3]:

-A morphism $f : X \to Y$ in $\mathbf{Simp(Gpd)}_*$ is a fibration if $\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & Y_0 \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\ O \arrow[r, "f"'] & O' \end{tikzcd}$ is a fibration of groupoids and, for any $p \in Obj(X)$, the induced morphism $X(p) \to Y(f(p))$ is a fibration of simplicial groups.

-The morphism f is a weak equivalence if f induces a 1-1 correspondence between the components of X and those of Y and, for any $p \in Obj(X)$, the induced morphism $X(p) \to Y(f(p))$ is a weak equivalence of simplicial groups.

-The morphism f is a cofibration if it is a retract of a free map.

In any groupoid $\begin{tikzcd} X_n \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \arrow[l, bend right, "I"'] \end{tikzcd}$, the composition of two morphisms $x, y \in X_n$ such that $s(x) = t(y)$ will be denoted by $x \circ y$.

2. Homotopy groupoids and simplicial homotopy identities

Given $X \in \mathbf{Simp(Gpd)}_*$, the *Moore complex* of X, NX, is defined as the following chain complex of groupoids:

$$N_*X = \left(\begin{tikzcd} \cdots N_nX \arrow[r, "\overline{d}_n"] \arrow[d, shift left] \arrow[d, shift right] & N_{n-1}X \cdots \arrow[d, shift left] \arrow[d, shift right] & N_2X \arrow[r, "\overline{d}_2"] \arrow[d, shift left] \arrow[d, shift right] & N_1X \arrow[r, "\overline{d}_1"] \arrow[d, shift left] \arrow[d, shift right] & N_0X \arrow[d, shift left] \arrow[d, shift right] \\ \cdots O \arrow[r, equal] & O \cdots & O \arrow[r, equal] & O \arrow[r, equal] & O \end{tikzcd} \right)$$

where $N_0X = X_0$, $N_nX \overset{s}{\underset{t}{\rightrightarrows}} O = Ker(d_0) \cap \cdots \cap Ker(d_{n-1})$ and $\bar{d}_n$ is the restriction of d_n.

Note that, as the face and degeneracy morphisms of X are the identity on objects, the groupoid $N_nX \overset{s}{\underset{t}{\rightrightarrows}} O$ is, for $n \geq 1$, the disjoint union of the vertex groups $Kerd_i(p)$, $p \in O$; in particular, $\overline{d}_n\overline{d}_{n+1}$ is always an identity so that $(NX, \overline{d})$ is, indeed, a chain complex of groupoids over O. Then, this construction (see [6]) gives a functor from $\mathbf{Simp(Gpd)}_*$ to the category of chain complexes of groupoids and, if O has only one element, it is clear that this construction reduces to the well-known Moore complex functor defined in $\mathbf{Simp(Gp)}$ (see [14]).

Recall that the homotopy groups of the underlying simplicial set of a simplicial group $G_\bullet$ (pointed by the identity element) can be obtained as the homology groups of the Moore complex of $G_\bullet$. In the same way, we can consider homotopy groupoids of any $X \in \mathbf{Simp(Gpd)}_*$, defined as follows:

Let us consider $B_n(X) \overset{s}{\underset{t}{\rightrightarrows}} O$, the groupoid whose set of morphisms is $\overline{d_{n+1}}(N_{n+1}(X))$, and $Z_n(X) \overset{s}{\underset{t}{\rightrightarrows}} O$, the groupoid whose set of morphisms is $Ker(\overline{d_n})$. It is clear that $B_n(X) \overset{s}{\underset{t}{\rightrightarrows}} O$ is a normal subgroupoid of $Z_n(X) \overset{s}{\underset{t}{\rightrightarrows}} O$ [9], and then we define, for all $n \geq 1$, the n-th *homotopy groupoid* of X, denoted by $\pi_n(X)$, as the quotient groupoid of $Z_n(X) \overset{s}{\underset{t}{\rightrightarrows}} O$ by $B_n(X) \overset{s}{\underset{t}{\rightrightarrows}} O$. Thus, $\pi_n(X)$ has O as the set of objects and $\frac{Z_n(X)}{\equiv}$ as the set of morphisms, where $x \equiv x'$ if there exist y, $y' \in B_n(X)$ such that $y' \circ x = x' \circ y$.

This construction determines a functor from $\mathbf{Simp(Gpd)}_*$ to the category of groupoids. Note that the group of automorphisms in $p \in O$ of the groupoid $\pi_n(X)$ is $\pi_n(X(p))$, which is the n-th homotopy group of the simplicial group of automorphisms in p, and, for $n \geq 1$, $\pi_n(X)$ is the disjoint union of the homotopy groups $\pi_n(X(p))$, $p \in O$.

Next we establish the notion of simplicial homotopy between morphisms in the category $\mathbf{Simp(Gpd)}_*$.

2.1. Definition. *Let $f, g : X \to Y$ be two morphisms in $\mathbf{Simp(Gpd)}_*$. A simplicial homotopy from f to g, denoted by $\beta : f \simeq g$, consists of a map $\beta : O \to Y_0$ such that $s\beta = f$ and $t\beta = g$ together with a family of maps $\beta_n^j : X_n \to Y_n$, $1 \leq j \leq n$, satisfying the following relations:*

a) $d_0\beta_n^1(x) = g_{n-1}d_0(x) \circ s_0^{n-1}\beta s(x)$; $d_n\beta_n^n(x) = s_0^{n-1}\beta t(x) \circ f_{n-1}d_n(x)$, $\forall x \in X_n$.

b) $d_i\beta_n^j = \begin{cases} \beta_{n-1}^{j-1}d_i & i < j \\ \beta_{n-1}^{j}d_i & i \geq j \end{cases}$; $s_i\beta_n^j = \begin{cases} \beta_{n+1}^{j+1}s_i & i < j \\ \beta_{n+1}^{j}s_i & i \geq j \end{cases}$.

c) Given $p \xrightarrow{x} q \xrightarrow{y} r \in X_n$, $\beta_n^j(y \circ x) = \beta_n^j(y) \circ (s_0^n\beta t(x))^{-1} \circ \beta_n^j(x)$.

2.2. Proposition. *Let $f, g : X \to Y$ be two morphisms in $\mathbf{Simp(Gpd)}_*$. Then, giving a simplicial homotopy from f to g is equivalent to giving a map $\alpha : O \to Y_0$ such that $s\alpha = f$ and $t\alpha = g$, together with a family of maps $\alpha_n^j : X_n \to Y_{n+1}$, $0 \leq j \leq n$, satisfying the following relations:*

a) $s\alpha_n^j = fs$; $t\alpha_n^j = gt$.

b) $d_0\alpha_n^0(x) = g_n(x) \circ s_0^n\alpha s(x)$; $d_{n+1}\alpha_n^n(x) = s_0^n\alpha t(x) \circ f_n(x)$, $\forall x \in X_n$.

c) $d_i\alpha_n^j = \begin{cases} \alpha_{n-1}^{j-1}d_i & i < j \\ d_i\alpha_n^{j+1} & i = j+1 \\ \alpha_{n-1}^{j}d_{i-1} & i > j+1 \end{cases}$; $s_i\alpha_{n-1}^j = \begin{cases} \alpha_n^{j+1}s_i & i \leq j \\ \alpha_n^{j}s_{i-1} & i > j \end{cases}$.

d) Given $p \xrightarrow{x} q \xrightarrow{y} r \in X_n$, $\alpha_n^j(y \circ x) = \alpha_n^j(y) \circ (s_0^{n+1}\alpha t(x))^{-1} \circ \alpha_n^j(x)$.

Proof. Let us suppose that α and α_n^j satisfy the above relations. Then, by putting $\beta = \alpha$ and, for each $1 \leq j \leq n$, $\beta_n^j = d_j\alpha_n^{j-1} : X_n \to Y_n$, it is straightforward to see that β is a simplicial homotopy from f to g. Conversely, if $\beta : f \simeq g$ is a simplicial homotopy, then $\alpha = \beta$ and $\alpha_n^j = \beta_{n+1}^{j+1}s_j$, $0 \leq j \leq n$, satisfy the above relations.

■

It is not very difficult to see that the functor $\overline{W}$ preserves simplicial homotopies.

Now, given $f,\ g : X \to Y$ two morphisms in $\mathbf{Simp(Gpd)}_*$ and $\beta : f \simeq g$ a simplicial homotopy, there is, for each $p \in O$, a morphism $\beta(p) : f(p) \to g(p)$, and then we can consider the morphism $g_\beta : X \to Y$ defined by $g_\beta(p) = f(p)$ and $g_\beta(x) = s_0^n\beta t(x)^{-1} \circ g(x) \circ s_0^n\beta s(x)$, $x \in X_n$. The next proposition shows the relationship between the induced morphisms in the homotopy groupoids by two morphisms that are simplicially homotopic.

