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ON FUNCTORS WHICH ARE LAX EPIMORPHISMS

JIŘÍ ADÁMEK, ROBERT EL BASHIR, MANUELA SOBRAL, JIŘÍ VELEBIL

Abstract. We show that lax epimorphisms in the category Cat are precisely the functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ for which the functor $P^* : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$ of composition with P is fully faithful. We present two other characterizations. Firstly, lax epimorphisms are precisely the "absolutely dense" functors, i.e., functors P such that every object B of $\mathcal{B}$ is an absolute colimit of all arrows $P(E) \longrightarrow B$ for E in $\mathcal{E}$. Secondly, lax epimorphisms are precisely the functors P such that for every morphism f of $\mathcal{B}$ the category of all factorizations through objects of $P[\mathcal{E}]$ is connected.

A relationship between pseudoepimorphisms and lax epimorphisms is discussed.

1. Introduction

What are the epimorphisms of Cat, the category of small categories and functors? No simple answer is known, and the present paper indicates that this may be a "wrong question", disregarding the 2-categorical character of Cat. Anyway, with strong epimorphisms we have more luck: as proved in [2], they are precisely those functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ such that every morphism of $\mathcal{B}$ is a composite of morphisms of the form Pf.

Our paper is devoted to lax epimorphisms in the 2-category Cat. We follow the concept of pseudoepimorphism (and pseudomonomorphism) as presented in [3]: a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ is called a *lax epimorphism* provided that for every pair $Q_1, Q_2 : \mathcal{B} \longrightarrow \mathcal{C}$ of functors and every natural transformation $u : Q_1 \cdot P \longrightarrow Q_2 \cdot P$ there exists a unique natural transformation $v : Q_1 \longrightarrow Q_2$ with $u = vP$. Briefly, P is a lax epimorphism if and only if the functor

$$(\_) \cdot P : [\mathcal{B}, \mathcal{C}] \longrightarrow [\mathcal{E}, \mathcal{C}]$$

is fully faithful, for every small category $\mathcal{C}$.

Our first observation is that, instead of all small categories $\mathcal{C}$, one can simply take Set. That is, P is a lax epimorphism if and only if

$$P^* = (\_) \cdot P : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$$

is fully faithful. This is what Peter Johnstone called "connected functors" in his lecture at the Cambridge PSSL meeting in November 2000. He has asked for a characterization

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of connected functors, which has inspired the present paper. We provide two characterizations. Recall from [10] that a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ is called dense if every object B of $\mathcal{B}$ is a colimit of the diagram of all arrows $P(E) \longrightarrow B$ (more precisely, B is a canonical colimit of the diagram $P/B \longrightarrow \mathcal{B}$ forgetting the codomain). Let us call a functor P *absolutely dense* if every object B of $\mathcal{B}$ is an absolute colimit of the diagram of all arrows $P(E) \longrightarrow B$. This property could also be called *locally final* because it is equivalent to saying that for every object B of $\mathcal{B}$ the inclusion functor

$$P/B \longrightarrow \mathcal{B}/B$$

is final.

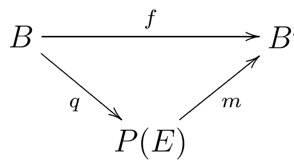
The main result of the paper is the following:

1.1. Theorem. *For a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between small categories the following conditions are equivalent:*

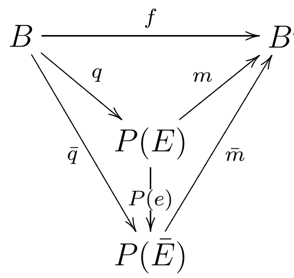
1. *P is a lax epimorphism.*
2. *The functor $P^* = (\_) \cdot P : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$ is fully faithful.*
3. *All the categories $f \mathbin{/\!\!/} P$ for morphisms f of $\mathcal{B}$ are connected.*
4. *P is absolutely dense.*

A category $\mathcal{C}$ is called connected if and only if the graph whose nodes are the objects and whose arrows are the pairs C, C' of objects with $\mathcal{C}(C, C') \neq \emptyset$ is connected (i.e., has precisely one component — thus, it is nonempty and every pair of nodes can be connected by a non-directed path).

In 3 above, the objects of $f \mathbin{/\!\!/} P$ are all triples (E, q, m) where E is an object of $\mathcal{E}$ and



is a commutative triangle in $\mathcal{B}$. Morphisms of $f \mathbin{/\!\!/} P$ from (E, q, m) to $(\bar{E}, \bar{q}, \bar{m})$ are all morphisms $e : E \longrightarrow \bar{E}$ of $\mathcal{E}$ such that the following diagram



commutes.

