AN EXTENSION OF THE VEKUA-BITSADZE METHOD FOR SOLVING EQUATIONS OF THE SHELLS

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I. Vekua has constructed several versions of the refined linear theory of thin and shallow shells, containing the regular process by means of the method of reduction of three-dimensional problems of elasticity to two-dimensional ones.

In the present paper by means of the I. Vekua method the system of differential equations for the nonlinear theory of non-shallow shells is obtained. Using the method I. Vekua and the method of a small parameter 2-D system of equations for the nonlinear and non-shallow shells is obtained. For any approximations of order N the complex representations of Vekua-Bitsadze type [2] of the general solutions are obtained.

Under thin and shallow shells I. Vekua meant three-dimensional shell-type elastic bodies satisfying the conditions

$$a_\alpha^\beta - x_3 b_\alpha^\beta \cong a_\alpha^\beta, \ \ -h(x^1,x^2) \le x_3 \le h(x^1,x^2) \ \ (\alpha,\beta=1,2), \qquad (*)$$

where a_α^β and b_α^β are mixed components of the metric tensor and the curvature tensor of the shell's midsurface, x_3 is the thickness coordinate and h is the semi-thickness, depending on curvilinear coordinates x^1, x^2.

In the sequel, under by non-shallow shells we mean elastic bodies not subject to assumption (*), i. e., such that

$$a_\alpha^\beta - x_3 b_\alpha^\beta \ncong a_\alpha^\beta \Rightarrow \ \ \left|x_3 b_\alpha^\beta\right| \le q < 1.$$

1. To construct the theory of shells we use the coordinate system which is normally connected with the midsurface S. This means that the radius-vector of any point of the domain Ω can be represented in the form [1]

$$\boldsymbol{R}(x^1,x^2,x^3) = \boldsymbol{r}(x^1,x^2) + x^3\boldsymbol{n}(x^1,x^2) \ \ (x^3 = x_3),$$

where $\boldsymbol{r}$ and $\boldsymbol{n}$ are radius-vector and the unit vector of the normal of the midsurface $S(x_3=0)$; x^1, x^2 are the Gaussian parameters of S.

Covariant and contravariant basis vectors $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ of the surface $\hat{S}(x_3 = \text{const})$ and the corresponding basis vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}^i$ of the midsurface $S(x_3=0)$ are connected by the following relations [1]:

$$\boldsymbol{R}_i = A_{i.}^{.j}\boldsymbol{r}_j = A_{ij}\boldsymbol{r}^j, \ \ \boldsymbol{R}^i = A_{.j}^{i.}\boldsymbol{r}^j = A^{ij}\boldsymbol{r}_j \ \ (i,j=1,2,3),$$

where

$$A_{\alpha.}^{\cdot\beta} = a_\alpha^\beta - x_3 b_\alpha^\beta, \quad A_{.\beta}^{\alpha\cdot} = \vartheta^{-1}[(1-2Hx_3)a_\beta^\alpha + x_3 b_\beta^\alpha], \quad A_3^i = A_i^3 = \delta_i^3,$$
$$\vartheta = 1 - 2Hx_3 + Kx_3^2, \quad \boldsymbol{R}_3 = \boldsymbol{R}^3 = \boldsymbol{r}_3 = \boldsymbol{r}^3 = \boldsymbol{n} \ \ (\alpha, \beta = 1, 2). \tag{1}$$

Here $(a_{\alpha\beta}, a^{\alpha\beta}, a_\beta^\alpha)$ and $(b_{\alpha\beta}, b^{\alpha\beta}, b_\beta^\alpha)$ are the components (co, contra, mixed) of the metric tensor and curvature tensor of the midsurface S. By H and K we denote a middle and Gaussian curvature of the surface S, where

$$2H = b_\alpha^\alpha = b_1^1 + b_2^2, \quad K = b_1^1 b_2^2 - b_2^1 b_1^2.$$

The main quadratic forms of the midsurface S have the form

$$\mathrm{I} = ds^2 = a_{\alpha\beta} dx^\alpha dx^\beta, \qquad \mathrm{II} = k_s ds^2 = b_{\alpha\beta} dx^\alpha dx^\beta, \tag{2}$$

where k_s is the normal curvature of the surface S, and

$$a_{\alpha\beta} = \boldsymbol{r}_\alpha \boldsymbol{r}_\beta, \quad b_{\alpha\beta} = -\boldsymbol{r}_\alpha \boldsymbol{n}_\beta, \quad k_s = b_{\alpha\beta} s^\alpha s^\beta, s^\alpha = \frac{dx^\alpha}{ds}.$$

Here and in the sequel, under a repeated indices we mean summation; note that the Greek indices range over 1, 2, while Latin indices range over 1, 2, 3.