2.3. Proposition. *Let $f,\ g : X \to Y$ be two morphisms in* $\mathbf{Simp(Gpd)}_*$. *If there exists a simplicial homotopy $\beta : f \simeq g$, then $\pi_n(f) = \pi_n(g_\beta)$, $n \geq 0$.*

Proof. Let $p \in O$ and consider the family of group morphisms $\bar{\beta}^j_n$, $1 \leq j \leq n$, where each $\bar{\beta}^j_n : X_n(p) \to Y_n(f(p))$ is given by $\bar{\beta}^j_n = s^n_0\beta(p)^{-1} \circ \beta^j_n$. It is straightforward to check that this family of morphisms determines a simplicial homotopy $\bar{\beta}$ between the simplicial group morphisms $f_p, (g_\beta)_p : X(p) \to Y(p)$. Thus, for each $n \geq 0$, $\pi_n(f_p) = \pi_n((g_\beta)_p)$, and so it is clear that $\pi_n(f) = \pi_n(g_\beta)$, for any $n \geq 1$.

In addition, $\pi_0(f) = \pi_0(g_\beta)$, since for any $x : p \to q \in X_0$ there exist $y, y' \in B_0(Y)$ such that $y' \circ f(x) = g_\beta \circ y$. In fact, if we consider $\bar{y} = s_0\beta(p)^{-1} \circ s_0 g(x)^{-1} \circ \beta^1_1(s_0 x) \in Y_1(f(p))$ then $d_0(\bar{y}) = Id_{f(p)}$ and $d_1(\bar{y}) = \beta(p)^{-1} \circ g(x)^{-1} \circ \beta(q) \circ f(x)$ and so the morphisms $y = \beta(p)^{-1} \circ g(x)^{-1} \circ \beta(q) \circ f(x)$ and $y' = Id_{f(q)}$ satisfy the required condition.

■

3. Path and cylinder constructions in simplicial groupoids

Let us start by recalling that in the closed model category $\mathbf{Simp(Gp)}$ of simplicial groups [16], given $H_\bullet \in \mathbf{Simp(Gp)}$, the simplicial group $H^I_\bullet$, whose n-simplices are:

$$(H^I_\bullet)_n = Hom_{\mathbf{Simp(Set)}}(\Delta[1] \times \Delta[n], H_\bullet) \cong$$

$$\cong \{(x_0, \ldots, x_n) \in (H_{n+1})^{n+1} \ /\ d_i x_i = d_i x_{i-1},\ 1 \leq i \leq n\}$$

and the face and degeneracy operators are:

$$d_i(x_0, \ldots, x_n) = (d_{i+1}x_0, \ldots, d_{i+1}x_{i-1}, d_i x_{i+1}, \ldots, d_i x_n),\ 0 \leq i \leq n,$$

$$s_i(x_0, \ldots, x_n) = (s_{i+1}x_0, \ldots, s_{i+1}x_i, s_i x_i, \ldots, s_i x_n),\ 0 \leq i \leq n\ ,$$

is a path space for $H_\bullet$ since there is a factorization of the diagonal morphism

$$H_\bullet \xrightarrow{\beta_\bullet} H^I_\bullet \xrightarrow{(\partial_0,\partial_1)} H_\bullet \times H_\bullet \quad ,$$

where $\beta_\bullet$, given by $\beta_n(x) = (s_0 x, \ldots, s_n x)$, is a weak equivalence and (∂_0, ∂_1), the morphism induced by $(\partial_0)_n(x_0, \ldots, x_n) = d_{n+1}x_n$ and $(\partial_1)_n(x_0, \ldots, x_n) = d_0 x_0$, is a fibration (see [7], [8]).

3.1. Definition. *Given $H \in \mathbf{Simp(Gpd)}_*$, consider the simplicial groupoid H^I whose set of objects is H_0 and the set of morphisms in dimension n is the set of $(n+1)$-uples of commutative squares:*

$$(H^I)_n = \left\{ \chi = \left(\begin{array}{ccc} p & \xrightarrow{s_0^{n+1}a} & q \\ \downarrow{\scriptstyle x_0} & & {\scriptstyle x'_0}\downarrow \\ p' & \xrightarrow[s_0^{n+1}b]{} & q' \end{array} ,\quad \ldots \quad , \begin{array}{ccc} p & \xrightarrow{s_0^{n+1}a} & q \\ \downarrow{\scriptstyle x_n} & & {\scriptstyle x'_n}\downarrow \\ p' & \xrightarrow[s_0^{n+1}b]{} & q' \end{array} \right) \Bigg/ \begin{array}{ll} a, b \in H_0 & d_i x_i = d_i x_{i-1} \\ x_i, x'_i \in H_{n+1} & d_i x'_i = d_i x'_{i-1} \\ p, q \in O & \end{array} \right\}$$

where, in each dimension, the source and target of the groupoid $(H^I)_n \overset{s}{\underset{t}{\rightrightarrows}} H_0$ *are defined by* $s(\chi) = a$ *and* $t(\chi) = b$*, and the face and degeneracy operators are given, in each dimension, by:*

$$d_i(\chi) = \left(\begin{tikzcd} p \arrow[r, "s_0^n a"] \arrow[d, "d_{i+1}x_0"] & q \arrow[d, "d_{i+1}x'_0"'] \\ p' \arrow[r, "s_0^n b"'] & q' \end{tikzcd}, \ldots, \begin{tikzcd} p \arrow[r, "s_0^n a"] \arrow[d, "d_{i+1}x_{i-1}"] & q \arrow[d, "d_{i+1}x'_{i-1}"'] \\ p' \arrow[r, "s_0^n b"'] & q' \end{tikzcd}, \begin{tikzcd} p \arrow[r, "s_0^n a"] \arrow[d, "d_i x_{i+1}"] & q \arrow[d, "d_i x'_{i+1}"'] \\ p' \arrow[r, "s_0^n b"'] & q' \end{tikzcd}, \ldots, \begin{tikzcd} p \arrow[r, "s_0^n a"] \arrow[d, "d_i x_n"] & q \arrow[d, "d_i x'_n"'] \\ p' \arrow[r, "s_0^n b"'] & q' \end{tikzcd} \right)$$

$$s_j(\chi) = \left(\begin{tikzcd} p \arrow[r, "s_0^{n+2} a"] \arrow[d, "s_{j+1}x_0"] & q \arrow[d, "s_{j+1}x'_0"'] \\ p' \arrow[r, "s_0^{n+2} b"'] & q' \end{tikzcd}, \ldots, \begin{tikzcd} p \arrow[r, "s_0^{n+2} a"] \arrow[d, "s_{j+1}x_j"] & q \arrow[d, "s_{j+1}x'_j"'] \\ p' \arrow[r, "s_0^{n+2} b"'] & q' \end{tikzcd}, \begin{tikzcd} p \arrow[r, "s_0^{n+2} a"] \arrow[d, "s_j x_j"] & q \arrow[d, "s_j x'_j"'] \\ p' \arrow[r, "s_0^{n+2} b"'] & q' \end{tikzcd}, \ldots, \begin{tikzcd} p \arrow[r, "s_0^{n+2} a"] \arrow[d, "s_j x_n"] & q \arrow[d, "s_j x'_n"'] \\ p' \arrow[r, "s_0^{n+2} b"'] & q' \end{tikzcd} \right).$$

The above definition determines a functor

$$(-)^I : \mathbf{Simp}(\mathbf{Gpd})_* \longrightarrow \mathbf{Simp}(\mathbf{Gpd})_*$$

and we actually have the following:

3.2. Proposition. *The simplicial groupoid* H^I *is a path space for any* H *in* $\mathbf{Simp}(\mathbf{Gpd})_*$*.*

Proof. We must prove that there exists a factorization of the diagonal morphism

$$\begin{tikzcd} H \arrow[r, "\beta"] \arrow[rr, bend right, "\Delta"'] & H^I \arrow[r, "(\partial_0,\partial_1)"] & H \times H \end{tikzcd}$$

such that morphism β is a weak equivalence and (∂_0, ∂_1) is a fibration.