For the special case of $f = \mathrm{id}_B$ we call the category $\mathrm{id}_B \mathbin{/\!\!/} P$ a *splitting fibre* of B. (Recall that a fibre of B is the category of all E in $\mathcal{E}$ with $P(E) = B$ and all morphisms $u : E \longrightarrow E'$ with $Pu = \mathrm{id}_B$.) Now a splitting fibre is the category of all split subobjects $B \longrightarrow P(E)$ for E in $\mathcal{E}$.) We conclude that every lax epimorphism has all splitting fibres connected. In his PSSL lecture, P. Johnstone announced that every extremal epimorphism with connected splitting fibres is "connected", i.e., is a lax epimorphism. We show a simple example demonstrating that this sufficient condition is not necessary.

Lax monomorphisms and epimorphisms (the latter also called full and faithful morphisms) in general 2-categories have already been studied by John Gray and Ross Street in the early 1970's, see [11], and the latter by other authors, see, e.g. [3] and [4]. In the latter reference Brian J. Day also calls lax epimorphisms in CAT Cauchy dense functors. However, the explicit characterization of lax epimorphisms in CAT we provide below is new.

How is the concept of lax epimorphism related to other epimorphism concepts? We will see easy examples demonstrating that

$$\text{regular epimorphism} \not\Rightarrow \text{lax epimorphism} \not\Rightarrow \text{epimorphism}$$

and so there seems to be no connection to the "strict" concepts. Next, every lax epimorphism is a pseudoepimorphism (defined as above, except that the natural transformation $u : Q_1 \cdot P \longrightarrow Q_2 \cdot P$ is supposed to be a natural isomorphism — and then so is v).

1.2. Open Problem. Is every pseudoepimorphism a lax epimorphism?

1.3. Remark. For functors between preorders we prove in Proposition 3.1 that the answer is affirmative.

2. A Characterization of Lax Epimorphisms

Proof of Theorem 1.1. 1.⇒2. This is trivial: given functors $Q_1, Q_2 : \mathcal{B} \longrightarrow \mathsf{Set}$, let $\mathcal{C}$ be a small full subcategory of Set containing both images so that we have codomain-restrictions $Q_1', Q_2' : \mathcal{B} \longrightarrow \mathcal{C}$. By 1., for every natural transformation $u' : Q_1' \cdot P \longrightarrow Q_2' \cdot P$ there is a unique $v' : Q_1' \longrightarrow Q_2'$ with $u' = v'P$. This is equivalent to having, for every natural transformation $u : Q_1 \cdot P \longrightarrow Q_2 \cdot P$ a unique $v : Q_1 \longrightarrow Q_2$ with $u = vP$.

2.⇔3. The functor $P^* = (\_) \cdot P$ has a left adjoint, viz, the functor

$$L : [\mathcal{E}, \mathsf{Set}] \longrightarrow [\mathcal{B}, \mathsf{Set}]$$

of left Kan extension along P. Therefore, P^* is full and faithful if and only if the counit

$$\varepsilon : L \cdot P^* \longrightarrow \mathrm{I}d$$

of the adjunction $L \dashv P^*$ is a natural isomorphism.

Since every object of $[\mathcal{B}, \mathsf{Set}]$ is a colimit of hom-functors, and since $L \cdot P^*$ preserves colimits, it follows that ε is a (pointwise) isomorphism if and only if the component of ε at every $\mathcal{B}(B, \_)$, for B an object of $\mathcal{B}$, is an isomorphism.

This component can be described as follows: express

$$L \cdot P^*\Big(\mathcal{B}(B, \_)\Big) = L\Big(\mathcal{B}(B, P\_)\Big)\mathcal{B} \longrightarrow \mathsf{Set}$$

as a coend

$$\int^E \mathcal{B}\Big(B, P(E)\Big) \times \mathcal{B}\Big(P(E), \_\Big)$$

then $\varepsilon_{\mathcal{B}(B, \_)}$ evaluated at B' in $\mathcal{B}$ is the morphism

$$\int^E \mathcal{B}\Big(B, P(E)\Big) \times \mathcal{B}\Big(P(E), B'\Big) \longrightarrow \mathcal{B}(B, B')$$

which sends an equivalence class of the pair

$$(m : B \longrightarrow P(E), q : P(E) \longrightarrow B')$$

to the composite

$$q \cdot m : B \longrightarrow B'.$$

We see that $\varepsilon_{\mathcal{B}(B, \_)}$ is a natural isomorphism if and only if for every morphism $f : B \longrightarrow B'$ the category $f \mathbin{/\!\!/} P$ is connected.

