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PSEUDOGROUPOIDS AND COMMUTATORS

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ABSTRACT. We develop a new approach to Commutator theory based on the theory of internal categorical structures, especially of so called pseudogroupoids. It is motivated by our previous work on internal categories and groupoids in congruence modular varieties.

0. Introduction

The purpose of this paper is to develop a new approach to *commutator theory.*

Since J. D. H. Smith [S] introduced the notion of *commutator* for a pair of congruences in a *congruence permutable* (*=Mal'tsev*) variety of universal algebras, there were many attempts to extend this notion to more general varieties and to simplify it. Accordingly, the important work of J. Hagemann and C. Hermann [HH] and of H. P. Gumm [G] has to be mentioned. The conclusion, also well supported by R. Freese and R. McKenzie [FMK], seems to be that the right level of generality in Commutator theory is the level of *congruence modular* varieties, where various possible definitions coincide and the commutator has "all nice properties". However there are various interesting investigations in non modular cases (see [K], P. Lipparini [L1], [L2] and references there).

Our viewpoint in this paper is that commutator theory should be based on the theory of internal categorical structures, this approach is motivated by the description of internal categories and groupoids in congruence modular varieties obtained in [JP]. In fact we use a new categorical structure which we call a *pseudogroupoid* — in contrast to *pregroupoids* (in the sense of A. Kock [Ko]) used in [P1] and [P2]; in the case of Mal'tsev varieties our approach is equivalent to the one developed in [P1].

Once the pseudogroupoids are introduced, the definition of the commutator becomes very simple.

The commutator $[\alpha, \beta]$ of congruences α and β on an algebra A is

$$[\alpha, \beta] = \big\{(x, y) \in A \times A | \eta(x) = \eta(y)\big\}, \tag{0.1}$$

where η is the canonical homomorphism from A to the free pseudogroupoid on the span

$$A/\alpha \longleftarrow A \longrightarrow A/\beta. \tag{0.2}$$

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This definition has a powerful conclusion: all properties of commutators are just the properties of the free-forgetful adjunction

$$\begin{pmatrix}\text{Spans}\\ \text{in } \mathbb{C}\end{pmatrix} \underset{\textit{forgetful}}{\overset{\textit{free}}{\rightleftarrows}} \begin{pmatrix}\text{Internal}\\ \text{pseudogroupoids}\\ \text{in } \mathbb{C}\end{pmatrix}, \tag{0.3}$$

where $\mathbb{C}$ is a ground variety.

On the other hand the notion of a pseudogroupoids is quite natural: as we explain in the Section 3 below it was "almost" introduced by J. D. H. Smith [S], and then again by E. W. Kiss [Ki].

The paper is organized as follows:

In the first four sections we are trying to describe the language of internal categorical structures and to show that they are really needed and even unexplicitly used all the time in Commutator theory — although the commutator itself is introduced only in the fifth section, first for spans in general categories, and then for congruences on an algebra, in fact via (0.1).

Our notion of commutator coincides with the "usual" one for congruence modular varieties, but seems to be useful also for the larger classes of varieties which we call Kiss, Gumm and Lipparini varieties since their definitions were suggested by the results of these authors. Even the largest class which we consider — the class of "abelianizable" varieties admits the "fundamental theorem on abelian algebras" and a good description of the largest commutator (i.e. $[\nabla_A, \nabla_A]$, where $\nabla_A = A \times A$ is the largest congruence on an algebra A). In the last section we show that a Kiss variety is congruence modular if (and only if) the commutator is preserved by surjective images, and deduce from the results of P. Lipparini [L1] that a Gumm variety is congruence modular if and only if the commutator is distributive (by "image" we mean the image in the categorical sense).

Since our commutator generalizes the "modular" one, it also generalizes the ordinary commutator for normal subgroups (and also $[K,K'] = KK' + K'K$ for ideals in a ring), however we give a simple direct proof — in order to show again that our generalization is very natural.

We would like to propose the following further questions and problems to be investigated:

1. Find out more about the relationship between the geometrical language of H. P. Gumm [G] and the categorical language.

2. What is the relationship with other known commutators in non-modular cases? (see also [KS])

3. Further development of commutator theory in general categories (we have introduced here only the definition and the very first properties — see Proposition 5.4).

4. Using the commutator investigate various congruence identities in the non-modular case and generalize them from varieties to exact, regular and possibly more general categories.

5. Further development of the theory of central extension [JK] in the case of pairs of varieties of the form ($\mathbb{C}$, Abelian algebras in $\mathbb{C}$), and extension of Homological methods of J. D. H. Smith [S] and J. Furtado-Coelho [F-C].

In the paper we use without proofs:

- elementary properties of limits, colimits, adjoint functors — much less that given in S. MacLane's book [ML];

- some motivations from [JP] — so we are asking our readers to read at least the introduction of that paper;

- some results in Commutator theory — either well known, or proved by E. W. Kiss [Ki], or by P. Lipparini [L1].

1. Rectangles and diamonds

Let $\mathbb{C}$ be a variety of universal algebras and α, β congruences on an algebra A in $\mathbb{C}$.

The composition $\alpha\beta$ is defined as

$$\alpha\beta = \big\{(a,b) \in A \times A | \exists x : a\alpha x\beta b\big\}, \tag{1.1}$$

and so $\alpha\beta = \beta\alpha$ can be expressed as

$$(\exists x : a\alpha x\beta b) \Longleftrightarrow (\exists y : a\beta y\alpha b). \tag{1.2}$$

This tells us that it is useful to work with four-tuples (a, x, y, b) for which

$$\begin{tikzcd} a \arrow[r, dash, "\beta"] \arrow[d, dash, "\alpha"'] & y \arrow[d, dash, "\alpha"] \\ x \arrow[r, dash, "\beta"'] & b \end{tikzcd} , \tag{1.3}$$

i.e. $a\alpha x\beta b$ and $a\beta y\alpha b$. The picture (1.3) will be called an *α-β-rectangle*; H. P. Gumm [G] also says that (a, x, y, b) is an α-β-parallelogram, and E. W. Kiss [Ki] says that (ax, yb) is an α-β-rectangle.

An *internal graph* $G =$

$$\begin{tikzcd} G_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] & G_0 \end{tikzcd} \tag{1.4}$$

in $\mathbb{C}$ consists of two algebras G_0 and G_1 in $\mathbb{C}$ and two homomorphisms d and c from G_1 to G_0. The elements of G_0 are called *objects*, or *points*, and the elements of G_1 are called *morphisms*, or *arrows* (of G); if $g \in G_1$ and $d(g) = u$, $c(g) = v$, then we write $g : u \longrightarrow v$.

If G is an internal graph in $\mathbb{C}$ and

$$\begin{array}{rcl} A & = & G_1 \\ \alpha & = & \big\{(f, g) \in G_1 | d(f) = d(g)\big\}, \\ \beta & = & \big\{(g, h) \in G_1 | c(g) = c(h)\big\}, \end{array} \tag{1.5}$$

then an α-β-rectangle

$$\begin{tikzcd} f \arrow[r, dash, "\beta"] \arrow[d, dash, "\alpha"'] & k \arrow[d, dash, "\alpha"] \\ g \arrow[r, dash, "\beta"'] & h \end{tikzcd} \tag{1.6}$$

becomes a "G-diamond"

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "f"] \arrow[dr, "g"'] & & \cdot \arrow[ul, "k"'] \arrow[dl, "h"] \\ & \cdot & \end{tikzcd} \tag{1.7}$$

— as in [JP]. Note that in this case A/α and A/β are canonically isomorphic to subalgebras in G_0, which is a useless additional condition on α and β. In order to avoid that condition we replace internal graphs by (internal) spans.

A *span* $S =$

$$\begin{tikzcd} S_0 & S_1 \arrow[l, "\pi"'] \arrow[r, "\pi'"] & S'_0 \end{tikzcd} \tag{1.8}$$

in $\mathbb{C}$ consists of three algebras S_0, S_0' and S_1 in $\mathbb{C}$, and two homomorphisms $\pi : S_1 \longrightarrow S_0$ and $\pi' : S_1 \longrightarrow S_0'$; if $g \in S_1$ and $\pi(g) = u$, $\pi'(g) = v$, we again write $g : u \longrightarrow v$.

Now instead of (1.5) we just say: given an algebra A in $\mathbb{C}$ and congruences α and β on A, we obtain the span

$$A/\alpha \longleftarrow A \longrightarrow A/\beta \tag{1.9}$$

of the canonical homomorphisms — and again an α-β-rectangle (1.6) is the same as an S-diamond (1.7).

Let S be an arbitrary span in $\mathbb{C}$. The set of all S-diamonds will be denoted by S_4, and for $x \in S_4$ sometimes we will write $x =$

$$\begin{tikzcd} & {} & \\ {} \arrow[ur, "x_{11}"] \arrow[dr, "x_{21}"'] & & {} \arrow[ul, "x_{12}"'] \arrow[dl, "x_{22}"] \\ & {} & \end{tikzcd}, \tag{1.10}$$

or just

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \tag{1.11}$$

which suggest to consider the diagram

$$\begin{tikzcd} S_1 \arrow[r, "\pi'"] \arrow[d, "\pi"'] & S_0' & S_1 \arrow[l, "\pi'"'] \arrow[d, "\pi"] \\ S_0 & S_4 \arrow[ul, "\pi_{11}"'] \arrow[ur, "\pi_{12}"] \arrow[dl, "\pi_{21}"] \arrow[dr, "\pi_{22}"'] & S_0 \\ S_1 \arrow[u, "\pi"] \arrow[r, "\pi'"'] & S_0' & S_1 \arrow[u, "\pi"'] \arrow[l, "\pi'"] \end{tikzcd} \tag{1.12}$$

where $\pi_{ij}(x) = x_{ij}$. Note that this is a diagram in $\mathbb{C}$ since S_4 obviously is a subalgebra in $S_1^4 = S_1 \times S_1 \times S_1 \times S_1$ and the maps π_{ij} are homomorphisms. Moreover the diagram (1.12) represents S_4 as the limits of

$$\begin{tikzcd} S_1 \arrow[r, "\pi'"] \arrow[d, "\pi"'] & S_0' & S_1 \arrow[l, "\pi'"'] \arrow[d, "\pi"] \\ S_0 & & S_0 \\ S_1 \arrow[u, "\pi"] \arrow[r, "\pi'"'] & S_0' & S_1 \arrow[u, "\pi"'] \arrow[l, "\pi'"] \end{tikzcd}. \tag{1.13}$$

Another way to present S_4 as a limit is to form the pullbacks

$$\begin{tikzcd} S_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & S_1 \arrow[d, "\pi"] \\ S_1 \arrow[r, "\pi"'] & S_0 \end{tikzcd} \;, \quad \begin{tikzcd} S_2' \arrow[r, "\pi_2'"] \arrow[d, "\pi_1'"'] & S_1 \arrow[d, "\pi'"] \\ S_1 \arrow[r, "\pi'"'] & S_0 \end{tikzcd} \;, \tag{1.14}$$

and then the diagram

$$\begin{tikzcd} S_1 & S_2' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] & S_1 \\ S_2 \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] & S_4 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & S_2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ S_1 & S_2' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & S_1 \end{tikzcd} \;, \tag{1.15}$$

which presents S_4 as the limit of

$$\begin{tikzcd} S_1 & S_2' \arrow[l, "\pi_1'"'] \arrow[r, "\pi_2'"] & S_1 \\ S_2 \arrow[u, "\pi_1"] \arrow[d, "\pi_2"'] & & S_2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ S_1 & S_2' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & S_1 \end{tikzcd} \;. \tag{1.16}$$

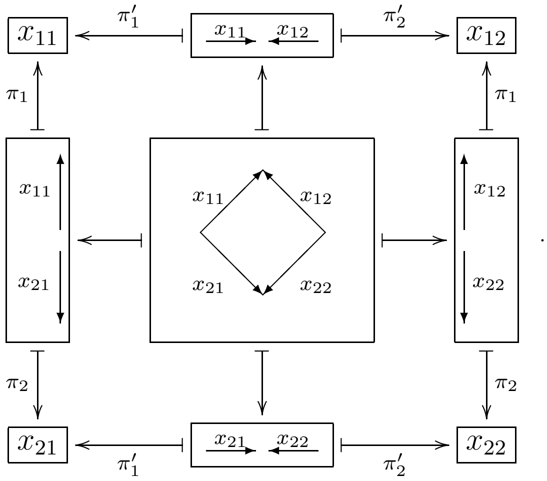
Since we decided to write the elements of S_4 as diamonds, it is convenient to write the elements of S_2 as vertical diagrams

$$\begin{tikzcd} \cdot \\ \cdot \arrow[u] \arrow[d] \\ \cdot \end{tikzcd}$$

and the elements of S_2' as horizontal diagrams

$$\begin{tikzcd} \cdot \arrow[r] & \cdot & \cdot \arrow[l] \end{tikzcd} \;;$$

in this notation the diagram (1.15) can be described (in term of elements) as



(1.17)

There is also a convenient matrix notation:

$$
\begin{tikzcd}
x_{11} & (x_{11}, x_{12}) \arrow[l, mapsto] \arrow[r, mapsto] & x_{12} \\
\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \arrow[u, mapsto] \arrow[d, mapsto] & \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \arrow[l, mapsto] \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix} \arrow[u, mapsto] \arrow[d, mapsto] \\
x_{21} & (x_{21}, x_{22}) \arrow[l, mapsto] \arrow[r, mapsto] & x_{22}
\end{tikzcd}
\; ; \tag{1.18}
$$

it simplifies (1.17) just as (1.11) simplifies (1.10).

2. Double equivalence relations

Let G be an internal graph in a category $\mathbb{C}$ with finite limits; just as in the case of a variety, G consists of two objects G_0 and G_1 in $\mathbb{C}$ and two morphisms d and c from G_1 to G_0. Such an internal graph G is said to be a *relation* if d and c are jointly monic, i.e. the morphism $\langle d, c\rangle : G_1 \longrightarrow G_0 \times G_0$ is a monomorphism. Let us also recall

2.1. Definition. *An internal graph G is said to be an (internal)* equivalence relation *if it is a relation and is*

(a) reflexive, i.e. there exists a morphism $e : G_0 \longrightarrow G_1$ with $de = 1_{G_0} = ce$;

(b) symmetric, i.e. there exists a morphism $i : G_1 \longrightarrow G_1$ with $di = c$ and $ci = d$;

(c) *transitive, i.e. there exists a morphism* $m : G_1 \times_{G_0} G_1 \longrightarrow G_1$*, where* $G_1 \times_{G_0} G_1$ *is constructed as the pullback*

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & G_1 \arrow[d, "c"] \\ G_1 \arrow[r, "d"'] & G_0 \end{tikzcd}, \tag{2.1}$$

such that $dm = dp_2$ *and* $cm = cp_1$.

2.2. Remark.

(a) It is often convenient to consider an equivalence relation as a special case of an (internal) groupoid, i.e. to describe it as a system $(G_0, G_1, d, c, e, m, i)$, where e, m, i (as in Definition 2.1) are however uniquely determined by d and c since $\langle d, c\rangle$ is a monomorphism;

(b) if $\mathbb{C}$ is a variety of universal algebras, then the internal equivalence relations in $\mathbb{C}$ are the same as "congruences".

Let $\mathrm{Eq}(\mathbb{C})$ be the category of equivalence relations in $\mathbb{C}$; a morphism $f : G \longrightarrow G'$ in $\mathrm{Eq}(\mathbb{C})$ is a diagram

$$\begin{tikzcd} G_1 \arrow[r, shift left, "d"] \arrow[r, shift right, "c"'] \arrow[d, "f_1"'] & G_0 \arrow[d, "f_0"] \\ G_1' \arrow[r, shift left, "d'"] \arrow[r, shift right, "c'"'] & G_0' \end{tikzcd} \tag{2.2}$$

in which $d'f_1 = f_0 d$ and $c'f_1 = f_0 c$.