Let us define $\beta : H \to H^I$ as follows:

- On objects, $\beta : O \to H_0$ is given by $\beta(p) = Id_p \ \ \forall p \in O$.
- On morphisms, $\beta : H_n \to (H^I)_n$ is given, for each $x : p \to q \in H_n$, by

$$\beta(x) = \left(\begin{tikzcd} p \arrow[r, equal, "s_0^{n+1} Id_p"] \arrow[d, "s_0 x"'] & p \arrow[d, "s_0 x"] \\ q \arrow[r, equal, "s_0^{n+1} Id_q"'] & q \end{tikzcd}, \quad \ldots \quad, \begin{tikzcd} p \arrow[r, equal, "s_0^{n+1} Id_p"] \arrow[d, "s_n x"'] & p \arrow[d, "s_n x"] \\ q \arrow[r, equal, "s_0^{n+1} Id_q"'] & q \end{tikzcd} \right).$$

It is straightforward to check that β is a morphism of simplicial groupoids. To verify that it is a weak equivalence, we first show that it induces a bijection between the components of H and those of H^I. The injectivity is clear because, if $\beta(p)$ and $\beta(q)$ are in the same component of $(H^I)_0$, then p y q are in the same component of H_0, since if the following square

$$\begin{tikzcd} p \arrow[r, equal] \arrow[d, "x"'] & p \arrow[d, "x"] \\ q \arrow[r, equal] & q \end{tikzcd}$$

is a morphism connecting them in $(H^I)_0$, then $x : p \to q$ is a morphism connecting p and q in H_0. To show the surjectivity it is enough to see that, given $x : p \to q$ with $p \in O$, then $\beta(p)$ and x are in the same component in $(H^I)_0$, but this is clear if we consider the following commutative square:

$$\begin{tikzcd} p \arrow[r, equal, "s_0 Id_p"] \arrow[d, equal] & p \arrow[d, "s_0 x"] \\ p \arrow[r, "s_0 x"'] & q \end{tikzcd} .$$

Second, we must show that the induced morphism $\beta(p) : H(p) \to H^I(\beta(p))$ is a weak equivalence of simplicial groups for all $p \in O$ and this is clear since we have:

$$(H^I(\beta(p)))_n = \left\{ \left(\begin{tikzcd} p \arrow[r, equal, "s_0^{n+1} Id_p"] \arrow[d, "x_0"'] & p \arrow[d, "x_0"] \\ p \arrow[r, equal, "s_0^{n+1} Id_p"'] & p \end{tikzcd} , \quad \dots \quad , \begin{tikzcd} p \arrow[r, equal, "s_0^{n+1} Id_p"] \arrow[d, "x_n"'] & p \arrow[d, "x_n"] \\ p \arrow[r, equal, "s_0^{n+1} Id_p"'] & p \end{tikzcd} \right) \Big/ \begin{array}{l} d_i x_i = d_i x_{i-1} \\ x_i \in H_{n+1} \end{array} \right\} \cong$$

$$\cong \left\{ (x_0, \dots, x_n) \Big/ \begin{array}{l} d_i x_i = d_i x_{i-1} \\ x_i \in (H(p))_{n+1} \end{array} \right\} = (H(p))^I ,$$

and so morphism $\beta(p)$ can be identified with the corresponding morphism in the construction of the path space in the category of simplicial groups (see the beginning of this section), and therefore it is a weak equivalence.

Next we define $(\partial_0, \partial_1) : H^I \to H \times H$ as follows:

- On objects, $(\partial_0, \partial_1) : H_0 \to O \times O$ is given by $(\partial_0, \partial_1)(a) = (s(a), t(a))$, $\forall a \in H_0$.

- On morphisms, $(\partial_0, \partial_1)_n : (H^I)_n \to H_n \times H_n$ is given by

$$(\partial_0, \partial_1)_n \left(\begin{tikzcd} p \arrow[r, "s_0^{n+1} a"] \arrow[d, "x_0"] & q \arrow[d, "x'_0"'] \\ p' \arrow[r, "s_0^{n+1} b"'] & q' \end{tikzcd} , \quad \dots \quad , \begin{tikzcd} p \arrow[r, "s_0^{n+1} a"] \arrow[d, "x_n"] & q \arrow[d, "x'_n"'] \\ p' \arrow[r, "s_0^{n+1} b"'] & q' \end{tikzcd} \right) = (d_{n+1} x_n, d_0 x'_0) .$$

The proof that (∂_0, ∂_1) is a morphism of simplicial groupoids is long and tedious but straightforward. For instance, in dimension 1, we have to prove that the following cube is commutative:

$$
\begin{tikzcd}
 & (H^I)_1 \arrow[rr, "(\partial_0,\partial_1)_1"] \arrow[dl, shift left, "d_1"'] \arrow[dl, shift right, "d_0"] \arrow[dd, shift left] \arrow[dd, shift right] & & H_1 \times H_1 \arrow[dl, shift left, "d_1"'] \arrow[dl, shift right, "d_0"] \arrow[dd, shift left] \arrow[dd, shift right] \\
(H^I)_0 \arrow[rr, "(\partial_0,\partial_1)_0"'] \arrow[dd, shift right, "t"'] \arrow[dd, shift left, "s"] & & H_0 \times H_0 \arrow[dd, shift left] \arrow[dd, shift right] & \\
 & H_0 \arrow[rr] & & O \times O \\
H_0 \arrow[rr] \arrow[ur, equal] & & O \times O \arrow[ur, equal] &
\end{tikzcd}
$$

and, for example, for the upper face and the morphism d_0 this is true because:

$$
(d_0(\partial_0,\partial_1)_1)\left(\begin{array}{ccc} p \xrightarrow{s_0^2 a} q & & p \xrightarrow{s_0^2 a} q \\ \downarrow{\scriptstyle x_0} \qquad {\scriptstyle x'_0}\downarrow & , & \downarrow{\scriptstyle x_1} \qquad {\scriptstyle x'_1}\downarrow \\ p' \xrightarrow[s_0^2 b]{} q' & & p' \xrightarrow[s_0^2 b]{} q' \end{array}\right) = d_0(d_2x_1, d_0x'_0) = (d_0d_2x_1, d_0d_0x'_0)
$$

$$
((\partial_0,\partial_1)_0 d_0)\left(\begin{array}{ccc} p \xrightarrow{s_0^2 a} q & & p \xrightarrow{s_0^2 a} q \\ \downarrow{\scriptstyle x_0} \qquad {\scriptstyle x'_0}\downarrow & , & \downarrow{\scriptstyle x_1} \qquad {\scriptstyle x'_1}\downarrow \\ p' \xrightarrow[s_0^2 b]{} q' & & p' \xrightarrow[s_0^2 b]{} q' \end{array}\right) = (\partial_0,\partial_1)_0\left(\begin{array}{c} p \longrightarrow q \\ \downarrow{\scriptstyle d_0x_1} \qquad \downarrow{\scriptstyle d_0x'_1} \\ p' \longrightarrow q' \end{array}\right) = (d_1d_0x_1, d_0d_0x'_1)
$$

and the simplicial identities and the relation $d_i x_i = d_i x_{i-1}$ imply $(d_0d_2x_1, d_0d_0x'_0) = (d_1d_0x_1, d_0d_0x'_1)$.

Finally, let us prove that the morphism (∂_0, ∂_1) is a fibration in $\mathbf{Simp}(\mathbf{Gpd})_*$. To do so, note first that the morphism of groupoids

$$
\begin{tikzcd}
(H^I)_0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "(\partial_0,\partial_1)"'] & H_0 \arrow[d, "(\partial_0,\partial_1)"] \\
H_0 \times H_0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O \times O
\end{tikzcd}
$$

is a fibration since, given $a : p \to q \in H_0$ and $(p \xrightarrow{x} p', q \xrightarrow{x'} q') \in H_0 \times H_0$, the element of $(H^I)_0$ represented by the commutative square

$$
\begin{tikzcd}
p \arrow[r, "s_0 a"] \arrow[d, "s_0 x"'] & q \arrow[d, "s_0 x'"] \\
p' \arrow[r, "s_0(x' a x^{-1})"'] & q'
\end{tikzcd}
$$

satisfies $(\partial_0, \partial_1)\left(\begin{smallmatrix} & \rightarrow & \\ \downarrow & & \downarrow \\ & \rightarrow & \end{smallmatrix}\right) = (x, x')$.

It remains to prove that, given $a : p \to q \in H_0$, the morphism $(\partial_0, \partial_1) : H^I(a) \to H(p) \times H(q)$ is a fibration of simplicial groups. Now, since the objects p and q are in the same component of H_0, $H(p) \times H(q) \cong H(p) \times H(p)$. Moreover, it is easy to identify $H^I(a)$ with $H(p)^I$ because an element of $(H^I(a))_n$ such as the following:

$$\left(\begin{array}{c} \begin{array}{ccc} p & \xrightarrow{s_0^{n+1}a} & q \\ {\scriptstyle x_0}\downarrow & & \downarrow{\scriptstyle x'_0} \\ p & \xrightarrow[s_0^{n+1}a]{} & q \end{array}, \begin{array}{ccc} p & \xrightarrow{s_0^{n+1}a} & q \\ {\scriptstyle x_1}\downarrow & & \downarrow{\scriptstyle x'_1} \\ p & \xrightarrow[s_0^{n+1}a]{} & q \end{array}, \ldots, \begin{array}{ccc} p & \xrightarrow{s_0^{n+1}a} & q \\ {\scriptstyle x_n}\downarrow & & \downarrow{\scriptstyle x'_n} \\ p & \xrightarrow[s_0^{n+1}a]{} & q \end{array} \end{array}\right)$$

determines $(x_0, x_1, \ldots, x_n) \in (H(p))^I_n$ and, conversely, given $(x_0, x_1, \ldots, x_n) \in (H(p))^I_n$, the morphisms $x'_0, \ldots, x'_n$ are determined by composition. Thus, the morphism $(\partial_0, \partial_1) : H^I(a) \to H(p) \times H(q)$ can be identified with the morphism of simplicial groups $H(p)^I \to H(p) \times H(p)$, which is a fibration since it is the corresponding morphism in the construction of the path space in the category **Simp**(**Gp**).