3.⇒4. Firstly note that the assumption in 3. is "self-dual", i.e., we deduce from 2.⇔3. that the functor

$$(\_) \cdot P^{op} : [\mathcal{B}^{op}, \mathsf{Set}] \longrightarrow [\mathcal{E}^{op}, \mathsf{Set}]$$

is full and faithful. Thus, the composite

$$\mathcal{B} \xrightarrow{\;Y\;} [\mathcal{B}^{op}, \mathsf{Set}] \xrightarrow{\;(\_)\cdot P^{op}\;} [\mathcal{E}^{op}, \mathsf{Set}]$$

of $(\_) \cdot P^{op}$ with the Yoneda embedding Y is full and faithful. This means that P is dense, and this implies that every object B of $\mathcal{B}$ can be expressed as a colimit of a diagram $P : \mathcal{E} \longrightarrow \mathcal{B}$ weighted by $\mathcal{B}(P\_, B) : \mathcal{E}^{op} \longrightarrow \mathsf{Set}$ (see Theorem 5.1 of [9]). To prove that P is absolutely dense, we verify that every functor $F : \mathcal{B} \longrightarrow \mathcal{X}$ (with $\mathcal{X}$ small) preserves the weighted colimits $B \cong \mathcal{B}(P\_, B) * P$. In fact, recall that $(\_) \cdot P^{op}$ is full and faithful and use $\mathcal{B}(P\_, B) = \mathcal{B}(\_, B) \cdot P^{op}$ to deduce the following isomorphisms

$$\begin{aligned}
\mathcal{X}\Big(\mathcal{B}(P_, B) * F \cdot P, X\Big) &\cong [\mathcal{E}^{op}, \mathsf{Set}]\Big(\mathcal{B}(P_, B), \mathcal{X}(F \cdot P_, X)\Big)\\
&\cong [\mathcal{E}^{op}, \mathsf{Set}]\Big(\mathcal{B}(_, B) \cdot P^{op}, \mathcal{X}(F_, X) \cdot P^{op}\Big)\\
&\cong [\mathcal{B}^{op}, \mathsf{Set}]\Big(\mathcal{B}(_, B), \mathcal{X}(F_, X)\Big)\\
&\cong \mathcal{X}(FB, X)
\end{aligned}$$

natural in every object X in $\mathcal{X}$.

4.⇒1. Since we assume that P is absolutely dense, this means that a left Kan extension $\mathrm{Lan}_P P \cong \mathrm{Id}_{\mathcal{B}}$ of P along itself is preserved by any functor $F : \mathcal{B} \longrightarrow \mathcal{X}$. Thus, for every pair $F, G : \mathcal{B} \longrightarrow \mathcal{X}$ we have isomorphisms

$$\begin{aligned}[\mathcal{B}, \mathcal{X}](F, G) &\cong [\mathcal{B}, \mathcal{X}](\mathrm{Lan}_P(F \cdot P), G) \\ &\cong [\mathcal{E}, \mathcal{X}](F \cdot P, G \cdot P)\end{aligned}$$

where the last isomorphism is induced by precomposing with P. But this means precisely that P is a lax epimorphism. ■

2.1. Examples. The following are examples of lax epimorphisms:

1. Coinserters: recall that a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$, together with a natural transformation $\alpha : P \cdot F \longrightarrow P \cdot G$ is called a *coinserter* in Cat of the pair $F, G : \mathcal{C} \longrightarrow \mathcal{E}$ if and only if the following two conditions are satisfied:

   (a) For every natural transformation $\beta : Q \cdot F \longrightarrow Q \cdot G$ with $Q : \mathcal{E} \longrightarrow \mathcal{D}$ there is a unique functor $H : \mathcal{B} \longrightarrow \mathcal{D}$ such that $H \cdot P = Q$ and $H\alpha = \beta$.

   (b) For every pair $H_1, H_2 : \mathcal{B} \longrightarrow \mathcal{D}$ of functors and every natural transformation $\gamma : H_1 \cdot P \longrightarrow H_2 \cdot P$ satisfying $(H_2\alpha)(\gamma F) = (\gamma G)(H_1\alpha)$ there is a unique natural transformation $\delta : H_1 \longrightarrow H_2$ with $\delta P = \gamma$.

   Thus, every coinserter is a lax epimorphism, since the second condition above is satisfied by every natural transformation $\gamma : H_1 \cdot P \longrightarrow H_2 \cdot P$.

2. The following functor between preordered sets is a lax epimorphism which is not a coinserter:

   $$\boxed{\bullet} \longrightarrow \boxed{\bullet \cong \bullet}$$

3. Categories of fractions: given a set Σ of morphisms in a small category $\mathcal{E}$, then the canonical functor

   $$P_\Sigma : \mathcal{E} \longrightarrow \mathcal{E}[\Sigma^{-1}]$$

   into the category of fractions is a lax epimorphism (see, e.g., Lemma 1.2 of [5]).