To construct the theory of non- shallow shells, it is necessary to obtain formulas for a family of surfaces $\hat{S}(x_3 = \text{const})$, analogous to (2) of the midsurface $S(x_3 = 0)$ which have the form [1]

$$\mathrm{I} = d\hat{s}^2 = g_{\alpha\beta} dx^\alpha dx^\beta, \qquad \mathrm{II} = k_{\hat{s}} d\hat{s}^2 = \hat{b}_{\alpha\beta} dx^\alpha dx^\beta, \tag{3}$$

where

$$g_{\alpha\beta} = a_{\alpha\beta} - 2x_3 b_{\alpha\beta} + x_3^2(2Hb_{\alpha\beta} - Ka_{\alpha\beta}), \ \hat{b}_{\alpha\beta} = (1 - 2Hx_3)b_{\alpha\beta} + x_3 K a_{\alpha\beta},$$

and $k_{\hat{s}}$ the normal curvature of the surface $\hat{S}$.

It is not now difficult to get the expression for the tangential normal $\hat{\boldsymbol{l}}$ of the surface $\hat{S}$ directed to $\hat{\boldsymbol{s}}$ [3]:

$$\hat{\boldsymbol{l}} = \hat{\boldsymbol{s}} \times \boldsymbol{n} = [(1 - x_3 k_s)\boldsymbol{l} - x_3 \tau_s \boldsymbol{s}]\frac{ds}{d\hat{s}}, \quad d\hat{s} = \sqrt{1 - 2x_3 k_s + x_3^2(k_s^2 + \tau_s^2)}ds,$$

where $\boldsymbol{s}$ and $\boldsymbol{l}$ are the unit vectors of the tangent and tangential normal on S, $d\hat{s}$ and ds are the linear elements of the surfaces $\hat{S}$ and S, and τ_s is the geodesic torsion of the surface S.

2. We write the equation of equilibrium of elastic shell-type bodies in a vector form

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} \boldsymbol{\sigma}^i}{\partial x^i} + \Phi = 0, \Rightarrow \nabla_i \boldsymbol{\sigma}^i + \Phi = 0, \tag{4}$$

where g is the discriminant of the metric quadratic form of the three-dimensional domain Ω, ∇_i are covariant derivatives with respect to the space coordinates x^i, Φ is on external force, $\boldsymbol{\sigma}^i$ are the contravariant constituents of the stress vector $\boldsymbol{\sigma}_{(\overset{*}{l})}$ acting on the area with the normal $\overset{*}{\boldsymbol{l}}$ and representable as the Cauchy formula as follows:

$$\boldsymbol{\sigma}_{(\overset{*}{l})} = \boldsymbol{\sigma}^i \overset{*}{l}_i, \quad \left(\overset{*}{l}_i = \overset{*}{\boldsymbol{l}} \boldsymbol{R}_i, \quad \right).$$

For the stress vector acting on the area with normal $\hat{\boldsymbol{l}}$, we obtain

$$\boldsymbol{\sigma}_{(\hat{\boldsymbol{l}})} = \boldsymbol{\sigma}^\alpha (\hat{\boldsymbol{l}} \boldsymbol{R}_\alpha) = \vartheta \boldsymbol{\sigma}^\alpha (\boldsymbol{l} \boldsymbol{r}_\alpha) \frac{ds}{d\hat{s}}. \tag{5}$$

The stress-strain relation for the geometrically nonlinear theory of elasticity has the form

$$\boldsymbol{\sigma}^i = \sigma^{ij} (\boldsymbol{R}_j + \partial_j \boldsymbol{U}) = E^{ijpq} e_{pq} (\boldsymbol{R}_j + \partial_j \boldsymbol{U}), \tag{6}$$

where σ^{ij} are contravariant components of the stress tensor, e_{ij} are covariant components of the strain tensor, $\mathbf{U}$ is the displacement vector, E^{ijpq} and e_{ij} are defined by the formulas:

$$E^{ijpq} = \lambda g^{ij} g^{pq} + \mu (g^{ip} g^{jq} + g^{iq} g^{ip}), \; e_{ij} = \frac{1}{2} (\boldsymbol{R}_i \partial_j \boldsymbol{U} + \boldsymbol{R}_j \partial_i \boldsymbol{U} + \partial_i \boldsymbol{U} \partial_j \boldsymbol{U}). \tag{7}$$

To reduce the three-dimensional problems of the theory of elasticity to the two-dimensional problems, it is necessary to rewrite the relation (4-7) in forms of the bases of the midsurface S of the shell Ω.