It is easy to see that since $\mathbb{C}$ has finite limits, $\mathrm{Eq}(\mathbb{C})$ also has finite limits. Therefore we can consider the equivalence relations in $\mathrm{Eq}(\mathbb{C})$ — we will call them *double equivalence relations* in $\mathbb{C}$. A double equivalence relation D in $\mathbb{C}$ can also be described as a diagram

$$\begin{tikzcd} D_4 \arrow[r, shift left, "q_1'"] \arrow[r, shift right, "q_2'"'] \arrow[d, shift right, "q_1"'] \arrow[d, shift left, "q_2"] & D_2 \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\ D_2' \arrow[r, shift left, "p_1'"] \arrow[r, shift right, "p_2'"'] & D_1 \end{tikzcd} \tag{2.3}$$

in which

$$\begin{array}{ll} p_1'q_1 = p_1q_1', & p_2'q_1 = p_1q_2' \\ p_1'q_2 = p_2q_1', & p_2'q_2 = p_2q_2', \end{array} \tag{2.4}$$

and each pair of parallel arrows forms an equivalence relation.

The identities (2.4) are equivalent to the commutativity of the diagram

$$
\begin{tikzcd}
D_1 & D'_2 \arrow[l, "p'_1"'] \arrow[r, "p'_2"] & D_1 \\
D_2 \arrow[u, "p_1"] \arrow[d, "p_2"'] & D_4 \arrow[l, "q'_1"'] \arrow[r, "q'_2"'] \arrow[u, "q_1"'] \arrow[d, "q_2"'] & D_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
D_1 & D'_2 \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & D_1
\end{tikzcd}
\tag{2.5}
$$

clearly similar to (1.15). Moreover, it is easy to check that any span S in $\mathbb{C}$ determines, via (1.15), a double equivalence relation in $\mathbb{C}$ which we will denote by $\mathrm{Eq}(S)$.

By the analogy with ordinary (internal) equivalence relations we introduce

2.3. Definition. *A double equivalence relation in* $\mathbb{C}$ *is said to be* effective *if it is of the form* $\mathrm{Eq}(S)$ *for some span* S *in* $\mathbb{C}$.

If $\mathbb{C}$ is a variety of universal algebras (or, more generally, an *exact* category) then every equivalence relation in $\mathbb{C}$ is *effective*, i.e. is of the form $\mathrm{Eq}(\varphi) =$

$$
\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "pr_1"] \arrow[r, shift right, "pr_2"'] & X
\end{tikzcd}
\tag{2.6}
$$

for some $\varphi : X \longrightarrow Y$ in $\mathbb{C}$. The situation with double equivalence relations is much more complicated: in some sense commutator theory is a theory of noneffective double equivalence relations. This viewpoint is suggested by the results of [JP] and the following

2.4. Example. Let $G = (G_0, G_1, d, c, e, m, i,)$ be an *internal groupoid* in $\mathbb{C}$. Recall that it is a diagram in $\mathbb{C}$ of the form

$$
\begin{tikzcd}
G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] \arrow[loop below, "i"] & G_0 \arrow[l, "e" description]
\end{tikzcd}
\tag{2.7}
$$

in which $de = 1_{G_0} = ce$, $dm = dp_2$, $cm = cp_1$, $di = c$, $ci = d$ (where $G_1 \times_{G_0} G_1$ together with the projections p_1 and p_2 is constructed as in 2.1), and the following diagrams commute:

$$
\begin{tikzcd}
G_1 \times_{G_0} G_1 \times_{G_0} G_1 \arrow[r, "m\times 1"] \arrow[d, "1\times m"'] & G_1 \times_{G_0} G_1 \arrow[d, "m"] \\
G_1 \times_{G_0} G_1 \arrow[r, "m"'] & G_1
\end{tikzcd}
,
\tag{2.8}
$$

$$
\begin{tikzcd}
G_1 \arrow[r, "\langle 1,ed\rangle"] \arrow[d, "\langle ec,1\rangle"'] \arrow[dr, equal] & G_1 \times_{G_0} G_1 \arrow[d, "m"] \\
G_1 \times_{G_0} G_1 \arrow[r, "m"'] & G_1
\end{tikzcd}
,
\tag{2.9}
$$

$$\begin{tikzcd} G_1 \arrow[r, "\langle 1,i \rangle"] \arrow[d, "\langle i,1 \rangle"'] \arrow[dr, shift left, "ec"] \arrow[dr, shift right, "ed"'] & G_1 \times_{G_0} G_1 \arrow[d, "m"] \\ G_1 \times_{G_0} G_1 \arrow[r, "m"'] & G_1 \end{tikzcd} \quad . \tag{2.10}$$

If $\mathbb{C}$ is a variety of universal algebras, we will use the same notation as in the Section 1 (and as for the ordinary groupoids), i.e. write $g : u \longrightarrow v$ if $g \in G_1$ and $d(g) = u$, $c(g) = v$, and also

$$m(f, g) = fg, \quad e(u) = 1_u, \quad i(g) = g^{-1}. \tag{2.11}$$

The diagrams (2.8), (2.9) and (2.10) express the associativity, the right and left unit law, and the right and left inverse law respectively.

A diamond (1.7) in G is said to be *commutative* if $fg^{-1} = kh^{-1}$. The set Comm(G) of all commutative diamonds form a subalgebra in G_4 which can be defined as any of the following two pullbacks:

$$\begin{tikzcd} \mathrm{Comm}(G) \arrow[r] \arrow[d] & G_2 \arrow[d, "m(1\times i)"] \\ G_2 \arrow[r, "m(1\times i)"'] & G_1 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Comm}(G) \arrow[r] \arrow[d] & G_2' \arrow[d, "m(i\times 1)"] \\ G_2' \arrow[r, "m(i\times 1)"'] & G_1 \end{tikzcd} \tag{2.12}$$

and therefore it is a well defined (sub-) double equivalence relation (of Eq(S), where $S = (G_0 \xleftarrow{d} G_1 \xrightarrow{c} G_0)$ is the underlying span of G), also in the case of an abstract category $\mathbb{C}$ with pullbacks.

It is easy to see that the following conditions are equivalent:

(a) the double equivalence relation determined by Comm(G) is effective;

(b) Comm$(G) = G_4$;

(c) G is a relation.

3. Pseudogroupoids

For a given diamond (1.7) in a groupoid G let us write

$$m(f, g, k, h) = fg^{-1}h; \tag{3.1}$$

in particular

$$m(f, g, fg^{-1}h, h) = fg^{-1}h. \tag{3.2}$$

So $m(f, g, k, h)$ does not depend on k; the reason why we involve k in the notation is that we are going to generalize the notion of groupoid in such a way that the composition above will be defined only if such a k does exist.

3.1. Definition. *A* pseudogroupoid *is a pair* (S, m) *in which* S *is a span and* $m : S_4 \longrightarrow S_1$ *a map written as*

$$m \begin{pmatrix} f & k \\ g & h \end{pmatrix} = m(f, g, k, h), \tag{3.3}$$

with:

(a) $m(f, g, k, h)$ *is parallel to* k*, i.e.* $\pi m(f, g, k, h) = \pi(k)$ $(= \pi(h))$ *and* $\pi' m(f, g, k, h) = \pi'(k)$ $(= \pi'(f))$*;*

(b) $m(f, g, k, h)$ *does not depend on* k*, i.e.* $m(f, g, k, h) = m(f, g, k', h)$ *if both sides are defined;*

(c) if $f = g$ *then* $m(f, g, k, h) = h$*;*

(d) if $g = h$ *then* $m(f, g, k, h) = f$*;*

(e) $m(m(x_1, x_2, y, x_3), x_4, t, x_5) = m(x_1, x_2, t, m(x_3, x_4, z, x_5))$ *for every diagram in* S *of the form*

$$\begin{tikzcd}
\cdot & & & \cdot \arrow[lll, "t"'] \arrow[d, "x_5"] \arrow[ddll, "z"] \\
\cdot \arrow[u, "x_1"] \arrow[dr, "x_2"'] & & & \cdot \\
& \cdot & \cdot \arrow[l, "x_3"] \arrow[uull, "y"] \arrow[ur, "x_4"'] &
\end{tikzcd} \tag{3.4}$$

This definition can be "internalized via Yoneda", i.e. we have an obvious notion of internal pseudogroupoid in an abstract category $\mathbb{C}$ with finite limits.

For, given a span S and an object C in $\mathbb{C}$, we construct the span $\hom_{\mathbb{C}}(C, S) =$

$$\hom_{\mathbb{C}}(C, S_0) \xleftarrow{\hom_{\mathbb{C}}(C,\pi)} \hom_{\mathbb{C}}(C, S_1) \xrightarrow{\hom_{\mathbb{C}}(C,\pi')} \hom_{\mathbb{C}}(C, S_0') \tag{3.5}$$

(in Sets). In this span $(\hom_{\mathbb{C}}(C, S))_4$ can be identified with $\hom_{\mathbb{C}}(C, S_4)$ and we introduce

3.2. Definition. *An* internal pseudogroupoid *in a category* $\mathbb{C}$ *with finite limits is a pair* (S, m) *in which* S *is a span in* $\mathbb{C}$ *and* $m : S_4 \longrightarrow S_1$ *(where* S_4 *is the limit of* (1.13)*) a morphism in* $\mathbb{C}$ *such that* $(\hom_{\mathbb{C}}(C, S), \hom_{\mathbb{C}}(C, m))$ *is a pseudogroupoid for every object* C *in* $\mathbb{C}$*.*

Note that if $\mathbb{C}$ is a variety of universal algebras, then an internal pseudogroupoid in $\mathbb{C}$ is just a pair (S, m) in which S is a span in $\mathbb{C}$ and $m : S_4 \longrightarrow S_1$ a homomorphism (i.e. a morphism in $\mathbb{C}$) making (S, m) a pseudogroupoid (in Sets).

Consider examples:

3.3. Example. Any groupoid G can be considered as a pseudogroupoid (S, m) in which S is the span

$$G_0 \xleftarrow{d} G_1 \xrightarrow{c} G_0 \tag{3.6}$$

and m is defined by (3.1). All the conditions of Definition 3.1 clearly hold; in particular (c), (d) and (e) become

$$gg^{-1}h = h, \quad fg^{-1}g = f \tag{3.7}$$

and

$$(x_1x_2^{-1}x_3)x_4^{-1}x_5 = x_1x_2^{-1}(x_3x_4^{-1}x_5) \tag{3.8}$$

respectively — which tells us that they play the roles of “Mal’tsev identities” and associativity.

Similarly any internal groupoid in a category $\mathbb{C}$ (with finite limits) determines an internal pseudogroupoid in $\mathbb{C}$.

3.4. Example. A *pregroupoid* is a pair (S, l) in which S is a span and $l : S_3 \longrightarrow S_1$, where

$$S_3 = \big\{(f, g, h) \in S_1 \times S_1 \times S_1 | \pi(f) = \pi(g), \pi'(g) = \pi'(h)\big\}, \tag{3.9}$$

is a map with:

(a) $\pi l(f, g, h) = \pi(h)$ and $\pi' l(f, g, h) = \pi'(f)$;

(b) if $f = g$ then $l(f, g, h) = h$;

(c) if $g = h$ then $l(f, g, h) = f$;

(d) $l(l(x_1, x_2, x_3), x_4, x_5) = l(x_1, x_2, l(x_3, x_4, x_5))$ for every diagram in S of the form

$$\cdot \xleftarrow{x_1} \cdot \xrightarrow{x_2} \cdot \xleftarrow{x_3} \cdot \xrightarrow{x_4} \cdot \xleftarrow{x_5} \cdot \qquad . \tag{3.10}$$

We immediately see that this is a special case of a pseudogroupoid: just put

$$m(f, g, k, h,) = l(f, g, h). \tag{3.11}$$

Conversely, if for every diagram in S of the form

$$\cdot \xleftarrow{f} \cdot \xrightarrow{g} \cdot \xleftarrow{h} \cdot \tag{3.12}$$

there exists k in S_1 with $\pi(k) = \pi(h)$ and $\pi'(k) = \pi'(f)$, then (3.11) can be used as the definition of l in terms of m. Thus $(S, l) \longmapsto (S, m)$ (where m is as in (3.11)) is an isomorphism between the category of pregroupoids and the category of pseudogroupoids in which every diagram (3.12) can be completed as above.

Again, the same can be repeated in the "internal context". The internal pregroupoids were introduced by A. Kock [Ko], and then used in [P1] to develop Commutator theory in Mal'tsev categories. Note that the notion of pregroupoid is clearly "between" the notions of groupoid and of pseudogroupoid: if G is a groupoid, we replace (3.1) by

$$l(f, g, h) = fg^{-1}h \tag{3.13}$$

and this clearly defines a pregroupoid.

3.5. Example. If S is a relation, i.e. the map $\langle \pi, \pi' \rangle : S_1 \longrightarrow S_0 \times S'_0$ is injective, then S has a unique pseudogroupoid structure; it is defined by

$$m(f, g, k, h) = k. \tag{3.14}$$

In the internal context this can be written as

$$m = \pi_{12} : S_4 \longrightarrow S_1, \tag{3.15}$$

where π_{12} is as in (1.12).

3.6. Example. Any pseudogroupoid (S, m) has an *opposite* (=*dual*) pseudogroupoid $(S, m)^{op} = (S^{op}, m^{op})$, in which the opposite span S^{op} is

$$S'_0 \xleftarrow{\pi'} S_1 \xrightarrow{\pi} S_0, \tag{3.16}$$

and m^{op} is defined by

$$m(f, g, k, h) = m^{op}(h, g, k, f) \tag{3.17}$$

— or

$$m = m^{op}\langle \pi_{22}, \pi_{21}, \pi_{12}, \pi_{11} \rangle \tag{3.18}$$

in the internal context.

Clearly this notion of opposite contains the notions of opposite groupoid and opposite (=inverse) relation.

3.7. Example. Recall that a variety $\mathbb{C}$ of universal algebras is said to be *congruence modular* if, for every C in $\mathbb{C}$, the lattice $\mathrm{Cong}(C)$ of congruences on C is modular. Let A be an algebra in a congruences modular variety $\mathbb{C}$ and α, β congruences on A. The *modular commutator* (we take this expression from the title of [Ki]; it is just the *commutator* in the common sense) $[\alpha, \beta]$ is defined by

$$[\alpha, \beta] = \big\{(a, b) \in A \times A | \big((a, a), (a, b)\big) \in \Delta_{\alpha,\beta}\big\} \tag{3.19}$$

where $\Delta_{\alpha,\beta}$ is the congruence on α generated by all

$$\Big\{\big((a,a),(b,b)\big)|(a,b)\in\beta\Big\}. \tag{3.20}$$

A four variable term q is said to be a Kiss difference term (E. W. Kiss [Ki] says "a 4-difference term") if

$$q(x,y,x,y)=x, \quad q(x,x,y,y)=y \tag{3.21}$$

are identities in $\mathbb{C}$ and

$$\big(q(a,b,c,d),q(a,b,c',d)\big)\in[\alpha,\beta] \tag{3.22}$$

whenever $\begin{pmatrix} a & c \\ b & d\end{pmatrix}, \begin{pmatrix} a & c' \\ b & d\end{pmatrix}\in S_4$ — where S is the span (1.9).

Among other things E. W. Kiss [Ki] proves that every congruence modular variety has such a term q, and that $[\alpha,\beta]=\Delta_A(=\{(a,a)|a\in A\})$ if and only if the following conditions hold:

(a) the map $m: S_4 \longrightarrow S_1 = A$ defined by

$$m\begin{pmatrix} a & c \\ b & d\end{pmatrix} = q(a,b,c,d) \tag{3.23}$$

is a homomorphism;

(b) $q(a,b,c,d)=q(a,b,c',d)$ whenever $\begin{pmatrix} a & c \\ b & d\end{pmatrix}$ and $\begin{pmatrix} a & c' \\ b & d\end{pmatrix}$ are in S_4.