4. Epimorphisms of small categories, which are one-to-one on objects, are lax epimorphisms. This is due to the second equivalent condition of Theorem 1.1 and Corollary 2.2 in [6].

A regular epimorphism in Cat need not be a lax epimorphism:

$$\boxed{\begin{matrix}\bullet \\ \bullet\end{matrix}} \longrightarrow \boxed{\bullet}$$

(See [2] for a characterization of regular epimorphisms as precisely those functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ which are surjective on objects and such that every morphism in $\mathcal{B}$ is a composite of morphisms in $P[\mathcal{E}]$.)

3. Pseudoepimorphisms

3.1. Proposition. *For functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ where $\mathcal{B}$ is a preordered set we have:*

P is a pseudoepimorphism if and only if it is a lax epimorphism.

Proof. Let P be a pseudoepimorphism. For every $x \le y$ in $\mathcal{B}$ we prove that the category $\mathcal{C} = (x \le y) \mathbin{/\!\!/} P$, which is the full subcategory of $\mathcal{E}$ on all E with $x \le P(E) \le y$, is connected.

Define a functor $F : \mathcal{B} \longrightarrow \mathsf{Set}$ on objects by

$$Fb = \begin{cases} 1+1, & \text{if } x \le b \le y \\ 1, & \text{if } x \le b \not\le y \\ \emptyset, & \text{otherwise} \end{cases}$$

and on morphisms by setting $Ff = id$ for every morphism $f : b \le b'$ with $Fb = Fb'$.

The category $\mathcal{C}$ is nonempty because otherwise we would have two natural isomorphisms $\beta_1, \beta_2 : F \longrightarrow F$ with $\beta_1 \ne \beta_2$ and $\beta_1 P = id = \beta_2 P$ ($\beta_1 = id$ and β_2 is the transposition of $1+1$), in contradiction to P being a pseudoepimorphism.

Let $\mathcal{C}_0$ be a connected component of $\mathcal{C}$, we will prove that $\mathcal{C}_0 = \mathcal{C}$. We have a natural isomorphism

$$\alpha : F \cdot P \longrightarrow F \cdot P$$

whose components are

$$\alpha_E = \begin{cases} id, & \text{if } E \text{ is not in } \mathcal{C}_0 \\ t, & \text{if } E \text{ is in } \mathcal{C}_0 \end{cases}$$

where $t : 1+1 \longrightarrow 1+1$ swaps the two copies of 1. The naturality squares

$$\begin{array}{ccc} FP(E) & \xrightarrow{\alpha_E} & FP(E) \\ {\scriptstyle FPh}\downarrow & & \downarrow{\scriptstyle FPh} \\ FP(E') & \xrightarrow[\alpha_{E'}]{} & FP(E') \end{array}$$

commute for all $h : E \longrightarrow E'$: this is obvious except for the case that E is in $\mathcal{C}_0$ and E' is in $\mathcal{C} \setminus \mathcal{C}_0$, or vice versa, but that case does not happen because $\mathcal{C}_0$ is a connected component of $\mathcal{C}$.

There exists a natural isomorphism $\beta : F \longrightarrow F$ with $\alpha = \beta P$. The component β_x is t because choosing any E in $\mathcal{C}_0$, the following square

$$\begin{array}{ccc} 1+1 & \xrightarrow{\beta_x} & 1+1 \\ {\scriptstyle id = F(x \longrightarrow P(E))}\downarrow & & \downarrow{\scriptstyle id = F(x \longrightarrow P(E))} \\ 1+1 & \xrightarrow[t = (\beta P)_E]{} & 1+1 \end{array}$$

commutes. This proves that $\mathcal{C} = \mathcal{C}_0$: by choosing any $E \in \mathcal{C} \setminus \mathcal{C}_0$ we would obtain, analogously, $\beta_x = id$, which is impossible. ∎

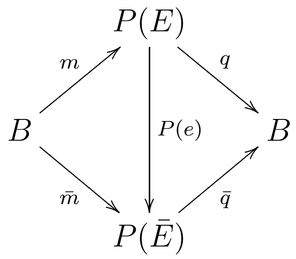
3.2. Remark. Given a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ and an object B of $\mathcal{B}$ we can form a *splitting fibre* of B: it is the category whose objects are all pairs of morphisms

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E) \quad \text{with } q \cdot m = \mathrm{id}$$

where E is an object of $\mathcal{E}$. And morphisms into

$$B \underset{\bar{q}}{\overset{\bar{m}}{\rightleftarrows}} p(\bar{E})$$

are those morphisms $e : E \longrightarrow \bar{E}$ in $\mathcal{E}$ for which the following diagram



commutes.