Finally, it is clear that $(\partial_0, \partial_1)\beta = \Delta$. ■

Let us recall now that if $G_\bullet$ is a cofibrant (i.e., free) simplicial group, the simplicial group $G_\bullet \otimes I$, whose n-simplices are $(G_\bullet \otimes I)_n = \coprod_{i=0}^{n+1} (G_n)_i$, where $(G_n)_i = G_n$, $0 \leq i \leq n$, is a cylinder object for $G_\bullet$ in the closed model category **Simp**(**Gp**) since there is a factorization of the codiagonal morphism

$$G_\bullet \coprod G_\bullet \xrightarrow{i_0+i_1} G_\bullet \otimes I \xrightarrow{\sigma} G_\bullet$$

where σ, which is the morphism induced by the identities, is a weak equivalence, and the morphism $i_0 + i_1$, which is induced by the first and last inclusions respectively, is a cofibration (see [7], [8]).

3.3. Definition. *Given $G \in$ **Simp**(**Gpd**)$_*$, consider the simplicial groupoid $G \otimes I$ whose set of objects is $O \vee O = O \times \{0, 1\}$ and the sets of morphisms are constructed as follows:*

Given $p \in O$, we denote $p_0 = (p, 0) \in O \times \{0\}$ and $p_1 = (p, 1) \in O \times \{1\}$. For any $x \in G_0$, $x : p \to q$, we consider two morphisms, $x^0 : p_0 \to q_0$ and $x^1 : p_0 \to q_0$, and the sets $G_0^0 = \{x^0 \ / \ x \in G_0\}$ and $G_0^1 = \{x^1 \ / \ x \in G_0\}$. Also, for any $p \in O$ we consider a morphism $I_p : p_0 \to p_1$ and its inverse $I_p^{-1} : p_1 \to p_0$.

The set $(G \otimes I)_0$ of morphisms in dimension zero, denoted by $G_0^0 \vee G_0^1 \vee O$, consists of these three classes of morphisms and all the words $\alpha_n \circ \alpha_{n-1} \circ \ldots \alpha_0$, where each α_i is any of the morphisms described above, such that $s(\alpha_i) = t(\alpha_{i-1})$. All these morphisms are subject to the relations $x^0 \circ y^0 = (x \circ y)^0$, $x^1 \circ y^1 = (x \circ y)^1$, $(Id_p)^0 = (Id_p)^1$.

Now, for any morphism $x : p \to q \in G_1$, we consider three morphisms $x^{00} : p_0 \to q_0$, $x^{01} : p_0 \to q_0$ and $x^{11} : p_0 \to q_0$ and the sets $G_1^{00} = \{x^{00} \ / \ x \in G_1\}$, $G_1^{01} = \{x^{01} \ / \ x \in G_1\}$

and $G_1^{11} = \{x^{11} \ / \ x \in G_1\}$*. Also, for any* $p \in O$*, we consider the morphisms* $I_p : p_0 \to p_1$ *and their inverses.*

Then, the set $(G \otimes I)_1$ *of morphisms in dimension one, denoted by* $G_1^{00} \vee G_1^{01} \vee G_1^{11} \vee O$*, consists of these four classes of morphisms and all the words* $\alpha_n \circ \alpha_{n-1} \circ \ldots \alpha_0$*, where each* α_i *is any of the morphisms described above, such that* $s(\alpha_i) = t(\alpha_{i-1})$*. All these morphisms are subject to the relations* $x^{00} \circ y^{00} = (x \circ y)^{00}$, $x^{01} \circ y^{01} = (x \circ y)^{01}$, $x^{11} \circ y^{11} = (x \circ y)^{11}$, $(Id_p)^{00} = (Id_p)^{01} = (Id_p)^{11} = Id_{p_0}$.

By iterating this process, we define $(G \otimes I)_n = \bigvee_{\tau \in (\Delta[1])_n} G_n^\tau \vee O$*. Thus, given* $x \in G_n$ *and* $\tau \in (\Delta[1])_n$*, we have* $x^\tau \in G_n^\tau \subset (G \otimes I)_n$*.*

Note that we have a groupoid

$$\begin{tikzcd} (G \otimes I)_0 \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & O \vee O \arrow[l, bend right, "Id"'] \end{tikzcd} ,$$

where $Id_{p_0} = (Id_p)^0 = (Id_p)^1$*;* $Id_{p_1} = I_p I_p^{-1}$*;* $s(x^0) = s(x^1) = s(I_p) = p_0$*;d* $s(I_p^{-1}) = p_1$*;* $t(x^0) = t(x^1) = t(I_q^{-1}) = q_0$ *and* $t(I_q) = q_1$*. In general, we have, in each dimension* n*, a groupoid*

$$\begin{tikzcd} (G \otimes I)_n \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & O \vee O \arrow[l, bend right, "Id"'] \end{tikzcd} .$$

Thus, we have constructed a simplicial groupoid where the face and degeneracy operators are defined by:

$d_i(x^\tau) = (d_i x)^{d_i \tau}$, $x^\tau \in G_n$, $0 \le i \le n$,
$d_i(I_p) = I_p$ *(in a smaller dimension),*
$s_i(x^\tau) = (s_i x)^{s_i \tau}$, $x^\tau \in G_n$, $0 \le i \le n-1$,
$s_i(I_p) = I_p$ *(in a higher dimension).*

The above definition determines a functor

$$(-) \otimes I : \mathbf{Simp(Gpd)}_* \longrightarrow \mathbf{Simp(Gpd)}_*$$

and we actually have the following:

3.4. Proposition. *The simplicial groupoid* $G \otimes I$ *is a cylinder object for any cofibrant object* G *in* $\mathbf{Simp(Gpd)}_*$*.*

Proof. We must prove that there exists a factorization of the codiagonal morphism

$$\begin{tikzcd} G \coprod G \arrow[r, "i_0+i_1"] \arrow[rr, bend right, "\nabla"'] & G \otimes I \arrow[r, "\sigma"] & G \end{tikzcd}$$

such that $i_0 + i_1$ is a cofibration and σ is a weak equivalence.

We define $\sigma : G \otimes I \to G$ by $\sigma(p_0) = \sigma(p_1) = p$, $p \in O$; $\sigma(x^\tau) = x$, $x^\tau \in G_n^\tau$; $\sigma(I_p) = Id_p$, $p \in O$.

The morphisms i_0, $i_1 : G \to G \otimes I$ are defined by $i_0(p) = p_0$, $p \in O$; $i_0(x) = x^{00\ldots0}$, $x \in G_n$; $i_1(p) = p_1$, $p \in O$; $i_1(x) = I_q x^{1\ldots11} I_p^{-1}$, $x \in G_n$.

It is clear that $\sigma i_0 = \sigma i_1 = 1_G$.

Moreover, if G is cofibrant, $i_0 + i_1$ is a free map, and thus a cofibration, since it is clearly one-to-one on objects and maps and it is straightforward to see that, if V is a base of G, then $W = \{W_n \ / \ n \in \mathbb{N}\}$ is a base of $G \otimes I$ where, with the same notation used in the construction of $G \otimes I$, $W_n = V_n^{00\ldots01} \cup V_n^{00\ldots011} \cup \ldots \cup V_n^{011\ldots1} \cup O$.

To prove that σ is a weak equivalence it is enough to see that it is a homotopy equivalence (see [17], theorem 1.3). To do so, below we construct a right homotopy from $i_0\sigma$ to $1_{G\otimes I}$. Note first that $(i_0\sigma)(p_0) = p_0$, $(i_0\sigma)(p_1) = p_1$, $(i_0\sigma)(I_p) = (Id_p)^0$, $p \in O$; $(i_0\sigma)(x^\tau) = x^{00\ldots0}$, $x^\tau \in (G \otimes I)_n$.