As follows from *Lemma* 3.8 below, in our language that criterion for $[\alpha,\beta]=\Delta_A$ simply says: $[\alpha,\beta]=\Delta_A$ if and only if (3.23) defines a pseudogroupoid structure on S.

3.8. Lemma. *Let $\mathbb{C}$ be a variety of universal algebras, q a four variable term in $\mathbb{C}$ satisfying* (3.21)*, S a span in $\mathbb{C}$ and $m: S_4 \longrightarrow S_1$ a homomorphism satisfying Definition 3.1(b) and* (3.23)*. Then (S,m) is an internal pseudogroupoid in $\mathbb{C}$.*

Proof. We have to show that (S,m) satisfies the conditions of Definition 3.1.

3.1(a):

$$\begin{aligned}\pi m(f,g,k,h) &= \pi\big(q(f,g,k,h)\big)\\ &= q\big(\pi(f),\pi(g),\pi(k),\pi(h)\big) =\\ &= q\big(\pi(f),\pi(f),\pi(k),\pi(k)\big) = \pi(k);\end{aligned}$$

$$\begin{aligned}\pi' m(f,g,k,h) &= \pi'\big(q(f,g,k,h)\big) =\\ &= q\big(\pi'(f),\pi'(g),\pi'(k),\pi'(h)\big) =\\ &= q\big(\pi'(k),\pi'(g),\pi'(k),\pi'(g)\big) = \pi'(k).\end{aligned}$$

3.1(c): if $\begin{pmatrix} f & k \\ g & h \end{pmatrix} \in S_4$ has $f = g$, then k is parallel to h, and we have $m(f,g,k,h) = m(f,f,h,h) = q(f,f,h,h) = h$.

3.1(d): if $\begin{pmatrix} f & k \\ g & h \end{pmatrix} \in S_4$ has $g = h$, then k is parallel to f, and we have $m(f,g,k,h) = m(f,g,f,g) = q(f,g,f,g) = f$.

3.1(e):

$$\begin{gathered} m\big(m(x_1,x_2,y,x_3),x_4,t,x_5\big) = \\ = m\big(m(x_1,x_2,y,x_3),m(x_4,x_4,x_4,x_4),m(x_1,x_2,t,z),m(x_4,x_4,x_5,x_5)\big) = \\ = q\big(m(x_1,x_2,y,x_3),m(x_4,x_4,x_4,x_4),m(x_1,x_2,t,z),m(x_4,x_4,x_5,x_5)\big) = \\ = m\big(q(x_1,x_4,x_1,x_4),q(x_2,x_4,x_2,x_4),q(y,x_4,t,x_5),q(x_3,x_4,z,x_5)\big) = \\ = m\big(x_1,x_2,m(y,x_4,t,x_5),m(x_3,x_4,z,x_5)\big) = \\ = m\big(x_1,x_2,t,m(x_3,x_4,z,x_5)\big). \end{gathered}$$

■

Now we are going to show that a pseudogroupoid can be described as a span equipped with an appropriate set of "commutative diamonds ". We use the set-theoretic context just for simplicity — in fact everything can be repeated in the internal context as we will see.

First we need

3.9. Lemma. *Given a diamond* (1.7) *in a pseudogroupoid, the following conditions are equivalent:*

(a) $m(f,g,k,h) = k$*;*

(b) $m(g,f,h,k) = h$*;*

(c) $m(h,k,g,f) = g$*;*

(d) $m(k,h,f,g) = f$*.*

Proof. Since we deal with an arbitrary diamond in an arbitrary pseudogroupoid, it suffices to prove (a)⇒(b). If (a) holds, then using 3.1(e) for

$$\begin{tikzcd} \cdot & & & \cdot \arrow[lll, "h"'] \arrow[d, "h"] \arrow[ddll, "k"] \\ \cdot \arrow[u, "g"] \arrow[dr, "f"'] & & & \cdot \\ & \cdot & \cdot \arrow[l, "f"] \arrow[ur, "g"'] \arrow[uulll, "g"] & \end{tikzcd} \tag{3.24}$$

we obtain

$$\begin{aligned} m(g,f,h,k) &= m\big(g,f,h,m(f,g,k,h)\big) = \\ &= m\big(m(g,f,g,f),g,h,h\big) = \\ &= m(g,g,h,h) = h. \end{aligned}$$

∎

This lemma suggests to introduce

3.10. Definition. *A diamond* (1.7) *in a pseudogroupoid is said to be* commutative *if it satisfies the equivalent conditions of Lemma 3.9.*

Consider the diagram

$$\begin{tikzcd} \mathrm{Comm}(S,m) \arrow[dr, "inclusion"] & & \\ & S_4 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & S_2 \arrow[d, shift left] \arrow[d, shift right] \\ & S_2' \arrow[r, shift left] \arrow[r, shift right] & S_1 \end{tikzcd} , \tag{3.25}$$

where $\mathrm{Comm}(S,m)$ is the set of commutative diamonds in a pseudogroupoid (S,m), and the square is the effective double equivalence relation $\mathrm{Eq}(S)$. This diagram determines a sub-double equivalence relation of $\mathrm{Eq}(S)$ — just as in the case of groupoids considered in 2.4. The proof is straightforward; however it is good to see how the "associativity" condition 3.1(e) helps to prove the transitivity of

$$\mathrm{Comm}(S,m) \rightrightarrows S_2 \tag{3.26}$$

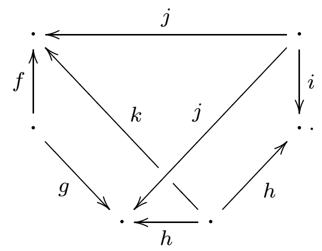
and

$$\mathrm{Comm}(S,m) \rightrightarrows S_2'. \tag{3.27}$$

Since (3.26) and (3.27) are dual to each other, let us consider only (3.26). The transitivity of (3.26) means that the commutativity of $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$ and $\begin{pmatrix} k & j \\ h & i \end{pmatrix}$ implies the commutativity of $\begin{pmatrix} f & j \\ g & i \end{pmatrix}$, i.e.

$$\big(m(f,g,k,h) = k, m(k,h,j,i) = j\big) \Longrightarrow m(f,g,j,i) = j. \tag{3.28}$$

We have $j = m(k, h, j, i) = m(m(f, g, k, h), h, j, i) = m(f, g, j, m(h, h, i, i)) = m(f, g, j, i)$ as desired; here 3.1(e) was applied to



(3.29)

Note that, again just as in 2.4, $\mathrm{Comm}(S, m)$ determines an effective double equivalence relation if and only if S is a relation. In other words it determines a thin double equivalence relation in the sense of

3.11. Definition. *A double equivalence relation D is said to be* thin *if it satisfies the following condition:*

Let $\overline{D}_4$ =

$$
\begin{tikzcd}
D_1 & D'_2 \arrow[l, "p'_1"'] \arrow[r, "p'_2"] & D_1 \\
D_2 \arrow[u, "p_1"] \arrow[d, "p_2"'] & \overline{D}_4 \arrow[l, "\overline{q'}_1"'] \arrow[r, "\overline{q'}_2"'] \arrow[u, "\overline{q}_1"'] \arrow[d, "\overline{q}_2"] & D_2 \arrow[u, "p_1"'] \arrow[d, "p_2"] \\
D_1 & D'_2 \arrow[l, "p'_1"] \arrow[r, "p'_2"'] & D_1
\end{tikzcd}
\tag{3.30}
$$

be the limit of the diagram (2.5) *with removed D_4, and $i : D_4 \longrightarrow \overline{D}_4$ the canonical injection; let α, β, γ be any three out of four canonical maps $\overline{D}_4 \longrightarrow D_1$. Then for every $x \in \overline{D}_4$ there exists a unique $y \in D_4$ such that x and $i(y)$ have the same images under α, β, γ.*

This in fact gives an alternative (equivalent) definition of a pseudogroupoid:

3.12. Theorem. *Let S be a span and C a subset of S_4 such that*

$$
\begin{tikzcd}
C \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & S_2 \arrow[d, shift left] \arrow[d, shift right] \\
S'_2 \arrow[r, shift left] \arrow[r, shift right] & S_1
\end{tikzcd}
\tag{3.31}
$$

is a thin sub-double equivalence relation of $\mathrm{Eq}(S)$. *Then there exists a unique pseudogroupoid structure $m : S_4 \longrightarrow S_1$ on S with* $\mathrm{Comm}(S, m) = C$.

PROOF. The assertion that (3.31) is thin means that for every diamond $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$ in S there exists a unique $k' \in S_1$ such that $\begin{pmatrix} f & k' \\ g & h \end{pmatrix}$ is in C — and so we must have $m(f,g,k,h) = k'$, which proves the uniqueness of m (provided it exists). On the other hand if we define m in this way, the conditions (a)–(e) of Definition 3.1 are satisfied:

3.1(a) and 3.1(b) hold by the definition of m.

3.1(c): $m(f,g,k,h) = m(f,f,h,h)$ by $f = g$ and 3.1(b), and $m(f,f,h,h) = h$ since $\begin{pmatrix} f & h \\ f & h \end{pmatrix}$ is in C because $C \rightrightarrows S_2'$ is reflexive.

3.1(d): $m(f,g,k,h) = m(f,g,f,g)$ by $g = h$ and 3.1(b), and $m(f,g,f,g) = f$ since $\begin{pmatrix} f & f \\ g & g \end{pmatrix}$ is in C because $C \rightrightarrows S_2$ is reflexive.

3.1(e): Given a diagram (3.4), we have to show that

$$\begin{pmatrix} m(x_1,x_2,y,x_3) & t' \\ x_4 & x_5 \end{pmatrix} \in C \Longrightarrow \begin{pmatrix} x_1 & t' \\ x_2 & m(x_3,x_4,z,x_5) \end{pmatrix} \in C \tag{3.32}$$

(where t' is an arrow parallel to t). That is, we have to show that

$$\begin{pmatrix} x_1 & y' \\ x_2 & x_3 \end{pmatrix}, \begin{pmatrix} y' & t' \\ x_4 & x_5 \end{pmatrix}, \begin{pmatrix} x_3 & z' \\ x_4 & x_5 \end{pmatrix} \in C \Longrightarrow \begin{pmatrix} x_1 & t' \\ x_2 & z' \end{pmatrix} \in C. \tag{3.33}$$

We have:

1° . Since $C \rightrightarrows S_2'$ is symmetric and $\begin{pmatrix} x_3 & z' \\ x_4 & x_5 \end{pmatrix}$ is in C, $\begin{pmatrix} x_4 & x_5 \\ x_3 & z' \end{pmatrix}$ also is in $\mathbb{C}$.

2° . Since $C \rightrightarrows S_2'$ is transitive and $\begin{pmatrix} y' & t' \\ x_4 & x_5 \end{pmatrix}, \begin{pmatrix} x_4 & x_5 \\ x_3 & z' \end{pmatrix}$ are in C, $\begin{pmatrix} y' & t' \\ x_3 & z' \end{pmatrix}$ is in C.

3° . Since $C \rightrightarrows S_2$ is transitive and $\begin{pmatrix} x_1 & y' \\ x_2 & x_3 \end{pmatrix}, \begin{pmatrix} y' & t' \\ x_3 & z' \end{pmatrix}$ are in C, $\begin{pmatrix} x_1 & t' \\ x_2 & z' \end{pmatrix}$ is in C as desired.

■

3.13. REMARK. If S is a span in a variety $\mathbb{C}$ of universal algebras and C a subalgebra in S_4 as in Theorem 3.12, then the corresponding $m : S_4 \longrightarrow S_1$ is a homomorphism and so we obtain an internal pseudogroupoid structure in $\mathbb{C}$.

Theorem 3.12 (+ Remark 3.13) suggests to consider the following example of the situation — in the original work of J. D. H. Smith [S] — where the notion of *pseudogroupoid* (in fact *pregroupoid*) was almost introduced.

3.14. Example. The Section 2.1 "Centrality in general" in [S] begins by:

> "For this section, let $\underline{\underline{T}}$ be any variety, not necessarily *Mal'tsev.*
>
> 211 Definition. *Let A be a $\underline{\underline{T}}$-algebra, let β, γ be congruences on A, and let $(\gamma|\beta)$ be a congruence on β. Then γ is said* to centralizeβ *by means of the* centralizing congruence $(\gamma|\beta)$ *iff the following conditions are satisfied:*
>
> *(C0):* $(x, y)(\gamma|\beta)(x', y') \Longrightarrow x\gamma x'$.
>
> *(C1):* $\forall (x, y) \in \beta, \pi^\circ : (x, y)^{(\gamma|\beta)} \longrightarrow x^\gamma ; (x', y') \longmapsto x'$ *bijects.*
>
> *(C2): The following three conditions are satisfied:*
>
> *(RR):* $\forall (x, y) \in \gamma, (x, x)(\gamma|\beta)(y, y)$.
>
> *(RS):* $(x, y)(\gamma|\beta)(x', y') \Longrightarrow (y, x)(\gamma|\beta)(y', x')$.
>
> *(RT):* $(x, y)(\gamma|\beta)(x', y')$ *and*
> $(y, z)(\gamma|\beta)(y', z') \Longrightarrow (x, z)(\gamma|\beta)(x', z')$.
>
> Conditions (RR), (RS) and (RT) respectively are known as *respect for the reflexivity, symmetry,* and *transitivity* of β. (C2) is called *respect for equivalence.*
>
> Intuitively, one thinks of the relation $(x, y)(\gamma|\beta)(x', y')$ as a parallelogram
>
> $$\begin{tikzcd} & y \arrow[r, two heads, "\gamma"] & y' \\ x \arrow[ur, dash, "\beta"] \arrow[r, two heads, "\gamma"] & x' \arrow[ur, dash, "\beta"'] \end{tikzcd} \quad ."$$

Let us translate this in our language. First we note that $\underline{\underline{T}}$ plays the same role as our $\mathbb{C}$ which now is supposed to be an arbitrary variety of universal algebras. Since the definition above begins with congruences β, γ (instead of our α, β in (1.9)), we take S to be the span

$$A/\beta \longleftarrow A \longrightarrow A/\gamma \tag{3.34}$$

(in $\mathbb{C}$) instead of (1.9). Then β and γ are the same as our $S_2 \rightrightarrows S_1$ and $S'_2 \rightrightarrows S_1$ respectively. Since $(\gamma|\beta)$ is a congruence on β, i.e. on S_2, let us write it as $(\gamma|\beta) = C \rightrightarrows S_2$. Since C is a subalgebra in $S_2 \times S_2$, the condition (C0) says that C is a subalgebra in S_4; together with (C2) this says that C determines a sub-double equivalence relation of $\mathrm{Eq}(S)$ (in particular (RR), (RS), and (RT) are just reflexivity, symmetry, and transitivity of $C \rightrightarrows S'_2$). Accordingly the parallelogram above corresponds to

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "y"] \arrow[dr, "x"'] & & \cdot \arrow[ul, "y'"'] \arrow[dl, "x'"] \\ & \cdot & \end{tikzcd} \in C. \tag{3.35}$$

The condition (C1) translates now as

$$\forall \begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "y"] \arrow[dr, "x"'] & & \cdot \arrow[dl, "x'"] \\ & \cdot & \end{tikzcd} \exists!\, y' \begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "y"] \arrow[dr, "x"'] & & \cdot \arrow[ul, "y'"'] \arrow[dl, "x'"] \\ & \cdot & \end{tikzcd}, \tag{3.36}$$

i.e. it says that for every $(y, x, x') \in S_3$ (see (3.9)) there exists a unique y' with (3.35). That is, it says that C is thin and, moreover, the corresponding (internal) pseudogroupoid is a pregroupoid.