Every lax epimorphism has all splitting fibres connected. In fact, these are just the categories $\mathrm{id}_B \mathbin{/\!\!/} P$.

3.3. Proposition. *Let $P : \mathcal{E} \longrightarrow \mathcal{B}$ have connected splitting fibres and let*

(∗) *all morphisms in $\mathcal{B}$ be composites of isomorphisms and morphisms in $P[\mathcal{E}]$.*

Then P is a lax epimorphism.

Proof. The functor $P^* : [\mathcal{B}, \mathsf{Set}] \longrightarrow [\mathcal{E}, \mathsf{Set}]$ is faithful. In fact, for distinct natural transformations $\alpha, \beta : F \longrightarrow G$ (where F, G are functors in $[\mathcal{B}, \mathsf{Set}]$) we find an object B with $\alpha_B \neq \beta_B$, and we choose an object

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E)$$

in the splitting fibre of B. Then $(\alpha P)_E \neq (\beta P)_E$. In fact: assuming the contrary, we get a contradiction:

$$\alpha_B = \alpha_B \cdot Fq \cdot Fm = Gq \cdot \alpha_E \cdot Fm = Gq \cdot \beta_E \cdot Fm = \beta_B.$$

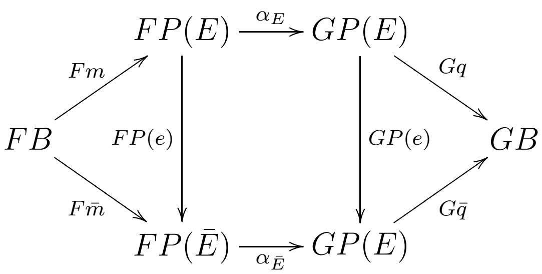
The functor P^* is full: consider functors $F, G : \mathcal{B} \longrightarrow \mathsf{Set}$ and a natural transformation $\alpha = FP \longrightarrow GP$. Define, for every object B of $\mathcal{B}$, the morphism $\beta_B : FB \longrightarrow GB$ as follows: choose

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E)$$

in the splitting fibre and put

$$\beta_B = Gq \cdot \alpha_E \cdot Fm.$$

This is independent of the choice: to show this, we use the connectedness of the splitting fibre and verify only that given a morphism as in 3.2, then $Gq \cdot \alpha_E \cdot Fm = G\bar{q} \cdot \alpha_{\bar{E}} \cdot F\bar{m}$:



Consequently,

$$\beta_{P(E)} = \alpha_E \quad \text{for each } E \text{ in } \mathcal{E}$$

because we choose $q = m = \mathrm{i}d$. To show that β_B is natural in B, it is sufficient — due to $(*)$ — to consider all isomorphisms and all morphisms in $P[\mathcal{E}]$.

Let $h : B \longrightarrow B'$ be an isomorphism. Given

$$B \underset{q}{\overset{m}{\rightleftarrows}} P(E)$$

in the splitting fibre of B, then

$$B' \underset{h\cdot q}{\overset{m\cdot h^{-1}}{\rightleftarrows}} P(E)$$

lies in the splitting fibre of B', thus,

$$\beta_{B'} = G(h \cdot q) \cdot \alpha_E \cdot F(m \cdot h^{-1})$$

which implies

$$Gh \cdot \beta_B = Gh \cdot Gq \cdot \alpha_E \cdot Fm = \beta_{B'} \cdot Fh.$$

Let $h : B \longrightarrow B'$ have the form $h = P(k)$ for $k : E \longrightarrow E'$ in $\mathcal{E}$. Since $P(E) = B$ and $P(E') = B'$, we conclude

$$Gh \cdot \beta_B = GP(k) \cdot \alpha_E = \alpha_{E'} \cdot FP(k) = \beta_{B'} \cdot Fh.$$

Thus, β is natural. ∎

3.4. Proposition. *For a functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between finite preorders the following conditions are equivalent:*

1. *P is a lax epimorphism.*

2. *P has connected splitting fibres and satisfies condition $(*)$ of Proposition 3.3.*

Proof. 1.⇒2. Every morphism in $\mathcal{B}$ is, since $\mathcal{B}$ is a finite preorder, a composite of isomorphism and coverings $x_0 \longrightarrow y_0$ (i.e., $x_0 < y_0$ and no x in $\mathcal{B}$ fulfils $x_0 < x < y_0$). Thus, it is sufficient to prove condition $(*)$ of Proposition 3.3 for every covering $x_0 \longrightarrow y_0$.