The right homotopy is a morphism $H \ : \ G \otimes I \ \to \ (G \otimes I)^I$ such that $\partial_0 H = i_0\sigma$, $\partial_1 H = 1_{G\otimes I}$ and it is given as follows:

- On objects:

$$H(p_0) = (\ p_0 \overset{Id}{=\!=} p_0\ ) \quad H(p_1) = (\ p_0 \xrightarrow{I_p} p_1\ )$$

- In dimension 0:

$$H(x^0) = \begin{tikzcd} p_0 \arrow[r, equal, "Id"] \arrow[d, "(s_0x)^{00}"'] & p_0 \arrow[d, "(s_0x)^{00}"] \\ q_0 \arrow[r, equal, "Id"'] & q_0 \end{tikzcd} ; \ H(x^1) = \begin{tikzcd} p_0 \arrow[r, equal, "Id"] \arrow[d, "(s_0x)^{01}"'] & p_0 \arrow[d, "(s_0x)^{01}"] \\ q_0 \arrow[r, equal, "Id"'] & q_0 \end{tikzcd} ; \ H(I_p) = \begin{tikzcd} p_0 \arrow[r, equal, "Id"] \arrow[d, equal, "Id"'] & p_0 \arrow[d, "I_p"] \\ p_0 \arrow[r, "I_p"'] & p_1 \end{tikzcd}$$

- In dimension 1:

$$H(x^{00}) = \left(\begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_0x)^{000}"'] & p_0 \arrow[d, "(s_0x)^{000}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} , \begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_1x)^{000}"'] & p_0 \arrow[d, "(s_1x)^{000}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} \right)$$

$$H(x^{01}) = \left(\begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_0x)^{001}"'] & p_0 \arrow[d, "(s_0x)^{001}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} , \begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_1x)^{001}"'] & p_0 \arrow[d, "(s_1x)^{001}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} \right)$$

$$H(x^{11}) = \left(\begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_0x)^{011}"'] & p_0 \arrow[d, "(s_0x)^{011}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} , \begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_1x)^{001}"'] & p_0 \arrow[d, "(s_1x)^{001}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} \right)$$

$$H(I_p) = \left(\begin{tikzcd} p_0 \arrow[r, "I_p"] \arrow[d, equal, "Id"'] & p_1 \arrow[d, equal, "Id"] \\ p_0 \arrow[r, "I_p"'] & p_1 \end{tikzcd} , \begin{tikzcd} p_0 \arrow[r, "I_p"] \arrow[d, equal, "Id"'] & p_1 \arrow[d, equal, "Id"] \\ p_0 \arrow[r, "I_p"'] & p_1 \end{tikzcd} \right)$$

- In general, for any element $x^{00\overset{(n-j+1}{\ldots}011\overset{(j}{\ldots}1} \in (G \otimes I)_n, \quad 0 \le j \le n+1$:

$$H(x^{00\overset{(n-j+1}{\cdots}011\overset{(j}{\cdots}1}) = (d_0, d_1, \ldots, d_k, \ldots, d_n)$$

where each d_k is a diagram of the form:

$$\begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_kx)^{00\overset{(n-j+2}{\cdots}011\overset{(j}{\cdots}1}"'] & p_0 \arrow[d, "(s_kx)^{00\overset{(n-j+2}{\cdots}011\overset{(j}{\cdots}1}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} \quad \text{if } \ 0 \le k \le n-j+1,$$

$$\begin{tikzcd} p_0 \arrow[r, equal] \arrow[d, "(s_kx)^{00\overset{(k+1}{\cdots}011\overset{(n-k+1}{\cdots}1}"'] & p_0 \arrow[d, "(s_kx)^{00\overset{(k+1}{\cdots}011\overset{(n-k+1}{\cdots}1}"] \\ q_0 \arrow[r, equal] & q_0 \end{tikzcd} \quad \text{if } \ n-j+2 \le k \le n+1.$$

Finally, it is straightforward to see that $\partial_0 H = i_0\sigma$ y $\partial_1 H = 1_{G\otimes I}$.

■

4. A simplicial description of $Ho(\mathbf{Simp}(\mathbf{Gpd})_*)$

The construction of path space in $\mathbf{Simp}(\mathbf{Gpd})_*$ given in Section 3, allows us to formulate the right homotopy relation between two morphisms in terms of the simplicial homotopy relations given in Proposition 2.2. In fact, if $f, g : G \to H$ are two morphisms in $\mathbf{Simp}(\mathbf{Gpd})_*$, a right homotopy from f to g is a morphism $\alpha : G \to H^I$ such that $\partial_0\alpha = f$ and $\partial_1\alpha = g$ and this means to give:

- A map $\alpha : O \to H_0$ such that $(\partial_0, \partial_1)\alpha = (f, g)$ and, since $(\partial_0, \partial_1)\alpha = (s\alpha, t\alpha)$, the map α must satisfy that $s\alpha = f$ and $t\alpha = g$.

- A map $\alpha_0 : G_0 \to H_0^I$ such that $(\partial_0, \partial_1)\alpha_0 = (f_0, g_0)$ and the following diagram is commutative:

$$\begin{tikzcd} G_0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\alpha_0"'] & O \arrow[d, "\alpha"] \\ H_0^I \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & H_0 \end{tikzcd} .$$

Now, if $x : p \to q \in G_0$, $\alpha_0(x) : \alpha(p) \to \alpha(q) \in H_0^I$ is a commutative square of the form

$$\begin{tikzcd} f(p) \arrow[r, "s_0\alpha(p)"] \arrow[d, "x_0"'] & g(p) \arrow[d, "x_0'"] \\ f(q) \arrow[r, "s_0\alpha(q)"] & g(q) \end{tikzcd} \quad \text{with } \ d_1x_0 = f_0(x) \text{ and } \ d_0x_0' = g_0(x),$$

and so, giving α_0 in these conditions is equivalent to giving a map

$$\begin{aligned} \alpha_0^0 : G_0 &\longrightarrow H_1 \\ x &\longmapsto s_0\alpha t(x) \circ x_0 \end{aligned}$$

such that: $\begin{cases} s\alpha_0^0(x) = fs(x),\ t\alpha_0^0(x) = gt(x), \\ d_1\alpha_0^0(x) = \alpha t(x)\circ f_0(x),\ d_0\alpha_0^0(x) = g_0(x)\circ \alpha s(x) \end{cases}$
(note that $d_0\alpha_0^0(x) = \alpha t(x)\circ d_0x_0 = \alpha t(x)\circ(\alpha t(x))^{-1}\circ d_0x_0'\circ\alpha s(x) = g_0(x)\circ\alpha s(x)$).

The condition that the pair (α_0,α) is a morphism of groupoids is clearly equivalent to α_0^0 satisfying the relation $\alpha_0^0(y\circ x) = \alpha_0^0(y)\circ(s_0\alpha t(x))^{-1}\circ\alpha_0^0(x)$, for any morphisms in G_0, $p \xrightarrow{x} q \xrightarrow{y} r$.

- A map $\alpha_1 : G_1 \to (H^I)_1$ such that $(\partial_0,\partial_1)\alpha_1 = (f_1, g_1)$ and the following diagrams are commutative:

$$\begin{array}{ccc} G_1 & \overset{s}{\underset{t}{\rightrightarrows}} & O \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha} \\ (H^I)_1 & \overset{s}{\underset{t}{\rightrightarrows}} & H_0 \end{array} \quad , \quad \begin{array}{ccc} G_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & G_0 \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_0} \\ (H^I)_1 & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & (H^I)_0 \end{array} .$$

Now, if $x : p \to q \in G_1$, $\alpha_1(x) : \alpha(p) \to \alpha(q) \in (H^I)_1$ and so it is of the form

$$\left(\begin{array}{ccc} f(p) & \xrightarrow{s_0^2\alpha(p)} & g(p) \\ \downarrow{\scriptstyle x_0} & & {\scriptstyle x_0'}\downarrow \\ f(q) & \xrightarrow[s_0^2\alpha(q)]{} & g(q) \end{array} , \begin{array}{ccc} f(p) & \xrightarrow{s_0^2\alpha(p)} & g(p) \\ \downarrow{\scriptstyle x_1} & & {\scriptstyle x_1'}\downarrow \\ f(q) & \xrightarrow[s_0^2\alpha(q)]{} & g(q) \end{array} \Big/ \begin{array}{l} d_1x_0 = d_1x_1 \\ d_1x_0' = d_1x_1' \end{array}\right)$$

with $d_2x_1 = f_1(x)$, $d_0x_0' = g_1(x)$, $\alpha_0(d_0x) = \left(\begin{array}{ccc} f(p) & \xrightarrow{s_0\alpha(p)} & g(p) \\ {\scriptstyle d_0x_1}\downarrow & & \downarrow{\scriptstyle d_0x_1'} \\ f(q) & \xrightarrow{s_0\alpha(q)} & g(q) \end{array}\right)$ and

$$\alpha_0(d_1x) = \left(\begin{array}{ccc} f(p) & \xrightarrow{s_0\alpha(p)} & g(p) \\ {\scriptstyle d_2x_0}\downarrow & & \downarrow{\scriptstyle d_2x_0'} \\ f(q) & \xrightarrow{s_0\alpha(q)} & g(q) \end{array}\right).$$