Summarizing, we can simply say that γ centralizes β by means of $(\gamma|\beta)$ in the sense of J. D. H. Smith [S] if and only if they form (as above) an internal pregroupoid.

4. Free internal and algebraic pseudogroupoids

The category $P(\mathbb{C})$ of internal pseudogroupoids in a variety $\mathbb{C}$ of universal algebras has "all standard properties" plus the existence of the commutative triangle

$$\begin{tikzcd} P(\mathbb{C}) \arrow[rr, "forgetful"] & & S(\mathbb{C}) \\ & R(\mathbb{C}) \arrow[ul, "inclusion"] \arrow[ur, "inclusion"'] & \end{tikzcd}, \tag{4.1}$$

where $S(\mathbb{C})$ is the category of spans in $\mathbb{C}$ and $R(\mathbb{C})$ the category of ("homomorphic") relations in $\mathbb{C}$; the left hand inclusion comes from Example 3.5, which says that every relation has a unique pseudogroupoid structure.

A morphism $\varphi : (S, m) \longrightarrow (T, n)$ in $P(\mathbb{C})$ can be displayed as in $S(\mathbb{C})$, i.e. as

$$\begin{tikzcd} S_0 \arrow[d, "\varphi_0"'] & S_1 \arrow[l, "\pi_s"'] \arrow[r, "\pi'_s"] \arrow[d, "\varphi_1"] & S'_0 \arrow[d, "\varphi'_0"] \\ T_0 & T_1 \arrow[l, "\pi_T"] \arrow[r, "\pi'_T"'] & T'_0 \end{tikzcd}; \tag{4.2}$$

if $\varphi_0, \varphi'_0, \varphi_1$ are inclusions we will say that(S, m) is a subpseudogroupoid in (T, n) and write $(S, m) \leqslant (T, n)$.

The set of subpseudogroupoids in (T, n) forms a complete lattice which we will denote by $\mathrm{Sub}(T, n)$. For any subset X in T_1 the subpseudogroupoid $\langle X\rangle_{(T,n)}$ generated by X is defined as the smallest subpseudogroupoid (S, m) in (T, n) with $X \subset T_1$. It is often convenient to write

$$\langle X\rangle_{(T,n)} = \langle X\rangle \tag{4.3}$$

and identify this pseudogroupoid, as well as other elements in $\mathrm{Sub}(T,n)$ with the corresponding subsets in T_1 — as one usually does in Universal Algebra.

Of course there is a standard procedure to construct $\langle X\rangle$ as the union of

$$X = \langle X\rangle^0 \subset \langle X\rangle^1 \subset \langle X\rangle^2 \subset \dots , \tag{4.4}$$

where $\langle X\rangle^{i+1}$ is the subalgebra in $\langle X\rangle$ generated by

$$\left\{ n(f,g,k,h) \mid f,g,k,h \in \langle X\rangle^i \text{and} \begin{pmatrix} f & k \\ g & h \end{pmatrix} \in T_4 \right\} \tag{4.5}$$

for $i = 0,1,2,\dots$. In particular $\langle X\rangle$ is "as large as X", i.e.

$$\operatorname{card} X \leqslant \operatorname{card}\langle X\rangle \leqslant \max\{\operatorname{card} X, \operatorname{card}\Omega, \aleph_0\}, \tag{4.6}$$

where Ω is the set of operators in the signature of $\mathbb{C}$.

Any morphism $\varphi : (S,m) \longrightarrow (T,n)$ has an image $\varphi(S,m) \in \mathrm{Sub}(T,n)$. However in general it is not simply the set-theoretic image $\varphi_1(S_1) \subset T_1$, but

$$\varphi(S,m) = \langle \varphi_1(S_1)\rangle. \tag{4.7}$$

Any $X \subset T_1$ determines a relation $\mathrm{Rel}(X) \subset T_0 \times T_0'$ by

$$\mathrm{Rel}(X) = \big\{(t,t') \in T_0 \times T_0' \mid \exists x \in X, \pi_T(x) = t,\ \pi_T'(x) = t'\big\}, \tag{4.8}$$

in particular so does T_1 itself and we will write

$$\mathrm{Rel}(T,n) = \mathrm{Rel}(T) = \mathrm{Rel}(T_1), \tag{4.9}$$

and similarly for spans.

4.1. Proposition.

(a) The category $P(\mathbb{C})$ is complete and cocomplete;

(b) the forgetful functor $P(\mathbb{C}) \longrightarrow S(\mathbb{C})$ has a left adjoint $F : S(\mathbb{C}) \longrightarrow P(\mathbb{C})$ such that the diagram

$$\begin{tikzcd} S(\mathbb{C}) \arrow[rr, "F"] \arrow[dr, "\mathrm{Rel}"'] & & P(\mathbb{C}) \arrow[dl, "\mathrm{Rel}"] \\ & R(\mathbb{C}) & \end{tikzcd} \tag{4.10}$$

commute.

Proof. The completeness is obvious, and the cocompleteness and existence of the left adjoint follows from the completeness, (4.6), and (4.7) (although it is also a special case of a well known results for so-called essentially algebraic theories). The commutativity of (4.10) (up to an isomorphism) follows from the commutativity of (4.1) which consists of the right adjoints of the functors involved in (4.10). ∎

Note also that since the diagram

$$\begin{tikzcd} P(\mathbb{C}) \arrow[r, "forgetful"] \arrow[d, "()^{op}"'] & S(\mathbb{C}) \arrow[d, "()^{op}"] \\ P(\mathbb{C}) \arrow[r, "forgetful"'] & S(\mathbb{C}) \end{tikzcd} \tag{4.11}$$

commutes (see *Example* 3.6), so does (up to a canonical isomorphism) also the diagram

$$\begin{tikzcd} S(\mathbb{C}) \arrow[r, "F"] \arrow[d, "()^{op}"'] & P(\mathbb{C}) \cdot \arrow[d, "()^{op}"] \\ S(\mathbb{C}) \arrow[r, "F"'] & P(\mathbb{C}) \end{tikzcd} \tag{4.12}$$

Given a span S in $\mathbb{C}$, we will write

$$F(S) = (S^\star, m) \tag{4.13}$$

and $S^\star$=

$$S_0 \xleftarrow{\pi^\star} S_1^\star \xrightarrow{\pi'^\star} S_0' \tag{4.14}$$

— since we can take

$$S_0^\star = S_0, (S^\star)_0' = S_0' \tag{4.15}$$

as follows from Proposition 4.1(b). The canonical morphism $S \longrightarrow S^\star$ will be written as

$$\begin{tikzcd} & S_1 \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] \arrow[dd, "\eta=\eta_S"] & \\ S_0 & & S_0' \\ & S_1^\star \arrow[ul, "\pi^\star"] \arrow[ur, "\pi'^\star"'] & \end{tikzcd} \tag{4.16}$$

In the pseudogroupoid $F(S)$ we have

$$\langle \eta(S_1) \rangle = S_1^\star \tag{4.17}$$

and so

$$S_1^\star = \bigcup_{n=0}^{\infty} \langle \eta(S_1) \rangle^n \tag{4.18}$$

It might happen that $\langle \eta(S_1) \rangle^{n+1} = \langle \eta(S_1) \rangle^n$ for some n, and then also $S_1^\star = \langle \eta(S_1) \rangle^n$, but if this is the case for every span S in $\mathbb{C}$, then we could say that $\mathbb{C}$ has the *dimension* $\dim(\mathbb{C}) \leqslant n$ — and $\dim(\mathbb{C}) = \infty$ if there is no such n. However we "understand" only the cases $\dim(\mathbb{C}) = 0$ and $\dim(\mathbb{C}) = \infty$. In order to describe the first of them we introduce

4.2. Definition. *An internal pseudogroupoid* (S, m) *in a variety* $\mathbb{C}$ *is said to be* algebraic *if* $\mathbb{C}$ *has a four variable term* q *such that*

$$m(f, g, k, h) = q(f, g, k, h) \tag{4.19}$$

for every S*-diamond* $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$. *We will also say that* (S, m) *is* q-algebraic.

We are going to use the free algebras in $\mathbb{C}$; the free algebra on a set $\{x_1, \dots, x_n\}$ will be denoted by $\mathbb{C}(x_1, \dots, x_n)$.

Consider the span $\mathbb{D}$=

$$\mathbb{C}(s_1, s_2) \xleftarrow{\pi} \mathbb{C}(x_{11}, x_{21}, x_{12}, x_{22}) \xrightarrow{\pi'} \mathbb{C}(t_1, t_2) \tag{4.20}$$

in $\mathbb{C}$ in which

$$\begin{tikzcd}
 & t_1 & \\
s_1 \arrow[ur, "x_{11}"] \arrow[dr, "x_{21}"'] & & s_2 \arrow[ul, "x_{12}"'] \arrow[dl, "x_{22}"] \\
 & t_2 &
\end{tikzcd} \tag{4.21}$$

is a diamond, i.e. π and π' are defined by $\pi(x_{ij}) = s_j$ and $\pi'(x_{ij}) = t_i$ respectively. This span could be called the *generic diamond* in $\mathbb{C}$; it has the following obvious universal property.

4.3. Proposition. *For every span* S *in* $\mathbb{C}$ *and every* S*-diamond* x, *there exists a unique morphism* $\mathbb{D} \longrightarrow S$ *in* $S(\mathbb{C})$ *which sends the diamond* (4.21) *to* x.

In other words there is a canonical bijection

$$S_4 \approx \hom_{S(\mathbb{C})}(\mathbb{D}, S), \tag{4.22}$$

and of course in the case $\mathbb{C}$ = Sets, (4.21) is just the picture of $\mathbb{D}$.

Using the universal property of $\mathbb{D}$ we will prove the following theorem which gives various equivalent conditions for $\dim(\mathbb{C}) = 0$:

4.4. Theorem.

(a) For a given four variable term q *in a variety* $\mathbb{C}$ *the following conditions are equivalent:*

(a_1) every internal pseudogroupoid in $\mathbb{C}$ *is* q*-algebraic;*

(a_2) *the term* q *considered as an element in* $\mathbb{D}_1 = \mathbb{C}(x_{11}, x_{21}, x_{12}, x_{22})$ *via* $q = q(x_{11}, x_{21}, x_{12}, x_{22})$ *satisfies*

$$\eta_{\mathbb{D}}(q) = m\big(\eta_{\mathbb{D}}(x_{11}), \eta_{\mathbb{D}}(x_{21}), \eta_{\mathbb{D}}(x_{12}), \eta_{\mathbb{D}}(x_{22})\big); \tag{4.23}$$

(a_3) *the identities* (3.21) *(recall they are* $q(x, y, x, y) = x$ *and* $q(x, x, y, y) = y$*) hold, and for every diagram of the form*

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "f"] \arrow[dr, "g"'] & & \cdot \arrow[ul, shift left, "k'"'] \arrow[ul, shift right, "k"] \arrow[dl, "h"] \\ & \cdot & \end{tikzcd} \tag{4.24}$$

in any span S *in* $\mathbb{C}$ *we have*

$$\eta_S\big(q(f, g, k, h)\big) = \eta_S\big(q(f, g, k', h)\big). \tag{4.25}$$

(b) *For a given variety* $\mathbb{C}$ *the following conditions are equivalent:*

(b_1) $\dim(\mathbb{C}) = 0$*;*

(b_2) *the homomorphism* $\eta_S : S_1 \longrightarrow S_1^{\star}$ *is surjective for every span* S *in* $\mathbb{C}$*;*

(b_3) *for every span* S *in* $\mathbb{C}$ *and every* S*-diamond* $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$ *there exists* $k' \in S_1$ *with* $\eta_S(k') = m\left(\eta_S(f), \eta_S(g), \eta_S(k), \eta_S(h)\right)$*;*

(b_4) *the homomorphism* $\eta_{\mathbb{D}} : \mathbb{D}_1 \longrightarrow \mathbb{D}_1^{\star}$ *(where* $\mathbb{D}$ *is the span* (4.20) *as above) is surjective;*

(b_5) $\mathbb{C}$ *has a four variable term* q *satisfying the equivalent conditions* (a_1)*–*(a_3)*.*

PROOF. In order to prove (a) we will prove (a_1)$\Leftrightarrow$(a_2), and (a_1)$\Leftrightarrow$(a_3).

(a_1)$\Rightarrow$(a_2). Since the pseudogroupoid $F(\mathbb{D}) = (\mathbb{D}^{\star}, m)$ must be q-algebraic, we have

$$\begin{aligned} & m\big(\eta_{\mathbb{D}}(x_{11}), \eta_{\mathbb{D}}(x_{21}), \eta_{\mathbb{D}}(x_{12}), \eta_{\mathbb{D}}(x_{22})\big) = \\ &= q\big(\eta_{\mathbb{D}}(x_{11}), \eta_{\mathbb{D}}(x_{21}), \eta_{\mathbb{D}}(x_{12}), \eta_{\mathbb{D}}(x_{22})\big) = \\ &\qquad = \eta_{\mathbb{D}}\big(q(x_{11}, x_{21}, x_{12}, x_{22})\big) = \eta_{\mathbb{D}}(q). \end{aligned}$$

$(a_2)\Rightarrow(a_1)$. Given a pseudogroupoid (S,m) and an S-diamond $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$, we take $\varphi : \mathbb{D} \longrightarrow S$ with

$$\begin{pmatrix} \varphi_1(x_{11}) & \varphi_1(x_{12}) \\ \varphi_1(x_{21}) & \varphi_1(x_{22}) \end{pmatrix} = \begin{pmatrix} f & k \\ g & h \end{pmatrix}, \tag{4.26}$$

which does exist by Proposition 4.3. Denoting the morphism $F(\mathbb{D}) \longrightarrow (S,m)$ induced by φ by ψ, we obtain

$$\begin{aligned} q(f,g,k,h) &= q\big(\varphi_1(x_{11}),\varphi_1(x_{21}),\varphi_1(x_{12}),\varphi_1(x_{22})\big) = \\ &= \varphi_1\big(q(x_{11},x_{21},x_{12},x_{22})\big) = \psi_1\eta_{\mathbb{D}}\big(q(x_{11},x_{21},x_{12},x_{22})\big) = \\ &= \psi_1\Big(m\big(\eta_{\mathbb{D}}(x_{11}),\eta_{\mathbb{D}}(x_{21}),\eta_{\mathbb{D}}(x_{12}),\eta_{\mathbb{D}}(x_{22})\big)\Big) = \\ &= m\big(\psi_1\eta_{\mathbb{D}}(x_{11}),\psi_1\eta_{\mathbb{D}}(x_{21}),\psi_1\eta_{\mathbb{D}}(x_{12}),\psi_1\eta_{\mathbb{D}}(x_{22})\big) = \\ &= m\big(\varphi_1(x_{11}),\varphi_1(x_{21}),\varphi_1(x_{12}),\varphi_1(x_{22})\big) = m(f,g,k,h), \end{aligned}$$

and so (S,m) is q-algebraic.

$(a_1)\Rightarrow(a_3)$. In order to prove the identity $q(x,y,x,y) = x$ consider the span

$$1 \longleftarrow \mathbb{C}(x,y) = \!= \mathbb{C}(x,y), \tag{4.27}$$

where 1 is a one element algebra. This is a relation in $\mathbb{C}$ and therefore an internal pseudogroupoid. Applying (4.19) to the diamond $\begin{pmatrix} x & x \\ y & y \end{pmatrix}$ we obtain

$$q(x,y,x,y) = m(x,y,x,y) = x,$$

and since $q(x,y,x,y) = x$ holds in the free algebra $\mathbb{C}(x,y)$, it is an identity in $\mathbb{C}$.