Assuming the contrary, we extend $\mathcal{B}$ to a category $\mathcal{C}$ by adding, for every pair of objects $x \cong x_0$ and $y \cong y_0$ a new morphism

$$r_{xy} : x \longrightarrow y.$$

The composition in $\mathcal{C}$ extends that of $\mathcal{B}$ by the following rules: given $x \cong x_0$ and $y \cong y_0$ then for every $z \leq x$ put

$$r_{xy} \cdot (z \longrightarrow x) = \begin{cases} r_{zy}, & \text{if } z \cong x \\ z \longrightarrow y, & \text{otherwise} \end{cases}$$

and for every $y \geq z$ put

$$(y \longrightarrow z) \cdot r_{xy} = \begin{cases} r_{xz}, & \text{if } y \cong z \\ x \longrightarrow z, & \text{otherwise} \end{cases}$$

Since $x_0 \longrightarrow y_0$ (thus, $x \longrightarrow y$) is not a composite of isomorphisms and morphisms in $P[\mathcal{E}]$, we have a well-defined functor $Q_1 : \mathcal{B} \longrightarrow \mathcal{C}$ with $Q_1(x \longrightarrow y) = r_{xy}$ for all $x \cong x_0$, $y \cong y_0$ and otherwise Q_1 is the identity function. Let $Q_2 : \mathcal{B} \longrightarrow \mathcal{C}$ denote the inclusion functor. Then $Q_1 \cdot P = Q_2 \cdot P$. But there exists no natural transformation $v : Q_1 \longrightarrow Q_2$ because the square

$$\begin{array}{ccc} x_0 & \overset{v_{x_0}}{=\!=\!=} & x_0 \\ {\scriptstyle x_0 \longrightarrow y_0} \downarrow & & \downarrow {\scriptstyle r_{x_0 y_0}} \\ y_0 & \underset{v_{y_0}}{=\!=\!=} & y_0 \end{array}$$

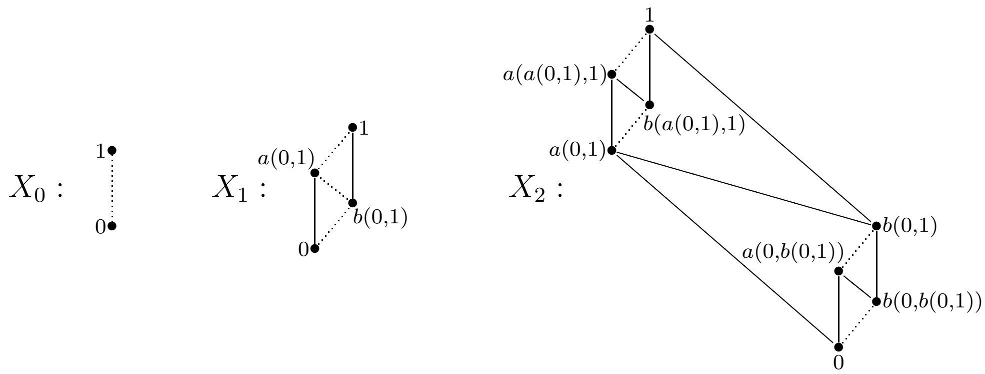
does not commute.

2.⇒1. This follows from Proposition 3.3. ■

3.5. Remark. The above condition $(*)$ together with surjectivity characterizes finite quotients (i.e., regular epimorphisms) in the category Top_0 of topological T_0 spaces. More precisely, if we identify a finite topological space with the induced order ($x \leq y$ if and only if x lies in the closure of y), then continuous functions are precisely the functors. And quotients are precisely the surjective functors satisfying $(*)$, as proved in [7] and [8].

3.6. Example. We exhibit a lax epimorphism $P : \mathcal{E} \longrightarrow \mathcal{B}$ between posets which does not satisfy condition $(*)$ of Proposition 3.3.

We define a set $X = \bigcup_{n\in\omega} X_n$ with two partial orderings $(\leq) \subseteq (\sqsubseteq)$ on it, where $(\leq) = \bigcup_{n\in\omega}(\leq_n)$ and $(\sqsubseteq) = \bigcup_{n\in\omega}(\sqsubseteq_n)$, by induction. The first three steps are illustrated below (where $\leq$ is indicated by full lines and $\sqsubseteq$ by dotted lines).