Thus, giving α_1 is equivalent to giving two maps $\begin{array}{l} \alpha_1^0,\ \alpha_1^1 : G_1 \longrightarrow H_2 \\ \qquad x \longmapsto s_0^2\alpha t(x)\circ x_0 \\ \qquad x \longmapsto s_0^2\alpha t(x)\circ x_1 \end{array}$

such that: $$\begin{cases} s\alpha_1^0(x) = fs(x),\ t\alpha_1^0(x) = gt(x),\ s\alpha_1^1(x) = fs(x),\ t\alpha_1^1(x) = gt(x)\ , \\ d_0\alpha_1^0(x) = s_0\alpha t(x)\circ d_0x_0 = d_0x_0'\circ s_0\alpha s(x) = g_1(x)\circ s_0\alpha s(x)\ , \\ d_1\alpha_1^0(x) = s_0\alpha t(x)\circ d_1x_0 = s_0\alpha t(x)\circ d_1x_1 = d_1\alpha_1^1(x)\ , \\ d_2\alpha_1^0(x) = s_0\alpha t(x)\circ d_2x_0 = \alpha_0^0 d_1(x)\ , \\ d_0\alpha_1^1(x) = s_0\alpha t(x)\circ d_0x_1 = \alpha_0^0 d_0(x)\ , \\ d_2\alpha_1^1(x) = s_0\alpha t(x)\circ d_2x_1 = s_0\alpha t(x)\circ f_1(x)\ . \end{cases}$$

Moreover, if $x : p \to q \in G_0$, then

$$s_0\alpha_0(x) = s_0 \left(\begin{array}{ccc} f(p) & \xrightarrow{s_0\alpha(p)} & g(p) \\ \downarrow{\scriptstyle x_0} & & \downarrow{\scriptstyle x'_0} \\ f(q) & \xrightarrow[s_0\alpha(q)]{} & g(q) \end{array} \right) = \left(\begin{array}{ccc} f(p) & \xrightarrow{s_0^2\alpha(p)} & g(p) \\ \downarrow{\scriptstyle s_1x_0} & & \downarrow{\scriptstyle s_1x'_0} \\ f(q) & \xrightarrow[s_0^2\alpha(q)]{} & g(q) \end{array} , \begin{array}{ccc} f(p) & \xrightarrow{s_0^2\alpha(p)} & g(p) \\ \downarrow{\scriptstyle s_0x_0} & & \downarrow{\scriptstyle s_0x'_0} \\ f(q) & \xrightarrow[s_0^2\alpha(q)]{} & g(q) \end{array} \right) = \alpha_1 s_0(x)$$

and so $s_0\alpha_0^0(x) = s_0^2\alpha t(x) \circ s_0x_0 = \alpha_1^1 s_0(x)$ and $s_1\alpha_0^0(x) = s_1s_0\alpha t(x) \circ s_1x_0 = s_0^2\alpha t(x) \circ s_1x_0 = \alpha_1^0 s_0(x)$.

Given $p \xrightarrow{x} q \xrightarrow{y} r \in G_1$, the condition that (α_1, α) is a morphism of groupoids is equivalent to α_1^0 and α_1^1 satisfying the relations $\alpha_1^0(y \circ x) = \alpha_1^0(y) \circ (s_0^2\alpha t(x))^{-1} \circ \alpha_1^0(x)$ and $\alpha_1^1(y \circ x) = \alpha_1^1(y) \circ (s_0^2\alpha t(x))^{-1} \circ \alpha_1^1(x)$.

By iterating these calculations and recalling the characterization given in Proposition 2.2, we can state the following:

4.1. Proposition. *Let G, $H \in \mathbf{Simp(Gpd)}_*$ and $f, g : G \to H$ two morphisms. Then, giving a morphism $\alpha : G \to H^I$ such that $(\partial_0, \partial_1)\alpha = (f, g)$ is equivalent to giving a simplicial homotopy $\alpha : f \simeq g$.*

In the same way as above, the construction of cylinder object in $\mathbf{Simp(Gpd)}_*$ given in Section 3, allows us to formulate the left homotopy relation between two morphisms in terms of the simplicial homotopy relations given in Definition 2.1. In fact, if $f, g : G \to H$ are two morphisms in $\mathbf{Simp(Gpd)}_*$, giving a morphism $\beta : G \otimes I \to H$ such that $\beta i_0 = f$ and $\beta i_1 = g$ is equivalent to giving the following data:

- A map $\bar\beta : O \vee O \to O'$ such that, for any $p \in O$, $\bar\beta(p_0) = f(p)$ and $\bar\beta(p_1) = g(p)$; thus, $\bar\beta$ is completely determined by the maps $f, g : O \to O'$.

- A map $\beta_0 : (G \otimes I)_0 \to H_0$ such that $\beta_0(i_0 + i_1) = f_0 + g_0$ and the following diagram

$$\begin{tikzcd} (G \otimes I)_0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_0"'] & O \vee O \arrow[d, "\bar\beta"] \\ H_0 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O' \end{tikzcd}$$

is commutative.

Now, given $x : p \to q \in G_0$, $i_0(x) = x^0$ and $\beta_0(x^0) = f_0(x) : f(p) \to f(q)$, then β_0 is determined on G_0^0 by f_0. If $p \in O$, $\beta_0(I_p)$ is a morphism in H_0 such that $s\beta_0(I_p) = f(p)$ and $t\beta_0(I_p) = g(p)$. Thus, giving β_0 on the morphisms I_p is equivalent to giving a map $\beta : O \to H_0$ such that $s\beta = f$ y $t\beta = g$. This map determines β_0 if we define $\beta_0(x^0) = f_0(x)$; $\beta_0(I_p) = \beta(p)$; $\beta_0(x^1) = \beta s(x)^{-1} \circ g_0(x) \circ \beta s(x)$.

- A map $\beta_1 : (G \otimes I)_1 \to H_1$ such that $\beta_1(i_0 + i_1) = f_1 + g_1$ and the following diagrams are commutative:

$$\begin{tikzcd}
(G\otimes I)_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_1"'] & O\vee O \arrow[d, "\bar\beta"] \\
H_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O'
\end{tikzcd}, \qquad
\begin{tikzcd}
(G\otimes I)_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "\beta_1"'] & (G\otimes I)_0 \arrow[l, bend right=40, "s_0"'] \arrow[d, "\beta_0"] \\
H_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & H_0 \arrow[l, bend right=40, "s_0"']
\end{tikzcd}$$

Now, if $p\in O$, it is clear that $\beta_1(I_p)=s_0\beta(p)$ and, given $x:p\to q\in G_1$, $\beta_1 i_0(x)=\beta_1(x^{00})=f_1(x):f(p)\to f(q)$ and $\beta_1 i_1(x)=s_0\beta t(x)\circ\beta_1(x^{11})s_0\circ\beta s(x)^{-1}=g_1(x):g(p)\to g(p)$. Thus, β_1 is determined by giving $\beta_1(x^{01}):f(p)\to f(q)$, satisfying that $d_0\beta_1(x^{01})=\beta_0(d_0(x^{01}))=\beta_0((d_0x)^1)=\beta t(x)^{-1}\circ g_0d_0(x)\circ\beta s(x)$ and $d_1\beta_1(x^{01})=\beta_0(d_1(x^{01}))=\beta_0((d_1x)^0)=f_0d_1(x)$, which is is equivalent to giving a map:

$$\begin{aligned}\beta_1^1: G_1 &\longrightarrow H_1\\ x &\longmapsto s_0\beta t(x)\circ\beta_1(x^{01}):f(p)\to g(q)\end{aligned}$$

satisfying that $\begin{cases} d_0\beta_1^1(x)=\beta t(x)\circ\beta t(x)^{-1}(x)\circ g_0d_0(x)\circ\beta s(x)=g_0(x)\circ d_0\beta s(x)\\ d_1\beta_1^1(x)=\beta t(x)\circ f_0d_1(x)\ .\end{cases}$

Given $p\xrightarrow{x}q\xrightarrow{y}r\ \in G_1$, the fact that $(\beta_1,\bar\beta)$ is a morphism of groupoids is equivalent to β_1^1 satisfying the following relation:

$$\beta_1^1(y\circ x)=s_0\beta t(y)\circ\beta_1((y\circ x)^{01})=s_0\beta t(y)\circ\beta_1(y^{01}x^{01})=s_0\beta t(y)\circ\beta_1(y^{01})\circ\beta_1(x^{01})=$$

$$=s_0\beta t(y)\circ\beta_1(y^{01})\circ s_0\beta t(x)^{-1}\circ s_0\beta t(x)\circ\beta_1(x^{01})=\beta_1^1(y)\circ(s_0\beta t(x))^{-1}\circ\beta_1^1(x).$$