Similarly, using the diamond $\begin{pmatrix} x & y \\ x & y \end{pmatrix}$ in

$$\mathbb{C}(x,y) = \!= \mathbb{C}(x,y) \longrightarrow 1 \tag{4.28}$$

we obtain $q(x,x,y,y) = y$.

For (4.25) we have

$$\begin{aligned} \eta_S\big(q(f,g,k,h)\big) &= q\big(\eta_S(f),\eta_S(g),\eta_S(k),\eta_S(h)\big) = \\ &= m\big(\eta_S(f),\eta_S(g),\eta_S(k),\eta_S(h)\big) = \\ &= m\big(\eta_S(f),\eta_S(g),\eta_S(k'),\eta_S(h)\big) = \\ &= q\big(\eta_S(f),\eta_S(g),\eta_S(k'),\eta_S(h)\big) = \eta_S\big(q(f,g,k',h)\big), \end{aligned}$$

i.e. (4.24) immediately follows from 3.1(b).

(a_3)⇒(a_1): For a given diamond $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$ in a pseudogroupoid (S, m) in $\mathbb{C}$, the identities (3.21) give $\pi\big(q(f,g,k,h)\big) = \pi(h), \pi'\big(q(f,g,k,h)\big) = \pi'(f)$, and then

$$\begin{pmatrix} f & q(f,g,k,h) \\ g & h \end{pmatrix} = q\left(\begin{pmatrix} f & f \\ g & g \end{pmatrix}, \begin{pmatrix} g & g \\ g & g \end{pmatrix}, \begin{pmatrix} f & k \\ g & h \end{pmatrix}, \begin{pmatrix} g & h \\ g & h \end{pmatrix}\right), \tag{4.29}$$

and so we have

$$\begin{aligned} m(f,g,k,h) &= m\big(f,g,q(f,g,k,h),h\big) = \\ &= m\left(q\left(\begin{pmatrix} f & f \\ g & g \end{pmatrix}, \begin{pmatrix} g & g \\ g & g \end{pmatrix}, \begin{pmatrix} f & k \\ g & h \end{pmatrix}, \begin{pmatrix} g & h \\ g & h \end{pmatrix}\right)\right) = \\ &= q\big(m(f,g,f,g), m(g,g,g,g), m(f,g,k,h), m(g,g,h,h)\big) = \\ &= q\big(f,g,m(f,g,k,h),h\big). \end{aligned}$$

On the other hand

$$\eta_S\Big(q\big(f,g,m(f,g,k,h),h\big)\Big) = \eta_S\big(q(f,g,k,h)\big)$$

by (4.25), and we obtain

$$\eta_S\big(m(f,g,k,h)\big) = \eta_S\big(q(f,g,k,h)\big).$$

Since (S, m) is an internal pseudogroupoid, η_S is a (split) monomorphism by a general property of adjoint functors. Therefore we conclude $m(f,g,k,h) = q(f,g,k,h)$ as desired.

(b): The conditions (b_1), (b_2) and (b_3) are clearly equivalent, and (b_2) implies (b_4). Therefore it suffices to prove (b_4)⇒(b_5) and (b_5)⇒(b_3).

(b_4)⇒(b_5). Since $\eta_{\mathbb{D}}$ is surjective, there exists $q \in \mathbb{D}_1$ satisfying (4.23). This means that there exists a four variable term q satisfying (a_2).

(b_5)⇒(b_3): If we take $k' = q(f,g,k,h)$, then

$$\begin{aligned} \eta_S(k') &= \eta_S\big(q(f,g,k,h)\big) = \\ &= q\big(\eta_S(f), \eta_S(g), \eta_S(k), \eta_S(h)\big) = \\ &= m\big(\eta_S(f), \eta_S(g), \eta_S(k), \eta_S(h)\big) \end{aligned}$$

— by (4.19). ■

4.5. Example. Recall that a three variable term p in a variety $\mathbb{C}$ is said to be a *Mal'tsev term* if

$$p(x,y,y) = x, \quad p(x,x,y) = y \tag{4.30}$$

are identities in $\mathbb{C}$. If such a term does exist, $\mathbb{C}$ is said to be a *Mal'tsev variety* (="*congruence permutable*" variety). In this case the four variable term q defined as

$$q(x,y,t,z) = p(x,y,z) \tag{4.31}$$

trivially satisfies 4.4(a_3) and so $\dim(\mathbb{C}) = 0$.

The formula (4.31) should be compared with the formula (3.11) in Example 3.4. In the Mal'tsev case every diagram (3.12) can be completed as

$$\begin{tikzcd}
& \cdot & \\
\cdot \arrow[ur, "f"] \arrow[dr, "g"'] & & \cdot \arrow[ul, "{p(f,g,h)}"'] \arrow[dl, "h"] \\
& \cdot &
\end{tikzcd} \tag{4.32}$$

since $\pi\big(p(f,g,h)\big) = p\big(\pi(f),\pi(g),\pi(h)\big) = p\big(\pi(f),\pi(f),\pi(h)\big) = \pi(h)$ and $\pi'\big(p(f,g,h)\big) = p\big(\pi'(f),\pi'(g),\pi'(h)\big) = p\big(\pi'(f),\pi'(g),\pi'(g)\big) = \pi'(f)$ — and therefore the pseudogroupoids are the same as the pregroupoids (this property is in fact equivalent to $\mathbb{C}$ being Mal'tsev). In the "pregroupoid version" of Theorem 4.4 we would have

$$l(f,g,h) = p(f,g,h) \tag{4.33}$$

instead of (4.19).

Note that the equivalence (a_1)$\Leftrightarrow$(a_3) in Theorem 4.4 gives the following characterization of Mal'tsev varieties: $\mathbb{C}$ is a Mal'tsev variety if and only if it has a three variable term p such that every internal pseudogroupoid in $\mathbb{C}$ is q-algebraic with q defined by (4.31).

Since most of varieties studied in classical Algebra are *Mal'tsev* varieties, let us also point out the following:

(a) If $\mathbb{C}$ is a variety of groups, possibly with an additional algebraic structure (say rings, modules, algebras) then we can take $p(x,y,z) = xy^{-1}z$ and so in every internal pseudogroupoid (=pregroupoid) (S,m) in $\mathbb{C}$ we have

$$m(f,g,k,h) = fg^{-1}h, \tag{4.34}$$

or

$$m(f,g,k,h) = f - g + h \tag{4.35}$$

if the notation is additive.

(b) More generally, if we have the quasigroup structure instead of the group structure, then

$$m(f,g,k,h) = \big(f/(g\backslash g)\big)\cdot(g\backslash h), \tag{4.36}$$

where $\cdot$, $/$, $\backslash$ are the multiplication, the right division, and the left division respectively.

4.6. Example. Suppose that $\mathbb{C}$ has a two variable term u, written as $u(x, y) = xy$, such that

$$(xy)y = xy, \quad xy = yx, \quad xx = x \tag{4.37}$$

are identities in $\mathbb{C}$. Then every internal pseudogroupoid in $\mathbb{C}$ is a relation. Indeed, if f and g are parallel arrows (i.e. $\pi(f) = \pi(g)$ and $\pi'(f) = \pi'(g)$) in an internal pseudogroupoid (S, m) in $\mathbb{C}$, then the third identity in (4.37) tells us that every arrow obtained from f and g by u (and its iterations) is also parallel to f and g, and

$$\begin{aligned} f = m(fg, fg, f, f) &= m(fg, gf, ff, ff) = \\ &= m(f, g, f, f)m(g, f, f, f) = hg, \end{aligned}$$

where $h = m(f, g, f, f)$, and then

$$fg = (hg)g = hg = f$$

— and similarly $gf = g$, which gives $f = fg = gf = g$.

From this and (3.14) we conclude that every internal pseudogroupoid in $\mathbb{C}$ is q-algebraic with

$$q(x, y, t, z) = t; \tag{4.38}$$

in particular $dim(\mathbb{C}) = 0$.

This of course applies to semilattices (with $u(x, y) = x \bigwedge y$ or $u(x, y) = x \bigvee y$), again possibly with an additional structure: lattices, Boolean and Heyting algebras and many other related varieties (although Boolean and Heyting algebras at the same time form Mal'tsev varieties!)

Now consider an example of $\dim(\mathbb{C}) = \infty$:

4.7. Example. Let A be a monoid and $\mathbb{C}$ the variety of A-sets. The monoid ring $\mathbb{Z}[A]$ being an A-module is an internal abelian group in $\mathbb{C}$ and so the span $S =$

$$1 \longleftarrow \mathbb{Z}[A] \longrightarrow 1 \tag{4.39}$$

with the usual $m(f, g, k, h) = f - g + h$ is an internal pseudogroupoid in $\mathbb{C}$. Since A is a subset in $\mathbb{Z}[A]$ we can consider the sequence (4.4) for $X = A$. Clearly $\langle X\rangle^{n+1} \neq \langle X\rangle^n$ for each $n = 0, 1, 2, \ldots$ and so $\dim(\mathbb{C}) = \infty$. In particular $\dim(\mathit{Sets}) = \infty$.

5. The commutator

Let us recall the notion of *subobject.*

Let $\mathbb{C}$ be a category and A an object in $\mathbb{C}$.

If $u : U \longrightarrow A$ and $v : V \longrightarrow A$ are monomorphisms then we write $(U, u) \leqslant (V, v)$ if there exists a morphism $\omega : U \longrightarrow V$ with $v\omega = u$; note that in this case ω is a uniquely determined monomorphism. We say that (U, u) is *equivalent* to (V, v) if $(U, u) \leqslant (V, v)$ and $(V, v) \leqslant (U, u)$; in this case the ω above is an isomorphism. The equivalence class of (U, u) written as $\langle U, u\rangle$ is called a *subobject* in A. The collection of all subobjects in A will be denoted by $\mathrm{Sub}(A)$; it has the induced *partial order* $\leqslant$, so that $\langle U, u\rangle \leqslant \langle V, v\rangle$ if and only if $(U, u) \leqslant (V, v)$.

Recall the following

5.1. Definition. *A category* $\mathbb{C}$ *is said to be* finitely well-complete *if it has*

(a) finite limits;

(b) all (even large) limits of diagrams which are collections of monomorphisms with the same codomain.

If $\mathbb{C}$ is finitely well-complete, then each $\mathrm{Sub}(A)$ is a (possibly large) complete lattice, and each morphism $\varphi : A \longrightarrow B$ induces the adjoint pair

$$\mathrm{Sub}(A) \underset{\varphi^\star}{\overset{\varphi_\star}{\rightleftarrows}} \mathrm{Sub}(B), \tag{5.1}$$

where the right adjoint $\varphi^\star$ sends $\langle V, v\rangle$ to the class of the pullback (="inverse image") of v along φ, and $\varphi_\star$ defined as the left adjoint of $\varphi^\star$ is called the direct image (i.e. $\varphi_\star\langle U, u\rangle$ is called the direct image of $\langle U, u\rangle$). Furthermore, each $\mathrm{Sub}(A \times A)$ contains the complete Λ-subsemilattice $\mathrm{ER}(A)$ of (internal) equivalence relations on A; the elements of $\mathrm{ER}(A)$ are the classes $\langle E, e\rangle$, where $e : E \longrightarrow A \times A$ is an equivalence relation (see Definition 2.1). And again $\varphi : A \longrightarrow B$ induces an adjunction

$$\mathrm{ER}(A) \underset{\varphi^\#}{\overset{\varphi_\#}{\rightleftarrows}} \mathrm{ER}(B) \tag{5.2}$$

such that the diagram

$$\begin{array}{ccc} \mathrm{Sub}(A \times A) & \xleftarrow{(\varphi\times\varphi)^\star} & \mathrm{Sub}(B \times B) \\ \cup & & \cup \\ \mathrm{ER}(A) & \xleftarrow[\varphi^\#]{} & \mathrm{ER}(B) \end{array} \tag{5.3}$$

commutes.

Recall also that every morphism $\varphi : A \longrightarrow B$ determines an equivalence relation $\mathrm{Eq}(\varphi)$ on A (see (2.6)).

5.2. Definition. *Let* $\mathbb{C}$ *be a finitely well-complete category and*

$$S = (\,S_0 \xleftarrow{\pi} S_1 \xrightarrow{\pi'} S_0'\,)$$

a span in $\mathbb{C}$*. The* commutator $[S]$ *of* S *is a congruence on* S_1 *defined as the intersection*

$$[S] = \bigwedge_{\varphi \in \Phi_S} \mathrm{Eq}(\varphi_1) \tag{5.4}$$

where Φ_S *is the collection of all morphisms from* S *to the underlying spans of pseudogroupoids.*

In particular

$$[S] = \Delta_{S_1} \tag{5.5}$$

if S has an internal pseudogroupoid structure.

Note also that if $\varphi = \varphi''\varphi'$ (in $\mathbb{C}$) then $\mathrm{Eq}(\varphi') \leqslant \mathrm{Eq}(\varphi)$, and this gives

5.3. Proposition. *If the forgetful functor* $P(\mathbb{C}) \longrightarrow S(\mathbb{C})$ *has a left adjoint written as* $S \longmapsto (S^\star, m)$ *and* η *is as in* (4.16) *then*

$$[S] = \mathrm{Eq}(\eta); \tag{5.6}$$

in particular, if $\mathbb{C}$ *is a variety of universal algebras, then*

$$[S] = \big\{(x, y) \in S_1 \times S_1 | \eta(x) = \eta(y)\big\}. \tag{5.7}$$

This proposition tells us that the properties of commutators, at least in the case of a variety of universal algebras should be deduced from the properties of the adjunction $S(C) \rightleftarrows P(C)$. However some of them can be obtained directly from the Definition 5.2 without the existence of the left adjoint of the forgetful functor $P(C) \longrightarrow S(C)$.

5.4. Proposition. *Let* $\mathbb{C}$ *be a finitely well-complete category, and* $S = (S_0 \xleftarrow{\pi} S_1 \xrightarrow{\pi'} S_0')$ *a span in* $\mathbb{C}$*. Then:*

(a) *For every morphism* $\chi : S \longrightarrow T$ *in* $S(\mathbb{C})$*,*

$$[S] \leqslant \chi_1^{\#}[T], \quad \chi_{1\#}[S] \leqslant [T]; \tag{5.8}$$

in particular, if $S_1 = T_1$ *and* $\chi_1 = 1_{S_1}$*, then*

$$[S] \leqslant [T] \tag{5.9}$$

(b) $[S^{op}] = [S]$*.*

(c) $[S] \leqslant \mathrm{Eq}(\pi) \wedge \mathrm{Eq}(\pi')$*.*

PROOF.

(a) For every $\varphi \in \Phi_T$ we have

$$\varphi\chi \in \Phi_S \tag{5.10}$$

and

$$\mathrm{Eq}(\varphi_1\chi_1) = \chi_1^{\#}\mathrm{Eq}(\varphi_1) \tag{5.11}$$

since the diagram

$$\begin{tikzcd} S_1 \times_{(\varphi_1\chi_1,\varphi_1\chi_1)} S_1 \arrow[r] \arrow[d] & T_1 \times_{(\varphi_1,\varphi_1)} T_1 \arrow[d] \\ S_1 \times S_1 \arrow[r, "\chi_1\times\chi_1"'] & T_1 \times T_1 \end{tikzcd} \tag{5.12}$$

is obviously a pullback. From (5.11) we obtain $[S] \leqslant \chi_1^{\#}\mathrm{Eq}(\varphi_1)$ and, since $\chi_1^{\#}$ preserves intersections, this gives the first inequality of (5.8). The second one is then obvious since $\chi_{1\#}$ is the left adjoint of $\chi_1^{\#}$.

(b) Follows from the facts that (since the diagram (4.11) commutes) the correspondence $(\varphi : S \longrightarrow P) \longmapsto (\varphi^{op} : S^{op} \longrightarrow P^{op})$ determines a bijection $\Phi_S \longrightarrow \Phi_{S^{op}}$, and obviously $\mathrm{Eq}(\varphi_1) = \mathrm{Eq}(\varphi_1^{op})$ for every $\varphi \in \Phi_S$.