First Step. $X_0 = \{0, 1\}$, $\leq_0$ is discrete, $\sqsubseteq_0$ is the chain $0 \sqsubseteq_0 1$.
Induction Step. Let Q_n be the set of all $\sqsubseteq_n$-coverings (i.e., pairs $x \sqsubseteq_n y$ with no element z satisfying $x \sqsubset_n z \sqsubset_n y$) which are not related by $\leq_n$. Let X_{n+1} be obtained by adding to X_n elements $a(x, y)$ and $b(x, y)$ for each $(x, y) \in Q_n$. Then $\leq_{n+1}$ is the reflective and transitive closure of $\leq_n$ extended by

$$x \leq_{n+1} a(x, y) \quad \text{and} \quad b(x, y) \leq_{n+1} y \quad \text{and} \quad b(x, y) \leq_{n+1} a(x, y)$$

for all $(x, y) \in Q_n$. And $\sqsubseteq_{n+1}$ is the transitive closure of $(\sqsubseteq_n) \cup (\leq_{n+1})$ extended by

$$a(x, y) \leq_{n+1} y \quad \text{and} \quad x \leq_{n+1} b(x, y)$$

for all $(x, y) \in Q_n$.

Claim. $\mathrm{id} : \langle X, \leq\rangle \longrightarrow \langle X, \sqsubseteq\rangle$ is a lax epimorphism.

That is, given a pair $x \sqsubseteq y$ in X, then the subposet V of $\langle X, \leq\rangle$ of all z with $x \sqsubseteq z \sqsubseteq y$ is connected. We prove this by induction on n with $x, y \in X_n$. If $n = 0$, the only interesting case is $x = 0$, $y = 1$ and $V = X$. The poset $\langle X, \leq\rangle$ is indeed connected, see Theorem 1.1 because $\langle X_1, \leq_1\rangle$ is connected, and every new element added to X_{k+1}, $k > 0$, is connected by $\leq_k$ to some element of X_k. For the induction case, observe that since $\langle X_n, \sqsubseteq_n\rangle$ is a finite poset, every morphism is a composite of coverings. And the elements we add at any stage later are only added in between coverings. (More precisely, given $z \in X_k$, for $k > n$, with $x \sqsubseteq_k z \sqsubseteq_k y$ there exists a covering $x' \sqsubseteq_n y'$ with $x \sqsubseteq_n x' \sqsubseteq_k z \sqsubseteq_k y' \sqsubseteq_n y$.) Thus, it is sufficient to prove that $\langle V, \leq\rangle$ is connected assuming that $x \sqsubseteq_n y$ is a covering. If $x \leq_n y$ then $V = \{x, y\}$ is connected. If $(x, y) \in Q_n$, the argument is as for 0, 1 at the beginning: $V \cap X_{n+1} = \{x, y, a(x, y), b(x, y)\}$ is connected, and every new element added to $V \cap X_{k+1}$, $k > n$, is connected by $\leq_k$ to some element of $V \cap X_k$.

Claim. The morphism $0 \longrightarrow 1$ of $\langle X, \sqsubseteq\rangle$ is not a composite of isomorphisms and morphisms in $P[\mathcal{E}]$ — in other words, $0 \not\leq 1$. This is clear.

4. Faithfulness of P^*

Whereas Theorem 1.1 characterizes functors P for which P^* is fully faithful, a characterization of those for which P^* is just faithful has been presented in [3], where they are called liberal. In the following we exhibit a simple direct proof of that characterization.

4.1. Proposition. *For every functor $P : \mathcal{E} \longrightarrow \mathcal{B}$ between small categories the following conditions are equivalent:*

1. *P^* is faithful.*
2. *P^* is conservative (i.e., reflects isomorphisms).*
3. *P^* is monadic.*
4. *Every object of $\mathcal{B}$ is a retract of an object in $P[\mathcal{E}]$.*

Proof. 1.⇒2. Since P^* is a right adjoint of $L : [\mathcal{E}, \mathsf{Set}] \longrightarrow [\mathcal{B}, \mathsf{Set}]$ (the functor of left Kan extension), faithfulness means that the counit is an epimorphism in $[\mathcal{B}, \mathsf{Set}]$; and since epimorphisms in $[\mathcal{B}, \mathsf{Set}]$ are regular, we conclude that the comparison functor $K : [\mathcal{E}, \mathsf{Set}] \longrightarrow [\mathcal{B}, \mathsf{Set}]^T$ of the monad T of that adjunction

$$\begin{array}{ccc} [\mathcal{E}, \mathsf{Set}] & \xrightarrow{\quad K \quad} & [\mathcal{B}, \mathsf{Set}]^T \\ & {\scriptstyle P^*} \searrow \quad \swarrow {\scriptstyle U} & \\ & [\mathcal{B}, \mathsf{Set}] & \end{array}$$

is full and faithful, thus, conservative. Since the forgetful functor $U : [\mathcal{B}, \mathsf{Set}]^T \longrightarrow [\mathcal{B}, \mathsf{Set}]$ is conservative, it follows that so is $P^* = U \cdot K$.