- A map $\beta_2:(G\otimes I)_2\to H_2$ such that $\beta_2(i_0+i_1)=f_2+g_2$ and the following diagrams are commutative:

$$\begin{tikzcd}
(G\otimes I)_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "\beta_2"'] & O\vee O \arrow[d, "\bar\beta"] \\
H_2 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & O'
\end{tikzcd}, \qquad
\begin{tikzcd}
(G\otimes I)_2 \arrow[r, shift left=2, "d_2"] \arrow[r] \arrow[r, shift right=2, "d_0"'] \arrow[d, "\beta_1"'] & (G\otimes I)_1 \arrow[l, bend right=50, "s_0"'] \arrow[l, bend right=30, "s_1"] \arrow[d, "\beta_0"] \\
H_2 \arrow[r, shift left=2, "d_2"] \arrow[r] \arrow[r, shift right=2, "d_0"'] & H_1 \arrow[l, bend right=50, "s_0"'] \arrow[l, bend right=30, "s_1"]
\end{tikzcd}.$$

Now, given $p\in O$, $\beta_2(I_p)=s_0^2\beta(p)$ and, if $x:p\to q\in G_2$, $\beta_2 i_0(x)=\beta_2(x^{000})=f_2(x)$ and $\beta_2 i_1(x)=\beta_2(I_qx^{111}I_p^{-1})=s_0^2\beta t(x)\circ\beta_2(x^{111})\circ s_0^2\beta s(x)^{-1}=g_2(x)$. Thus, β_2 is determined by $\beta_2(x^{001})$ and $\beta_2(x^{011})$ satisfying that:

$$d_0\beta_2(x^{001})=\beta_1d_0(x^{001})=\beta_1((d_0x)^{01})=s_0\beta t(x)^{-1}\circ\beta_1^1d_0(x),$$

$$d_1\beta_2(x^{001})=\beta_1d_1(x^{001})=\beta_1((d_1x)^{01})=s_0\beta t(x)^{-1}\circ\beta_1^1d_1(x),$$

$$d_2\beta_2(x^{001})=\beta_1d_2(x^{001})=\beta_1((d_2x)^{00})=f_1d_2(x)\circ d_0\beta_2(x^{011})$$

$$
\begin{aligned}
&= \beta_1 d_0(x^{011}) = \beta_1((d_0x)^{11}) = s_0\beta t(x)^{-1} \circ g_1 d_0(x) \circ s_0\beta s(x),\\
d_1\beta_2(x^{011}) &= \beta_1 d_1(x^{011}) = \beta_1((d_1x)^{01}) = s_0\beta t(x)^{-1} \circ \beta_1^1 d_1(x),\\
d_2\beta_2(x^{011}) &= \beta_1 d_2(x^{011}) = \beta_1((d_2x)^{01}) = s_0\beta t(x)^{-1} \circ \beta_1^1 d_2(x),
\end{aligned}
$$

which is equivalent to giving two maps:

$$
\begin{array}{rcl}
\beta_2^1 : G_2 & \longrightarrow & H_2\\
x & \longmapsto & s_0^2\beta t(x) \circ \beta_2(x^{001})
\end{array}
\qquad \text{and} \qquad
\begin{array}{rcl}
\beta_2^2 : G_2 & \longrightarrow & H_2\\
x & \longmapsto & s_0^2\beta t(x) \circ \beta_2(x^{011})
\end{array}
$$

satisfying the following relations:

$$
\begin{aligned}
d_0\beta_2^1(x) &= d_0 s_0^2\beta t(x) \circ d_0\beta_2(x^{011}) = s_0\beta t(x) \circ s_0\beta t(x)^{-1} \circ g_1 d_0(x) \circ s_0\beta s(x) =\\
&= g_1 d_0(x) \circ s_0\beta s(x),\\
d_1\beta_2^1(x) &= d_1 s_0^2\beta t(x) \circ d_1\beta_2(x^{011}) = s_0\beta t(x) \circ s_0\beta t(x)^{-1} \circ \beta_1^1 d_0(x) = \beta_1^1 d_0(x),\\
d_2\beta_2^1(x) &= d_2 s_0^2\beta t(x) \circ d_2\beta_2(x^{011}) = s_0 d_1 s_0\beta t(x) \circ f_1 d_2(x) = s_0\beta t(x) \circ f_1 d_2(x),\\
d_0\beta_2^2(x) &= d_0 s_0^2\beta t(x) \circ d_0\beta_2(x^{001}) = s_0\beta t(x) s_0\beta t(x)^{-1} \circ \beta_1^1 d_0(x) = \beta_1^1 d_0(x),\\
d_1\beta_2^2(x) &= d_1 s_0^2\beta t(x) \circ d_1\beta_2(x^{001}) = s_0\beta t(x) \circ s_0\beta t(x)^{-1} \circ \beta_1^1 d_1(x) = \beta_1^1 d_1(x),\\
d_2\beta_2^2(x) &= d_2 s_0^2\beta t(x) \circ d_2\beta_2(x^{001}) = s_0 d_1 s_0\beta t(x) \circ f_1 d_2(x) = s_0\beta t(x) f_1 d_2(x).
\end{aligned}
$$

Moreover, if $x : p \to q \in G_1$, then $d_0\beta_2^2 s_0(x) = \beta_1^1(x)$ and so $\beta_2^2 s_0(x) = s_0\beta_1^1(x)$, and $d_1\beta_2^1 s_0(x) = \beta_1^1(x)$ and so $\beta_2^1 s_0(x) = s_1\beta_1^1(x)$.

For any $p \xrightarrow{x} q \xrightarrow{y} r \in G_2$, the fact that $(\beta_2, \bar\beta)$ is a morphism of groupoids is equivalent to β_2^1 and β_2^2 satisfying following relations:

$$
\begin{aligned}
\beta_2^1(y \circ x) &= s_0^2\beta t(y) \circ \beta_2((y \circ x)^{011}) = s_0^2\beta t(y) \circ \beta_2(y^{011}x^{011}) = s_0^2\beta t(y) \circ \beta_2(y^{011}) \circ \beta_2(x^{011})\\
&= s_0^2\beta t(y) \circ \beta_2(y^{011})(s_0^2\beta t(x))^{-1} \circ s_0^2\beta t(x) \circ \beta_2(x^{011}) = \beta_2^1(y) \circ (s_0^2\beta t(x))^{-1} \circ \beta_2^1(x),
\end{aligned}
$$

$$
\begin{aligned}
\beta_2^2(y \circ x) &= s_0^2\beta t(y) \circ \beta_2((y \circ x)^{001}) = s_0^2\beta t(y) \circ \beta_2(y^{001}x^{001}) = s_0^2\beta t(y) \circ \beta_2(y^{001}) \circ \beta_2(x^{001}) =\\
&= s_0^2\beta t(y) \circ \beta_2(y^{001}) \circ (s_0^2\beta t(x))^{-1} \circ s_0^2\beta t(x) \circ \beta_2(x^{001}) = \beta_2^2(y) \circ (s_0^2\beta t(x))^{-1} \circ \beta_2^2(x).
\end{aligned}
$$

By iterating these calculations and recalling the definition of homotopy given in 2.1, we can state that:

4.2. Proposition. *Let $f, g : G \to H$ be two morphisms in $\mathbf{Simp(Gpd)}_*$. Then, giving a morphism $\beta : G \otimes I \to H$ such that $\beta(i_0 + i_1) = f + g$ is equivalent to giving a simplicial homotopy $\beta : f \simeq g$.*

Now, as a consequence of Propositions 4.1, 4.2 and 2.2, we have the following:

4.3. Corollary. *The functor $(-) \otimes I : \mathbf{Simp(Gpd)}_* \to \mathbf{Simp(Gpd)}_*$ is left adjoint to the functor $(-)^I : \mathbf{Simp(Gpd)}_* \to \mathbf{Simp(Gpd)}_*$.*

Lastly we have:

4.4. Theorem. *The homotopy category $Ho(\mathbf{Simp}(\mathbf{Gpd})_*)$ is equivalent to the category whose objects are the cofibrant (i.e. free) simplicial groupoids and whose morphisms are simplicial homotopy classes of simplicial groupoid morphisms.*

Proof. If G is a cofibrant simplicial groupoid and H is any simplicial groupoid, Propositions 3.4 and 4.2 assure that the left homotopy relation in $Hom_{\mathbf{Simp}(\mathbf{Gpd})_*}(G, H)$ coincides with the simplicial homotopy relation but this one coincides with the right homotopy relation according to Propositions 3.2 and 4.1. Since $\mathbf{Simp}(\mathbf{Gpd})_*$ is a closed model category we have then that $Hom_{Ho(\mathbf{Simp}(\mathbf{Gpd})_*)}(G, H) = \{$left $\equiv$ right homotopy classes of morphisms from G to $H\} = [G, H]$ the set of simplicial homotopy classes of simplicial morphisms from G to H. ∎

5. Homotopy constructions in the pointed case

Let us denote by $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$ the category of pointed simplicial groups. This is the category whose objects are pairs (X, p), with $X \in \mathbf{Simp}(\mathbf{Gpd})_*$ and $p \in O$ a fixed object of X, and given (X, p), $(Y, p') \in (\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$, the morphisms $f : (X, p) \to (Y, p')$ are morphisms $f : X \to Y \in \mathbf{Simp}(\mathbf{Gpd})_*$ such that $f(p) = p'$.