(c) Consider the commutative diagram

$$\begin{tikzcd} S_0 \arrow[d, "1_{S_0}"'] & S_1 \arrow[l, "\pi"'] \arrow[r, "\pi'"] \arrow[d, "\langle\pi,\pi'\rangle"] & S_0' \arrow[d, "1_{S_0'}"] \\ S_0 & S_0 \times S_0' \arrow[l, "pr_1"] \arrow[r, "pr_2"'] & S_0' \end{tikzcd}; \tag{5.13}$$

its bottom line being a product diagram is a relation and therefore an internal pseudogroupoid in $\mathbb{C}$.

Therefore it determines an element φ in Φ_S (with $\varphi_1 = \langle\pi,\pi'\rangle$). Hence

$$[S] \leqslant \mathrm{Eq}(\varphi_1) = \mathrm{Eq}\langle\pi,\pi'\rangle = \mathrm{Eq}(\pi) \wedge \mathrm{Eq}(\pi').$$

∎

If $\mathbb{C}$ is a variety of universal algebras with $\dim(\mathbb{C}) = 0$ then the free pseudogroupoid $F(S) = (S^\star, m)$ on a span S in $\mathbb{C}$ can be described as $S^\star =$

$$S_0 \xleftarrow{\pi^\star} S_1/[S] \xrightarrow{\pi'^\star} S'_0 \tag{5.14}$$

with m defined by q (i.e. by (4.19)), where q is as in Theorem 4.4. From this and Lemma 3.8 we obtain the following

5.5. Theorem. *Let S be a span in a variety $\mathbb{C}$ with $\dim(\mathbb{C}) = 0$, and q any four variable term in $\mathbb{C}$ satisfying the equivalent conditions of Theorem 4.4. Then $[S]$ is the smallest congruence on S_1 such that :*

(a) the composition

$$S_4 \xrightarrow{q|_{S_4}} S_1 \longrightarrow S_1/[S] \tag{5.15}$$

of the restriction of q on $S_4 \subset S_1 \times S_1 \times S_1 \times S_1$ and the canonical homomorphism $S_1 \longrightarrow S_1/[S]$ is a homomorphism;

(b) $(q(f, g, k, h), q(f, g, k', h)) \in [S]$ for every diagram of the form (4.24) in S.

5.6. Corollary. *Let S be a span in a Mal'tsev variety $\mathbb{C}$ and p any Mal'tsev term in $\mathbb{C}$. Then $[S]$ is the smallest congruence on S_1 such that the composition*

$$S_3 \xrightarrow{p|_{S_3}} S_1 \longrightarrow S_1/[S] \tag{5.16}$$

of the restriction of p on $S_3 \subset S_1 \times S_1 \times S_1$ (see (3.9)) and the canonical homomorphism $S_1 \longrightarrow S_1/[S]$ is a homomorphism.

Consider two principal examples:

5.7. Example. Let $\mathbb{C}$ be any variety of groups with p defined as in 4.5(a) and S a span in $\mathbb{C}$. Let K and K' be the kernels

$$\begin{aligned} K &= \{x \in S_1 | \pi(x) = 1\}, \\ K' &= \{x \in S_1 | \pi'(x) = 1\}. \end{aligned} \tag{5.17}$$

Since K and K' are normal subgroups in S_1, so is their (ordinary) commutator $[K, K']$. We claim that

$$[S] = [K, K'], \tag{5.18}$$

i.e. $[S]$ is the congruence on S_1 corresponding to the normal subgroup $[K, K']$. In order to prove this we have to show that $[K, K']$ is the smallest normal subgroup H in S_1 such that

$$f_1 f_2 (g_1 g_2)^{-1} h_1 h_2 H = f_1 g_1^{-1} h_1 f_2 g_2^{-1} h_2 H \tag{5.19}$$

for every $(f_1, g_1, h_1), (f_2, g_2, h_2) \in S_3$.

Note that (5.19) is equivalent to

$$f_2 g_2^{-1} g_1^{-1} h_1 H = g_1^{-1} h_1 f_2 g_2^{-1} H \tag{5.20}$$

and therefore to

$$[f_2 g_2^{-1}, g_1^{-1} h_1] \in H. \tag{5.21}$$

Since $\pi(f_2) = \pi(g_2)$ and $\pi'(g_1) = \pi'(h_1)$, we know that $f_2 g_2^{-1}$ is in K and $g_1^{-1} h_1$ is in K'. Therefore (5.21) holds for $H = [K, K']$.

Conversely, suppose that H satisfies (5.21). If k is an element in K and k' in K', then we can take $(f_1, g_1, h_1) = (1, 1, k'), (f_2, g_2, h_2) = (k, 1, 1)$ — and (5.21) gives $[k, k'] \in H$. That is $[K, K']$ is contained in H.

5.8. Example. Let $\mathbb{C}$ be a variety of rings (not necessarily associative, with or without identity) with p defined by $p(x, y, z) = x - y + z$ and S a span in $\mathbb{C}$. Let K, K' be the kernels

$$\begin{aligned} K &= \{x \in S_1 | \pi(x) = 0\}, \\ K' &= \{x \in S_1 | \pi'(x) = 0\}. \end{aligned} \tag{5.22}$$

We claim that

$$[S] = KK' + K'K, \tag{5.23}$$

i.e. $[S]$ is the congruence on S_1 corresponding to the ideal $KK' + K'K$. In order to prove that we have to show that $KK' + K'K$ is the smallest ideal I is S_1 such that

$$f_1 + f_2 - (g_1 + g_2) + (h_1 + h_2) + I = f_1 - g_1 + h_1 + f_2 - g_2 + h_2 + I \tag{5.24}$$

and

$$f_1 f_2 - g_1 g_2 + h_1 h_2 + I = (f_1 - g_1 + h_1)(f_2 - g_2 + h_2) + I \tag{5.25}$$

for every (f_1, g_1, h_1), $(f_2, g_2, h_2) \in S_3$. However since the addition is commutative, (5.24) holds trivially, and we have to consider only (5.25).

Note that (5.25) is equivalent to

$$-g_1 g_2 + I = -f_1 g_2 + f_1 h_2 - g_1 f_2 + g_1 g_2 - g_1 h_2 + h_1 f_2 - h_1 g_2 + I, \tag{5.26}$$

which itself is equivalent to

$$(f_1 - g_1)(g_2 - h_2) + (g_1 - h_1)(f_2 - g_2) \in I. \tag{5.27}$$

Since $\pi(f_i) = \pi(g_i)$ and $\pi'(g_i) = \pi'(h_i)$ we know that $f_i - g_i$ is in K and $g_i - h_i$ in $K'(i = 1, 2)$. Therefore (5.27) holds for $I = KK' + K'K$.

Conversely, suppose that I satisfies (5.27). If k is an element in K and k' in K', then we can take $(f_1, g_1, h_1) = (k, 0, 0)$, $(f_2, g_2, h_2) = (k', k', 0)$ — and (5.27) gives $kk' \in I$; or, we can take $(f_1, g_1, h_1) = (k', k', 0), (f_2, g_2, h_2) = (k, 0, 0)$ — and (5.27) gives $k'k \in I$. That is $KK' + K'K$ is contained in I.

These two examples together with Example 3.7 suggest to define the *commutator* $[\alpha, \beta]$ of two congruences α and β on an algebra A in a variety $\mathbb{C}$ as

$$[\alpha, \beta] = [S], \tag{5.28}$$

where $S = (A/\alpha \longleftarrow A \longrightarrow A/\beta)$ is the span (1.9); then we get the following:

5.9. Theorem. *Let $\mathbb{C}$ be a congruence modular variety of universal algebras. Then:*

(a) If q is a Kiss difference term in $\mathbb{C}$, then every internal pseudogroupoid in $\mathbb{C}$ is q-algebraic; in particular $\dim(\mathbb{C}) = 0$.

(b) The commutator $[\alpha, \beta]$, of congruences α and β on any algebra A in $\mathbb{C}$, defined by (5.28) *coincides with the modular (i.e. "ordinary") commutator $[\alpha, \beta]$ defined by* (3.19).

Proof.

(a) Let (S, m) be an arbitrary internal pseudogroupoid in $\mathbb{C}$ and $\alpha = \mathrm{Eq}(\pi), \beta = \mathrm{Eq}(\pi')$ the corresponding congruences on S_1; let γ be their modular commutator. Since the set $C \subset S_4$ of commutative S-diamonds forms an (internal) sub-double equivalence relation of $\mathrm{Eq}(S)$, it contains $\Delta_{\alpha,\beta}$. Therefore by (3.19) we conclude

$$(x, y) \in \gamma \Longrightarrow \begin{pmatrix} x & x \\ x & y \end{pmatrix} \in C \Longrightarrow x = m(x, x, x, y) \Longrightarrow x = y,$$

i.e. $\gamma = \Delta_{S_1}$. Together with (3.22) this tells us that

$$q(f, g, k, h) = q(f, g, k', h) \tag{5.29}$$

for every diagram of the form (4.24). Since this is true for all pseudogroupoids, and in particular for the free ones, the pair $(\mathbb{C}, q)$ satisfies the condition 4.4(a_3) — which proves (a).

(b) Given an algebra A in $\mathbb{C}$ and congruences α and β on A consider the span

$$A/\alpha \longleftarrow A/[\alpha, \beta] \longrightarrow A/\beta, \tag{5.30}$$

where $[\alpha, \beta]$ is the modular commutator of α and β. Denoting the canonical homomorphism $A \longrightarrow A/[\alpha, \beta]$ by φ, we can rewrite (5.30) as

$$\big(A/[\alpha, \beta]\big)/\varphi_{\#}\alpha \longleftarrow A/[\alpha, \beta] \longrightarrow \big(A/[\alpha, \beta]\big)/\varphi_{\#}\beta \tag{5.31}$$

and for the modular commutator $[\varphi_{\#}\alpha, \varphi_{\#}\beta]$ we have

$$[\varphi_{\#}\alpha, \varphi_{\#}\beta] = \Delta_{A/[\alpha,\beta]}, \tag{5.32}$$

as follows from [G, Corollary 6.17]. Hence, from the remark at the end of Example 3.7 we conclude that the span (5.30) has a (unique) internal pseudogroupoid structure.

By the universal property of the free internal pseudogroupoid we then conclude that the modular commutator $[\alpha, \beta]$ contains the one defined by (5.28).

In order to prove the converse we have to prove that

$$(a, b) \in [\alpha, \beta] \Longrightarrow \eta_S(a) = \eta_S(b), \tag{5.33}$$

where $S = (A/\alpha \longleftarrow A \longrightarrow A/\beta)$ as above.

Let $C_{\alpha,\beta}$ be the set of *potentially commutative* S-diamonds, i.e. those S-diamonds $\begin{pmatrix} f & k \\ g & h \end{pmatrix}$ for which

$$\begin{pmatrix} \eta_S(f) & \eta_S(k) \\ \eta_S(g) & \eta_S(h) \end{pmatrix} \tag{5.34}$$

is a commutative diamond in the free internal pseudogroupoid $F(S)$. Then clearly $C_{\alpha,\beta}$ determines an (internal) sub-double equivalence relation of $\mathrm{Eq}(S)$ and so $\Delta_{\alpha,\beta} \subset C_{\alpha,\beta}$. Hence if (a, b) is in $[\alpha, \beta]$, then $\begin{pmatrix} a & a \\ a & b \end{pmatrix}$ is in $C_{\alpha,\beta}$ and so $\eta_S(a) = q(\eta_S(a), \eta_S(a), \eta_S(a), \eta_S(b)) = \eta_S(b)$.

∎

5.10. Remark.

(a) If we assume the existence of a left adjoint to the forgetful functor $P(\mathbb{C}) \longrightarrow S(\mathbb{C})$, then (5.6) can be used as the definition of the commutator; in this situation $\mathbb{C}$ does not need to be finitely well-complete, but we still need the finite limits. If in addition $\mathbb{C}$ has coequalizers of effective equivalence relations, then every commutator $[S]$ can be written as $[\alpha, \beta]$ for appropriate effective equivalence relations α and β. For, given a span $S = (S_0 \xleftarrow{\pi} S_1 \xrightarrow{\pi'} S_0')$ in $\mathbb{C}$, we construct the new span $\tilde{S} =$

$$S_1/\mathrm{Eq}(\pi) \longleftarrow S_1 \longrightarrow S_1/\mathrm{Eq}(\pi'), \tag{5.35}$$

and it is easy to see that the free internal pseudogroupoid $F(\tilde{S})$ satisfies the universal property as need for $F(S)$; therefore

$$[S] = [\mathrm{Eq}(\pi), \mathrm{Eq}(\pi')]. \tag{5.36}$$

More generally, for every morphism $\chi = (\chi_0, \chi_1, \chi_0')\colon\ S \longrightarrow T$ in $S(\mathbb{C})$ in which $S_1 = T_1,\ \chi_1 = 1_{S_1}$, and χ_0 and χ_0' are monomorphisms (which gives $S_4 = T_4$), we have

$$[S] = [T] \tag{5.37}$$

(b) The "new" approach to commutators described in (a) easily gives all properties listed in Proposition 5.4.

If we replace $[S]$ by $[\alpha, \beta]$, they should be written as (we omit here (5.8)):

$$(\alpha_1 \leqslant \alpha_2, \beta_1 \leqslant \beta_2) \Longrightarrow [\alpha_1, \beta_1] \leqslant [\alpha_2, \beta_2], \tag{5.38}$$

$$[\alpha, \beta] = [\beta, \alpha], \tag{5.39}$$

$$[\alpha, \beta] \leqslant \alpha \wedge \beta. \tag{5.40}$$

(c) In the proof of 5.9(b), we used the important property (5.32) of the modular commutator. Does this property hold for our commutator also in non-modular cases? We know only two simple cases: the trivial case where all commutators are trivial (for example for $\mathbb{C} = Sets$), and the case of a variety $\mathbb{C}$ with $\dim(\mathbb{C}) = 0$, where (5.32) follows from (5.5) and is equivalent to the isomorphism

$$FF \approx F \tag{5.41}$$

for the free functor $F:\ S(\mathbb{C}) \longrightarrow P(\mathbb{C})$.

(d) Let α, β be congruences on an algebra A in an arbitrary variety $\mathbb{C}$ and $C_{\alpha,\beta}$ the algebra of potentially commutative diamonds as in the proof of 5.9(b). From (5.7) we conclude

$$[\alpha, \beta] = \left\{ (a, b) \in A \times A | \begin{pmatrix} a & a \\ a & b \end{pmatrix} \in C_{\alpha,\beta} \right\}, \tag{5.42}$$

which is similar to (3.19) (in which $\begin{pmatrix} a & a \\ a & b \end{pmatrix}$ was written as $((a, a), (a, b))$ since $\Delta_{\alpha,\beta}$ was considered as a congruence on α. Another known formula for the modular commutator mentioned in [Ki] is

$$[\alpha, \beta] = \Big\{ (a, b) \in A \times A | \exists c \in A, \big((a, c), (b, c)\big) \in \Delta_{\alpha,\beta} \Big\}, \tag{5.43}$$

and again, we can replace it in the general case by

$$[\alpha,\beta] = \left\{(a,b) \in A \times A | \exists c \in A, \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in C_{\alpha,\beta}\right\}. \tag{5.44}$$

One might notice that (5.43) and (5.44) would better agree with (3.19) and (5.42) respectively if we replace $\begin{pmatrix} a & b \\ c & c \end{pmatrix}$ by $\begin{pmatrix} c & a \\ c & b \end{pmatrix}$ (and $((a,c),(b,c))$ by $((c,c),(a,b))$); these modified formulas would also be true of course.