2.⇒3. This is clear from Beck's Theorem: P^* preserves coequalizers (in fact, colimits) and has a left adjoint.

3.⇒4. This is analogous to the proof of 2.⇔3. in Theorem 1.1: here ε is an epitransformation (because P^* is faithful) and we conclude that $\mathrm{id}_B : B \longrightarrow B$ has a preimage under $\varepsilon_{\mathcal{B}(B,\_)}$, i.e., there are

$$B \xrightarrow{m} P(E) \xrightarrow{q} B$$

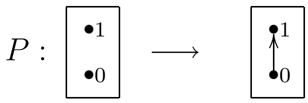
with $q \cdot m = \mathrm{id}_B$.

4.⇒1. Let $\alpha, \beta : F \longrightarrow G$ be different morphisms of $[\mathcal{B}, \mathsf{Set}]$. We are to prove $\alpha P \neq \beta P$. Given an object B with $\alpha_B \neq \beta_B$, find

$$B \xrightarrow{m} P(E) \xrightarrow{q} B$$

with $q \cdot m = \mathrm{id}_B$. Since Fq is a split epimorphism, we conclude that $\alpha_B \cdot Fq \neq \beta_B \cdot Fq$, or, equivalently, $Gq \cdot \alpha_{P(E)} \neq Gq \cdot \beta_{P(E)}$, thus $(\alpha P)_E \neq (\beta P)_E$. ∎

4.2. Remark. If P is a pseudoepimorphism, then P^* is obviously faithful. But the converse does not hold, e.g., for the embedding



which is certainly no pseudoepimorphism, P^* is faithful.

4.3. Remark.

(a) Recall that P^* is fully faithful if and only if each of the categories $f \mathbin{/\!\!/} P$ is connected, i.e., if and only if every morphism f of $\mathcal{B}$ has a factorization through some object of $P[\mathcal{E}]$ unique up to the equivalence $\approx$ of Theorem 1.1:

$$(E, m, q) \approx (\bar{E}, \bar{m}, \bar{q}) \quad \text{if and only if} \quad \bar{m} = P(e) \cdot m,\ q = \bar{q} \cdot P(e)$$

for some morphism e in $\mathcal{E}$. Now P^* is faithful if and only if each of the categories $f \mathbin{/\!\!/} P$ is nonempty, i.e., if and only if every morphism of $\mathcal{B}$ has a factorization through some object of $P[\mathcal{E}]$. In fact, given $f : B \longrightarrow B'$, choose a retraction $q : P(E) \longrightarrow B$ then, given $m : B \longrightarrow P(E)$ with $q \cdot m = \mathrm{i}d$, we factorize

$$f \equiv B \xrightarrow{m} P(E) \xrightarrow{f \cdot q} B'$$

(b) We can also characterize functors such that P^* is full. These are precisely the functors $P : \mathcal{E} \longrightarrow \mathcal{B}$ such that for every object B in $\mathcal{B}$ there exists an object E_0 in $\mathcal{E}$ and morphisms

$$B \xrightarrow{m_0} P(E_0) \xrightarrow{q_0} B$$

with the following property: given morphisms

$$B \xrightarrow{m} P(E) \xrightarrow{q} X$$

in $\mathcal{B}$ then

$$(E, m, q) \approx (E_0, m_0, q \cdot m \cdot q_0).$$

In fact, in the adjunction $L \dashv P^*$ we have P^* full if and only if ε is componentwise a split monomorphism (see 19.4 in [1]). This is the case if and only if the components

$$\varepsilon_{\mathcal{B}(B,\,\_)} : G \longrightarrow \mathcal{B}(B,\ \_)$$

(see the proof of Theorem 1.1) are, for all objects B in $\mathcal{B}$, split monomorphisms. To give a natural transformation $\alpha : \mathcal{B}(B,\ \_) \longrightarrow G$ with $\alpha \cdot \varepsilon_{\mathcal{B}(B,\,\_)} = \mathrm{i}d$ means precisely to give $(E_0, m_0, q_0) = \alpha_B(\mathrm{id}_B)$ as above.

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Institute of Theoretical Computer Science, Technical University, Braunschweig, Germany
Department of Mathematics, Charles University, Prague, Czech Republic
Department of Mathematics, University of Coimbra, Coimbra, Portugal
Faculty of Electrical Engineering, Technical University, Prague, Czech Republic
Email: `adamek@iti.cs.tu-bs.de`
`bashir@karlin.mff.cuni.cz`
`sobral@mat.uc.pt`
`velebil@math.feld.cvut.cz`

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