If $*$ denotes the zero object in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$, it is clear that this category is isomorphic to the category $(*, \mathbf{Simp}(\mathbf{Gpd})_*)$ of objects under $*$. Thus, $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$ inherits a closed model structure from that of $\mathbf{Simp}(\mathbf{Gpd})_*$ (see [16]) where $f : (X, p) \to (Y, p')$ is a fibration (respectively cofibration or weak equivalence) if $f : X \to Y$ is a fibration (resp. cofibration or weak equivalence) in $\mathbf{Simp}(\mathbf{Gpd})_*$.

Below we see that path space and cylinder constructions in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$ can be done by using the constructions of these objects in $\mathbf{Simp}(\mathbf{Gpd})_*$ which we have shown in Section 3. This allows us to define loop and suspension functors in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$.

5.1. Proposition. *The pointed simplicial groupoid (H^I, Id_p) is a path space for any (H, p) in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$.*

Proof. We have the following factorization of the diagonal morphism

$$(H, p) \xrightarrow{\beta} (H^I, Id_p) \xrightarrow{(\partial_0, \partial_1)} (H \times H, (p, p)) \ ,$$

where the morphisms β and (∂_0, ∂_1), defined as in Proposition 3.2, are clearly a weak equivalence and a fibration in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$ respectively. ∎

Now we can consider the loop functor

$$\overline{\Omega} : (\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p}) \to (\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$$

defined by $\overline{\Omega}((H, p)) = Ker((H^I, Id_p) \xrightarrow{(\partial_0, \partial_1)} (H \times H, (p, p))$. This functor induces the corresponding one in the homotopy category $H_0((\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p}))$.

Now, let us consider the groupoid interval $\mathcal{I}$, which is the groupoid with only two objects 0 and 1 and one morphism between them. We also denote by $\mathcal{I}$ the simplicial groupoid constant $\mathcal{I}$ in any dimension, and let us consider, for any $G \in \mathbf{Simp}(\mathbf{Gpd})_*$ and $p \in O$, the morphism of simplicial groupoids $u : \mathcal{I} \to G \otimes I$ defined, in each dimension, by: $u(0) = p_0$, $u(1) = p_1$, $u(0 \to 1) = I_p$. This morphism induces another one in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$, $u : (\mathcal{I}, 0) \to (G \otimes I, p_0)$, and we make the following:

5.2. Definition. *Given $(G, p) \in (\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$, we define $(\overline{G \otimes I}, p_0)$ by means of the following pushout diagram:*

$$\begin{tikzcd}
(\mathcal{I}, 0) \arrow[r, "u"] \arrow[d, "f"'] & (G \otimes I, p_0) \arrow[d, "\gamma"] \\
* \arrow[r, "i"] & (\overline{G \otimes I}, p_0)
\end{tikzcd}$$

5.3. Proposition. *For any cofibrant object $G \in \mathbf{Simp}(\mathbf{Gpd})_*$ and any $p \in O$, the pointed simplicial groupoid $(\overline{G \otimes I}, p_0)$ is a cylinder object for (G, p).*

Proof. Let us consider the morphisms $\sigma : G \otimes I \to G$ and $i_0, i_1 : G \to G \otimes I$ defined in Proposition 3.4 and the morphisms they induce in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$, $\sigma : (G \otimes I, p_0) \to (G, p)$ and $i_0 : (G, p) \to (G \otimes I, p_0)$.

The morphisms σ and $0 : * \to (G, p)$ induce a morphism $\sigma' : (\overline{G \otimes O}, p_0) \to (G, p)$ according to the following commutative diagram:

$$\begin{tikzcd}
(\mathcal{I}, 0) \arrow[r, "u"] \arrow[d, "0"'] & (G \otimes I, p_0) \arrow[d, "\gamma"] \arrow[ddr, "\sigma"] & \\
* \arrow[r] \arrow[drr, "0"'] & (\overline{G \otimes I}, p_0) \arrow[dr, dotted, "\sigma'"] & \\
& & (G, p)
\end{tikzcd}$$

Moreover, we consider the morphism $i'_0 : (G, p) \to (\overline{G \otimes I}, p_0)$ given as the composition $i'_0 = \gamma i_0$, so that, in each dimension and for any $q \in O$, we have $i'_0(q) = \gamma(i_0(q)) = \gamma(q_0) = q_0$, and given $x : q \to r$ a morphism in G, we have $i'_0(x) = \gamma i_0(x) = \gamma(x^{00\ldots0}) = x^{00\ldots0}$.

Now, considering the morphism $\gamma i_1 : G \to \overline{G \otimes I}$, we have the following induced morphism in $(\mathbf{Simp}(\mathbf{Gpd})_*, \mathbf{p})$, $i'_1 : (G, p) \to (\overline{G \otimes I}, p_0)$. In each dimension, and for any $q \in O$, we have $i'_1(q) = q_0$ and, given $x : q \to r$ a morphism in G,then $i'_1(x) = I_q x^{11\ldots1} I_r^{-1}$.

Then we have the following factorization of the codiagonal morphism

$$(G, p) \coprod (G, p) \xrightarrow{i'_0 + i'_1} (\overline{G \otimes I}, p_0) \xrightarrow{\sigma'} (G, p)$$

since:

$\sigma' i'_0(q) = \sigma'(q_0) = \sigma(q_0) = q,$

$\sigma' i'_0(x) = \sigma'(x^{00\ldots 0}) = \sigma(x^{00\ldots 0}) = x,$
$\sigma' i'_1(q) = \sigma'(q_0) = \sigma(q_0) = q,$
$\sigma' i'_1(x) = \sigma'(I_q x^{11\ldots 1} I_r^{-1}) = \sigma(I_q x^{11\ldots 1} I_r^{-1}) = x.$

The proof that $i'_0 + i'_1$ is a cofibration is exactly similar to the proof of this fact given in Proposition 3.4. Also, σ' is a homotopy equivalence and therefore a weak equivalence. The homotopy $H' : (\overline{G \otimes I}, p_0) \to ((\overline{G \otimes I})^I, Id_{p_0})$ is defined similarly to the homotopy $H : G \otimes I \to (G \otimes I)^I$ given in the proof of Proposition 3.4 since we can identify the set of objects of the pointed simplicial groupoid $((\overline{G \otimes I})^I, Id_{p_0})$ with the set $\{G_0^0 \vee G_0^1 \vee O / p_0 = p_1; I_p = Id_{p_0} = Id_{p_1}\}$, (see Definition 3.3) and, moreover, $((\overline{G \otimes I})^I)_n$ can be identified with the set

$$\left\{ \left(\begin{array}{ccc} q & \xrightarrow{s_0^{n+1} a} & r \\ \downarrow x_0 & & x'_0 \downarrow \\ q' & \xrightarrow[s_0^{n+1} b]{} & r' \end{array} , \quad \ldots \quad , \begin{array}{ccc} q & \xrightarrow{s_0^{n+1} a} & r \\ \downarrow x_n & & x'_n \downarrow \\ q' & \xrightarrow[s_0^{n+1} b]{} & r' \end{array} \right) \Bigg/ \begin{array}{ll} a, b \in (\overline{G \otimes I})_0 & d_i x_i = d_i x_{i-1} \\ x_i, x'_i \in (\overline{G \otimes I})_{n+1} & d_i x'_i = d_i x'_{i-1} \\ q, r \in O & \end{array} \right\} .$$

■

The above construction allows us to define a suspension functor

$$\overline{\Sigma} : (\mathbf{Simp(Gpd)}_*, \mathbf{p}) \to (\mathbf{Simp(Gpd)}_*, \mathbf{p})$$

given by

$$\overline{\Sigma}((G,p)) = Coker((G,p) \coprod (G,p) \xrightarrow{i'_0 + i'_1} (\overline{G \otimes I}, p_0)) .$$

This functor is left adjoint to the loop functor $\overline{\Omega}$ in $(\mathbf{Simp(Gpd)}_*, \mathbf{p})$ and it induces the corresponding suspension functor in the homotopy category $H_0((\mathbf{Simp(Gpd)}_*, \mathbf{p}))$.

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