All these suggest that in the modular case

$$\Delta_{\alpha,\beta} = C_{\alpha,\beta} \tag{5.45}$$

— not just $\Delta_{\alpha,\beta} \subset C_{\alpha,\beta}$ as we observed in the proof of 5.9(b). In fact this is true: if $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is in $C_{\alpha,\beta}$, then $\eta(q(a,b,c,d)) = \eta(c)$, i.e. $(q(a,b,c,d),c) \in [\alpha,\beta]$ and then $\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ((a,b),(c,d))$ is in $\Delta_{\alpha,\beta}$ by [Ki, Theorem 3.8(ii)]. That is, (5.45) easily follows from the results of [Ki] and Theorem 5.9.

(e) It is of course well known that the modular commutator in the cases of groups and rings becomes the ordinary one as described in the examples 5.7 and 5.8 respectively. However our purpose in those examples was to show that such description can be easily deduced directly from our definition of the commutator.

6. Kiss, Gumm, Lipparini, and abelianizable varieties

In this section $\mathbb{C}$ denotes an arbitrary variety of universal algebras, A an algebra in $\mathbb{C}$, α and β congruences on A, and $[\alpha,\beta]$ their commutator defined by (5.28) (see also (5.7)); $\nabla_A = A \times A$ will denote the largest congruence on A.

6.1. Definition. *A variety $\mathbb{C}$ is said to be a*

(a) Kiss variety *if it has a four variable term q with*

$$\begin{gathered} q(a,b,a,b) = a, q(a,a,b,b) = b, \\ \big(q(a,b,c,d), q(a,b,c',d)\big) \in [\alpha,\beta] \end{gathered} \tag{6.1}$$

for every $a,b,c,c',d \in A \in \mathbb{C}$ and congruences α,β on A with

$$\begin{tikzcd} & \cdot & \\ \cdot \arrow[ur, "a"] \arrow[dr, "b"'] & & \cdot \arrow[ul, "c"', shift right] \arrow[ul, "c'"', shift left] \arrow[dl, "d"] \\ & \cdot & \end{tikzcd} \tag{6.2}$$

in $A/\alpha \leftarrow A \rightarrow A/\beta$ (i.e. with $a\alpha b$, $c\alpha d$, $c'\alpha d$, $a\beta c$, $a\beta c'$, $b\beta d$);

(b) Gumm variety *if it has a three variable term p with*

$$p(a,b,b) = a, \big(p(a,a,b),b\big) \in [\alpha,\alpha] \tag{6.3}$$

for every $a,b \in A \in \mathbb{C}$ and congruence α on A with $a\alpha b$;

(c) Lipparini variety *if it has a three variable term p with*

$$\big(p(a,b,b),a\big), \big(p(a,a,b),b\big) \in [\alpha,\alpha] \tag{6.4}$$

for every a,b,A,α as in (b);

(d) abelianizable variety *if it has a three variable term p with*

$$\big(p(a,b,b),a\big), \big(p(a,a,b),b\big) \in [\nabla_A,\nabla_A] \tag{6.5}$$

for every $a,b, \in A \in \mathbb{C}$.

Of course $\mathbb{C}$ is a Kiss variety if and only if it satisfies the equivalent conditions (b_1)–(b_5) of Theorem 4.4, i.e. if and only if $\dim(\mathbb{C}) = 0$.

In particular we can say that $\mathbb{C}$ is a Kiss variety if and only if every internal pseudogroupoid in $\mathbb{C}$ is q-algebraic — and we can take the same q as in 6.1(a). It turns out that each of the types of varieties introduced in Definition 6.1 can be described as a variety in which certain internal pseudogroupoids are algebraic.

6.2. Theorem. *A variety in $\mathbb{C}$ is a*

(a) Gumm variety if and only if it has a four variable term q such that every internal pseudogroupoid $(S,m) = (S_0 \xleftarrow{\pi} S_1 \xrightarrow{\pi'} S_0, m)$ in $\mathbb{C}$ with

$$\mathrm{Eq}(\pi) \leqslant \mathrm{Eq}(\pi') \tag{6.6}$$

is q-algebraic;

(b) Lipparini variety if and only if it has a four variable term q such that every internal pseudogroupoid (S,m) in $\mathbb{C}$ with

$$\mathrm{Eq}(\pi) = \mathrm{Eq}(\pi') \tag{6.7}$$

is q-algebraic;

(c) abelianizable variety if and only if it has a four variable term q such that every internal pseudogroupoid (S,m) in $\mathbb{C}$ with

$$\mathrm{Eq}(\pi) = \nabla_{S_1} = \mathrm{Eq}(\pi') \tag{6.8}$$

is q-algebraic.

PROOF. We have to reconsider the proof of $(a_3)\Rightarrow(a_1)$ of Theorem 4.4. In fact the arguments used there prove the following (stronger) assertion:

($\star$) Let q be a four variable term in $\mathbb{C}$ and (S,m) an internal pseudogroupoid in $\mathbb{C}$ such that $q(f,g,f,g)=f$, $q(f,f,g,g)=g$, $(q(f,g,k,h),\, q(f,g,k',h))\in[S]$ for every diagram of the form (4.24) in S. Then (S,m) is q-algebraic.

We will use this in the proofs of (a), (b) and (c).

(a) Let $\mathbb{C}$ be a Gumm variety, and (S,m) an internal pseudogroupoid in $\mathbb{C}$ satisfying (6.6). We define q by the usual formula (4.31) (although now p is just a three variable term satisfying (6.3)), and we have

$$q(f,g,f,g)=p(f,g,g)=f,$$
$$q(f,g,k,h)=p(f,g,h)=q(f,g,k',h),$$

where f,g,k,k',h are as in ($\star$). So in order to prove that (S,m) is q-algebraic, we need to prove only $q(f,f,g,g)=g$, i.e. $p(f,f,g)=g$.

Since (S,m) is an internal pseudogroupoid, we have $[S]=\Delta_{S_1}$ (see (5.5)), after that $\mathrm{Eq}(\pi)\leqslant\mathrm{Eq}(\pi')$ gives

$$\big[\mathrm{Eq}(\pi),\mathrm{Eq}(\pi)\big]\leqslant\big[\mathrm{Eq}(\pi),\mathrm{Eq}(\pi')\big]=[S]=\Delta_{S_1},$$

and since $(p(f,f,g),g)\in[\mathrm{Eq}(\pi),\mathrm{Eq}(\pi)]$ by (6.3), we obtain $p(f,f,g)=g$.

Conversely, suppose that $\mathbb{C}$ has a four variable term q such that every internal pseudogroupoid (S,m) in $\mathbb{C}$ satisfying (6.6) is q-algebraic. We define p by

$$p(x,y,z)=q(x,y,x,z) \tag{6.9}$$

and we have to prove (6.3).

Since the span

$$A = A \longrightarrow 1 \tag{6.10}$$

is a relation, it has an internal pseudogroupoid structure; moreover, that pseudogroupoid satisfies (6.6) and so is q-algebraic. Since $\begin{pmatrix} a & b \\ a & b\end{pmatrix}$ is a diamond in the span (6.10), we obtain

$$p(a,b,b)=q(a,b,a,b)=m(a,b,a,b)=a.$$

The span $S=$

$$A/\alpha \longleftarrow A \longrightarrow A/\alpha \tag{6.11}$$

might not have an internal pseudogroupoid structure, but we can consider the free internal pseudogroupoid $F(S)$ which still satisfies (6.6) since $F(S)^{op} = F(S^{op}) = F(S)$, and since $\begin{pmatrix} a & a \\ a & b \end{pmatrix}$ is a diamond in (6.11) we have

$$\begin{aligned}\eta_S\big(p(a,a,b)\big) &= p\big(\eta_S(a),\eta_S(a),\eta_S(b)\big) = \\ &= q\big(\eta_S(a),\eta_S(a),\eta_S(a),\eta_S(b)\big) = \\ &= m\big(\eta_S(a),\eta_S(a),\eta_S(a),\eta_S(b)\big) = \\ &= m\big(\eta_S(a),\eta_S(a),\eta_S(b),\eta_S(b)\big) = \eta_S(b),\end{aligned}$$

i.e. $(p(a,a,b),b) \in [\alpha,\alpha]$.

(b) Let $\mathbb{C}$ be a Lipparini variety and (S,m) an internal pseudogroupoid in $\mathbb{C}$ satisfying (6.7). We define q by (4.31) again and we have $q(f,g,k,h) = q(f,g,k',h)$ as above. Since again $[\mathrm{Eq}(\pi),\mathrm{Eq}(\pi)] = \Delta_{S_1}$, (6.4) gives

$$\begin{aligned} q(f,g,f,g) &= p(f,g,g) = f, \\ q(f,f,g,g) &= p(f,f,g) = g, \end{aligned}$$

and so (S,m) is q-algebraic by $(\star)$.

Conversely, suppose $\mathbb{C}$ has a four variable term q such that every internal pseudogroupoid (S,m) in $\mathbb{C}$ satisfying (6.7) is q-algebraic. We define p by (6.9) again and using the span (6.11) we obtain

$$\begin{aligned}\eta_S\big(p(a,b,b)\big) &= p(\eta_S(a),\eta_S(b),\eta_S(b)) = \\ &= q\big(\eta_S(a),\eta_S(b),\eta_S(a),\eta_S(b)\big) = \\ &= m\big(\eta_S(a),\eta_S(b),\eta_S(a),\eta_S(b)\big) = \eta_S(a),\end{aligned}$$

$\eta_S(p(a,a,b)) = \eta_S(b)$ (as in the proof of (c) above), i.e. we obtain (6.4).

The proof (c) is again similar.

■

6.3. Remark.

(a) As we see from the proof, we could add "with q defined by $q(x,y,t,z) = p(x,y,z)$, where p is as in the definition" to any of the conditions (a), (b), (c) of Theorem 6.2; in particular we could take q to be independent of the third variable. However if we require q to be independent of the third variable in the definition of Kiss variety, then we obtain the definition of Mal'tsev variety.

(b) Let $\underset{\sim}{M}$, $\underset{\sim}{CM}$, $\underset{\sim}{K}$, $\underset{\sim}{G}$, $\underset{\sim}{L}$, $\underset{\sim}{A}$, and $\underset{\sim}{V}$ be the collections of Mal'tsev, congruence modular, Kiss, Gumm, Lipparini, abelianizable, and all varieties respectively. Then clearly

$$\underset{\sim}{M} \subset \underset{\sim}{CM} \subset \underset{\sim}{K} \subset \underset{\sim}{G} \subset \underset{\sim}{L} \subset \underset{\sim}{A} \subset \underset{\sim}{V} \tag{6.12}$$

and each inclusion seems to be strict. In particular: $\underset{\sim}{M} \neq \underset{\sim}{CM}$ is well known; the variety considered in Example 4.6 (and also, say, the variety of commutative idempotent semigroups) is in $\underset{\sim}{K}$, but not in $\underset{\sim}{CM}$; we do not know how to prove $\underset{\sim}{K} \neq \underset{\sim}{G}$, and $\underset{\sim}{G} \neq \underset{\sim}{L}$, although we think $\underset{\sim}{K} \neq \underset{\sim}{G}$ should be related to Problem 3.11 in [Ki][1], and $\underset{\sim}{G} \neq \underset{\sim}{L}$ should follow from the results of [L1]; Example 7.5 below shows that $\underset{\sim}{L} \neq \underset{\sim}{A}$; the category of sets is in $\underset{\sim}{V}$ but not in $\underset{\sim}{A}$, and so $\underset{\sim}{A} \neq \underset{\sim}{V}$.

In addition, let $\underset{\sim}{N}$ be the collection of varieties in which any algebra A is *neutral*, that is

$$[\alpha, \beta] = \alpha \wedge \beta \tag{6.13}$$

for every two congruences α and β on A. Consider the intersections of $\underset{\sim}{N}$ with the "classes" involved in (6.12). The classes $\underset{\sim}{N} \cap \underset{\sim}{M}$ and $\underset{\sim}{N} \cap \underset{\sim}{CM}$ are well known and important: a variety is said to be *arithmetical* if it belongs to $\underset{\sim}{N} \cap \underset{\sim}{M}$; a variety $\mathbb{C}$ is in $\underset{\sim}{N} \cap \underset{\sim}{CM}$ if and only if it is *congruence distributive*, i.e. the lattice $\mathrm{ER}(A) = \mathrm{Cong}(a)$ of congruences on A is distributive for every algebra A in $\mathbb{C}$. The other intersections are just $\underset{\sim}{N}$ since $\underset{\sim}{N} \subset \underset{\sim}{K}$(with q defined by (4.38)). Note that $\underset{\sim}{N} \cap \underset{\sim}{CM} \neq \underset{\sim}{N}$ by 4.6.

(c) The conditions on p used in 6.1(b)–(d) say that it is an F-G-difference term in the sense of [L1] for certain F and G, and so various results of [L1] can be applied to what we call Gumm, Lipparini, and abelianizable varieties.

Let (S, m) be an internal pseudogroupoid (in a variety $\mathbb{C}$) satisfying (6.6). Then for every $(f, g, h) \in S_3$, $\begin{pmatrix} f & f \\ g & h \end{pmatrix}$ is an S-diamond, and therefore (S, m) is an internal pregroupoid. Together with Remark 6.3(a) this suggests to introduce

6.4. Definition. *Let p be a three variable term in a variety $\mathbb{C}$. An internal pregroupoid (S, l) in $\mathbb{C}$ is said to be p-algebraic if* (4.33) *holds (i.e. $l(f, g, h) = p(f, g, h)$) for every $(f, g, h) \in S_3$.*

— and rewrite Theorem 6.2 as

6.5. Theorem. *A variety $\mathbb{C}$ is a*

(a) Gumm variety if and only if it has a three variable term p such that every internal pregroupoid in $\mathbb{C}$ satisfying (6.6) *is p-algebraic;*

(b) Lipparini variety if and only if it has a three variable term p such that every internal pregroupoid in $\mathbb{C}$ satisfying (6.7) *is p-algebraic;*

[1]Lipparini gave a surprising solution of the Kiss's problem, which translated in our language implies $\underset{\sim}{K} = \underset{\sim}{G}$.

(c) abelianizable variety if and only if it has a three variable term p such that every internal pregroupoid in $\mathbb{C}$ satisfying (6.8) *is p-algebraic.*

Instead of Theorem 5.5 we have

6.6. Theorem. *Let p be a three variable term in a variety $\mathbb{C}$ and S a span in $\mathbb{C}$ such that one of the following conditions hold:*

(i) $\mathbb{C}$ is a Gumm variety and S satisfies (6.6)*;*

(ii) $\mathbb{C}$ is a Lipparini variety and S satisfies (6.7)*;*

(iii) $\mathbb{C}$ is an abelianizable variety and S satisfies (6.8)*.*

Then $[S]$ is the smallest congruence on S_1 such that:

(a) $(p(f,g,g),f) \in [S]$ if $\pi(f) = \pi(g)$ and $(p(g,h,h),h) \in [S]$ if $\pi'(g) = \pi'(h)$ (of course $(p(f,g,g),f) \in [S]$ can be omitted in the case (i));

(b) the composition

$$S_3 \xleftarrow{p|S_3} S_1 \longrightarrow S_1/[S] \tag{6.14}$$

of the restriction of p on $S_3 \subset S_1 \times S_1 \times S_1$ and the canonical homomorphism $S_1 \longrightarrow S_1/[S]$ is a homomorphism.

6.7. Remark. If p was a Mal'tsev term, then 6.6(a) can be omitted, which agree with Corollary 5.6. If $\mathbb{C}$ is, say, a variety of semilattices (see Example 4.6) and (S,m) satisfies (6.8) then we can take any three variable term as p, for example $p(x,y,z) = x \wedge y \wedge z$; in this case 6.6(a) is relevant, but 6.6(b) can be omitted!

7. Abelian algebras and abelianization

Let (S,m) be an internal pseudogroupoid in a category $\mathbb{C}$ with finite limits, in which S has the form

$$1 \longleftarrow A \longrightarrow 1, \tag{7.1}$$

where 1 is a terminal object and A an arbitrary object in $\mathbb{C}$. Of course such a pseudogroupoid is a pregroupoid and in fact simply an *internal herd*, i.e. an object A together with a morphism

$$l : A \times A \times A \longrightarrow A \tag{7.2}$$

satisfying the identities

$$\begin{gathered} l(x,y,y) = x, \qquad l(x,x,y) = y, \\ l\big(l(x_1,x_2,x_3),x_4,x_5\big) = l\big(x_1,x_2,l(x_3,x_4,x_5)\big) \end{gathered} \tag{7.3}$$

(written "in terms of elements") as in 3.4. Here m and l are related by (3.11) and we will write

$$(S, m) = (A, l). \tag{7.4}$$

Let us recall from [FMK, Definition 5.4]:

7.1. Definition. *An algebra A in a variety $\mathbb{C}$ is said to be* affine*, if $\mathbb{C}$ has a three variable term p and A has an abelian group structure such that*

(a) $p(a, b, c) = a - b + c$ for all $a, b, c \in A$

(b) $t(a_1 - b_1 + c_1, \dots , a_n - b_n + c_n) = t(a_1, \dots , a_n) - t(b_1, \dots , b_n) + t(c_1, \dots , c_n)$ for each n variable $(n = 0, 1, \dots)$ term t in $\mathbb{C}$ and $a_1, \dots , a_n, b_1, \dots , b_n, c_1, \dots , c_n \in A$.

From this definition we easily conclude

7.2. Proposition. *An algebra A in a variety $\mathbb{C}$ is affine if and only if $\mathbb{C}$ has a three variable term p such that the formula*

$$l(a, b, c) = p(a, b, c) \tag{7.5}$$

(just the special case of (4.33)*!) defines an internal herd structure on A.*

Following the usual terminology we introduce

7.3. Definition. *An algebra A in a variety $\mathbb{C}$ is said to be* abelian *if $[\nabla_A, \nabla_A] = \Delta_A$.*

The following theorem generalizes the "fundamental theorem on abelian algebras":

7.4. Theorem.

(a) Every affine algebra in any variety is abelian;

(b) every abelian algebra in an abelianizable variety is affine with p as in 6.1(d).

Proof.

(a) follows from Proposition 7.2 and the fact that $[S] = \Delta_{S_1}$ for every internal pseudogroupoid (S, m).

(b) follows from Theorem 6.5(c) and Proposition 7.2 (we fix any element e in A and define $a+b = p(a, e, b)$; the commutativity of $+$ follows from 7.1(b) applied to $t = p$).

■

For a given algebra A in a variety $\mathbb{C}$ and congruences α and β on A we consider again the span $S = (A/\alpha \longleftarrow A \longrightarrow A/\beta)$(see (1.9)) and define

$$\begin{gathered} M_{\alpha,\beta} = \text{the subalgebra of } S_4 \text{ generated by all} \\ \begin{pmatrix} a & a \\ a' & a' \end{pmatrix} \text{ and } \begin{pmatrix} b & b' \\ b & b' \end{pmatrix} \text{ with } a\alpha a' \text{and } b\beta b' \end{gathered} \tag{7.6}$$

That is, our $M_{\alpha,\beta}$ is the same $M(\alpha, \beta)$ in [FMK, Definition 3.2(1) and Proposition 3.3(1)] and [L1, 1.1], and clearly

$$M_{\alpha,\beta} \subset \Delta_{\alpha,\beta} \subset C_{\alpha,\beta} \tag{7.7}$$

(see also Remark 5.10(d), where the relationship between $\Delta_{\alpha,\beta}$ and $C_{\alpha,\beta}$ is discussed).

According [FMK] and [L1] we say that α *centralizes* β modulo a congruence γ on A if

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in M_{\alpha,\beta} \text{ and } a\gamma c \text{ implies } b\gamma d \tag{7.8}$$

(we do not need to consider here the well known relationship between this notion and the one introduced in [S] see Example 3.14); we also write $C(\alpha, \beta; \gamma)$ in this case. The smallest congruence γ with this property is denoted by $C(\alpha, \beta)$.

This $C(\alpha, \beta)$ is one of "several commutators" in [FMK], which coincide in the case of a congruence modular variety; in [L1] $C(\alpha, \beta)$ is written as $[\alpha, \beta]$ (which we will not do in this paper!) and used as *the commutator*. Since $M_{\alpha,\beta} \subset C_{\alpha,\beta}$ and our commutator $[\alpha, \beta]$ obviously satisfies (7.8), we have

$$C(\alpha, \beta) \leqslant [\alpha, \beta]. \tag{7.9}$$

In particular this tells us that if a variety has a *weak difference term* in the sense of [L1, Definition 2.1], i.e. satisfies the condition similar to 6.1(c) but with $C(\alpha, \alpha)$ instead of $[\alpha, \alpha]$, then it is a Lipparini variety and so Theorem 7.4 above extends the corresponding result (namely 5.9(i)⇔(ii) of [L1]).

Consider

7.5. Example. Let $\mathbb{C}$ be a variety determined by a binary operator written $(x, y) \longmapsto xy$ and a 0-ary operator 1 with a set of identities including

$$1x = x = x1 \tag{7.10}$$

If (G, l) is an internal herd in $\mathbb{C}$, then

$$a \cdot b = l(a, 1, b) \tag{7.11}$$

defines a group structure on G with the identity 1 and

$$(ab) \cdot (cd) = (a \cdot c)(b \cdot d) \tag{7.12}$$

(since l must be a homomorphism) for all a, b, c, d in G. It is well known that this gives

$$a \cdot b = ab = b \cdot a \tag{7.13}$$

for all a, b in G, and so $(G, \cdot) = (G \text{ with } (x, y) \longmapsto xy)$ is an abelian group.

Such a $\mathbb{C}$ might not be abelianizable. For example the variety of all monoids *is not*, since there the free internal pseudogroupoid on a span (7.1) is just

$$1 \longleftarrow K(A) \longrightarrow 1 \tag{7.14}$$

where $K(A)$ is the Grothendieck group of A.

So let us require a new identity, say

$$u = v, \tag{7.15}$$

where u and v are one variable terms. In the case of monoids (7.15) would be equivalent to

$$x^n = x^m \tag{7.16}$$

for some n and m, and the same argument with the Grothendieck group tells us that $\mathbb{C}$ is abelianizable if and only if $n \neq m$. Furthermore, all abelian algebras are trivial if and only if $|n - m| = 1$.

This gives many varieties which are abelianizable but not Lipparini varieties (in particular they do not have a weak difference term in the sense of [L1]). For, we fix any identity (7.15) with, say $n > m > 0$ in the corresponding identity (7.16), and consider an algebra A in which

$$ab = \begin{cases} b & \text{if } a = 1, \\ a & \text{if } a \neq 1; \end{cases} \tag{7.17}$$

clearly such A satisfies the identity above. Let α be the congruence on A defined by

$$\alpha = \{(1, 1)\} \cup \Big(\big(A \backslash \{1\}\big) \times \big(A \backslash \{1\}\big) \Big). \tag{7.18}$$

We claim that the span $S = A/\alpha \longleftarrow A \longrightarrow A/\alpha$ has an internal pseudogroupoid structure: just note that S_4 satisfies (7.17) and any pseudogroupoid structure (in Set) $m : S_4 \longrightarrow S_1 = A$ with $m(1) = 1$ is a homomorphism. Therefore $[\alpha, \alpha] = \Delta_A$ and it is easy to see that there is no p satisfying (6.4) for all such A.

Consider the adjunction

$$(\text{Groups}) \underset{\textit{inclusion}}{\overset{\textit{abelianization}}{\rightleftarrows}} (\text{Abelian groups}) \tag{7.19}$$

The *abelianization* functor can be described as

$$G \longmapsto G/[G,G] = G/[\nabla_G, \nabla_G], \tag{7.20}$$

and $[G,G]$ can be described as the smallest normal subgroup in G such that the composition

$$G \times G \xrightarrow{(x,y)\mapsto xy} G \longrightarrow G/[G,G] \tag{7.21}$$

is a homomorphism — or, equivalently the composition

$$G \times G \times G \xrightarrow{(x,y,z)\mapsto xy^{-1}z} G \longrightarrow G/[G,G] \tag{7.22}$$

is a homomorphism, which is a special case of Theorem 6.6 of course.

In general, Theorem 6.6 describes the *abelianization* functor involved in the adjunction

$$\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "\text{abelianization:} A\mapsto A/[\nabla_A,\nabla_A]"] \arrow[dd, "\wr"'] & \begin{pmatrix}\text{Abelian}\\ \text{Algebras}\\ \text{in } \mathbb{C}\end{pmatrix} = \begin{pmatrix}\text{Affine}\\ \text{algebras}\\ \text{in } \mathbb{C}\end{pmatrix} \arrow[l, shift left, "\text{inclusion}"] \arrow[d, "\wr"] \\
 & (\text{Internal herds in } \mathbb{C}) \arrow[d, "\wr"] \\
\begin{pmatrix}\text{Spans } S \text{ in } \mathbb{C}\\ \text{with } S_0 = 1 = S_0'\end{pmatrix} \arrow[r, shift left, "\text{free}"] & \begin{pmatrix}\text{Internal pseudogroupoids}\\ (S,m) \text{ in } \mathbb{C} \text{ with } S_0 = 1 = S_0'\end{pmatrix} \arrow[l, shift left, "\text{forgetful}"]
\end{tikzcd} \; . \tag{7.23}$$

That is, "the abelianizable varieties admit a good abelianization!"

8. Two characterizations of congruence modular varieties

The purpose of this section is to prove

8.1. Theorem.

(a) A Kiss variety $\mathbb{C}$ is congruence modular if and only if the commutator in $\mathbb{C}$ is preserved by surjective images, i.e. for any surjective homomorphism $\varphi : A \longrightarrow A'$ in $\mathbb{C}$ and congruences α, β on A,

$$\varphi_\#[\alpha, \beta] = [\varphi_\#\alpha, \varphi_\#\beta]. \tag{8.1}$$

(b) A Gumm variety $\mathbb{C}$ is congruence modular if and only if the commutator in $\mathbb{C}$ is distributive, i.e.

$$[\alpha_1 \vee \alpha_2, \beta] = [\alpha_1, \beta] \vee [\alpha_2, \beta] \tag{8.2}$$

for any algebra A in $\mathbb{C}$ and congruences α_1, α_2, β on A.

PROOF. Both (8.1) and (8.2) in the modular case are well known. So we need to prove only the "if" parts in (a) and (b).

(a) It is well known that a variety $\mathbb{C}$ is congruence modular if and only if it satisfies the following

SHIFTING LEMMA. *For any algebra A in $\mathbb{C}$ and congruences α, β, γ on A with $\alpha \wedge \beta \leqslant \gamma$,*

$$(a, b) \in \gamma \Longrightarrow (c, d) \in \gamma \tag{8.3}$$

for every diamond $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *in the span* $S = (A/\alpha \longleftarrow A \longrightarrow A/\beta)$.

In order to prove that (8.1) implies the Shifting lemma we take $\varphi : A \longrightarrow A'$ to be the canonical homomorphism $A \longrightarrow A/\gamma$. We have

$$[\varphi_{\#}\alpha, \varphi_{\#}\beta] = \varphi_{\#}[\alpha, \beta] \leqslant \varphi_{\#}(\alpha \wedge \beta) \leqslant \varphi_{\#}\gamma = \Delta_{A'}.$$

On the other hand if $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a diamond in S with $(a, b) \in \gamma$, then since

$$\big(q(a, b, c, d), c\big) \in \alpha \wedge \beta \leqslant \gamma$$

(where q is as in 6.1(a)) and $\varphi(a) = \varphi(b)$ we obtain

$$\begin{aligned}
\eta\big(\varphi(c)\big) &= \eta\Big(\varphi\big(q(a, b, c, d)\big)\Big) = \\
&= q\big(\eta\varphi(a), \eta\varphi(b), \eta\varphi(c), \eta\varphi(d)\big) = \\
&= q\big(\eta\varphi(a), \eta\varphi(a), \eta\varphi(c), \eta\varphi(d)\big) = \\
&= m\big(\eta\varphi(a), \eta\varphi(a), \eta\varphi(c), \eta\varphi(d)\big) = \\
&= m\big(\eta\varphi(a), \eta\varphi(a), \eta\varphi(d), \eta\varphi(d)\big) = \eta\varphi(d),
\end{aligned}$$

and so $(\varphi(c), \varphi(d)) \in [\varphi_{\#}\alpha, \varphi_{\#}\beta]$; here η and m were as in Proposition 5.3, but using for the span

$$A'/\varphi_{\#}\alpha \longleftarrow A' \longrightarrow A'/\varphi_{\#}\beta \tag{8.4}$$

Since $(\varphi(c), \varphi(d)) \in [\varphi_{\#}\alpha, \varphi_{\#}\beta] = \Delta_{A'}$, we obtain $\varphi(c) = \varphi(d)$, i.e. (c, d) is in γ as desired.

(b) is just a special case of [L1, Theorem 3.2(i)].

■

8.2. Remark.

(a) If φ is the canonical homomorphism $A \longrightarrow A/[\alpha,\beta]$ then (8.1) holds in any Kiss variety $\mathbb{C}$ (see Remark 5.10(c)).

(b) As we see from the proof it suffices to require (8.1) in the case $[\varphi_\#\alpha, \varphi_\#\beta] = \Delta_{A'}$. On the other hand (8.2) can be replaced by the stronger condition of infinite distributivity since it holds in congruence modular varieties.

References

[F-C] J. Furtado-Coelho, *Homology and Generalized Baer Invariants*, Journal of Algebra 40 n2, 1976, 596–600

[FMK] R. Freese and R. McKenzie, *Commutator theory for congruence modular varieties*, London Math. Soc., *Lecture Notes Series*, vol. 125, Cambridge University Press, 1987

[G] H. P. Gumm, *Geometrical methods in congruence modular algebra*, Mem. AMS 45, No. 286, 1983

[HH] J. Hagemann and C. Harrmann, *A concrete ideal multiplication for algebraic systems and its relationship to congruence distributivity*, Arch. der Math. 32, 1979, 234–245

[JK] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, Journal of Pure and Applied Algebra 97, 1994, 135–161

[JP] G. Janelidze and M. C. Pedicchio, *Internal Categories and groupoids in congruence modular varieties*, Journal of Algebra, 193(1997), 552–570

[K] K. Kearnes, *An order-theoretic property of the commutator*, International Journal of Algebra and Computation 3, 1993, 491–533

[Ki] E. W. Kiss, *Three remarks on the modular commutator*, Algebra Universalis 29, 1992, 455–476

[Ko] A. Kock, *Fibre bundles in general categories*, Journal of Pure and Applied Algebra 56, 1989, 233–245

[KS] K. Kearnes, A. Szendrei, *The relationship Between Two Commutators*, preprint 1997

[L1] P. Lipparini, *Commutator theory without join-distributivity*, Trans. AMS 346, No. 1, 1994, 177–202

[L2] P. Lipparini, *A characterization of varieties with a difference term*, Canad. Math.Bull. 39(1996), 308–315

[L3] P. Lipparini, *A Kiss 4-difference term from a ternary one*, preprint

[ML] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, New York-Heidelberg-Berlin, 1971

[P1] M. C. Pedicchio, *A categorical approach to commutator theory*, Journal of Algebra 177, 1995, 647–657

[P2] M. C. Pedicchio, *Some remarks on internal pregroupoids*, Communications in Algebra, 26(6), 1998, 1737–1744

[S] J. D. H. Smith, *Mal'tsev Varieties*, *Lecture Notes in Mathematics* 554, Berlin, 1976

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