The relation (4) can be written as

$$\frac{1}{\sqrt{a}} \frac{\partial \sqrt{a} \vartheta \boldsymbol{\sigma}^\alpha}{\partial x^\alpha} + \frac{\partial \vartheta \boldsymbol{\sigma}^3}{\partial x^3} + \vartheta \Phi = 0, \quad (a = a_{11} a_{22} - a_{12}^2). \tag{8}$$

From (1), (6), (7) we obtain

$$\boldsymbol{\sigma}^i = A^i_{i_1} A^p_{p_1,} M^{i_1 j_1 p_1 q_1} [(\boldsymbol{r}_{q_1} \partial_p \boldsymbol{U}) + \frac{1}{2} A^q_{q_1} (\partial_p \boldsymbol{U} \partial_q \boldsymbol{U})] (\boldsymbol{r}_{j_1} + A^j_{j_1} \partial_j \boldsymbol{U}), \tag{9}$$

$$M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu (a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1}) \;\; (a^{ij} = \boldsymbol{r}^i \boldsymbol{r}^j)$$

3. The isometric system of coordinates on the surface S is of special interest, for in this system we can obtain basic equations of the theory of shells in a complex form which in turn allows one to construct for a rather wide class of problems complex representations of general solutions by means of analytic functions of one variable $z = x^1 + ix^2$.

The main quadratic forms in the system of coordinates are of the type

$$\mathrm{I} = ds^2 = \Lambda(z, \bar{z}) dz d\bar{z}, \quad \mathrm{II} = k_s ds^2 = \frac{1}{2} \Lambda \left[\bar{Q} dz^2 + 2H dz d\bar{z} + Q d\bar{z}^2 \right],$$

$$Q = 0.5(b_1^1 - b_2^2 + 2ib_2^1), \quad \Lambda(z, \bar{z}) > 0.$$

Introducing the well-known differential operators $2\partial z = \partial_1 - i\partial_2$, $2\partial_{\bar{z}} = \partial_1 + i\partial_2$ and the notations $\tau^{i}_{.j} = \vartheta\boldsymbol{\sigma}^i\boldsymbol{r}_j$, $X_i = \vartheta\boldsymbol{\Phi}\boldsymbol{r}_i$, for the geometrically nonlinear theory of non-shallow shells from (8) and (9) we obtain the following complex form both for the system of equations of equilibrium and Hooke's law:

$$\begin{aligned}
&\frac{1}{\Lambda}\frac{\partial\Lambda\boldsymbol{\tau}^+\boldsymbol{r}_+}{\partial z} + \frac{\partial\bar{\boldsymbol{\tau}}^+\boldsymbol{r}_+}{\partial\bar{z}} - \Lambda(H\tau_3^{\cdot+} + Q\bar{\tau}_3^{+}) + \frac{\partial\tau^3_{.+}}{\partial x_3} + X_+ = 0,\\
&\frac{1}{\Lambda}\left(\frac{\partial\Lambda\tau_3^+}{\partial z} + \frac{\partial\Lambda\bar{\tau}_3^+}{\partial\bar{z}}\right) + H(\tau_1^1 + \tau_2^2) + Re\left(\bar{Q}\boldsymbol{\tau}^+\boldsymbol{r}_+\right) + \frac{\partial\tau_3^3}{\partial x_3} + X_3 = 0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\boldsymbol{\tau}^+ =& \vartheta\left\{\lambda\Theta + \mu\left[\boldsymbol{R}^+\partial_z\boldsymbol{U} + \left(\bar{\boldsymbol{R}}_+ + 2\partial_z\boldsymbol{U}\right)\partial^{\bar{z}}\boldsymbol{U}\right]\right\}\left(\boldsymbol{R}^+ + 2\partial^{\bar{z}}\boldsymbol{U}\right)\\
&+ \mu\vartheta\left\{\left[\boldsymbol{R}^+\partial_{\bar{z}}\boldsymbol{U} + (\boldsymbol{R}_+ + 2\partial_{\bar{z}}\boldsymbol{U})\,\partial^{\bar{z}}\boldsymbol{U}\right]\left(\bar{\boldsymbol{R}}^+ + 2\partial^{z}\boldsymbol{U}\right)\right.\\
&+ \left.\left[\boldsymbol{R}^+\partial_3\boldsymbol{U} + (\boldsymbol{n} + \partial_3\boldsymbol{U})\,\partial^{\bar{z}}\boldsymbol{U}\right](\boldsymbol{n} + \partial_3\boldsymbol{U})\right\},\\
\boldsymbol{\tau}^3 =& \vartheta\left\{\left[\lambda\boldsymbol{\Theta} + 2\mu(\boldsymbol{n}\partial^3\boldsymbol{U} + \frac{1}{2}\partial_3\boldsymbol{U}\partial^3\boldsymbol{U})\right](\boldsymbol{n} + \partial_3\boldsymbol{U})\right.\\
&+\mu\left[(\frac{1}{2}\bar{\boldsymbol{R}}_+\partial_3\boldsymbol{U} + \boldsymbol{n}\partial_z\boldsymbol{U} + \partial_z\boldsymbol{U}\partial_3\boldsymbol{U})(\boldsymbol{R}^+ + 2\partial^{\bar{z}}\boldsymbol{U})\right.\\
&+ \left.\left.(\frac{1}{2}\boldsymbol{R}_+\partial_3\boldsymbol{U} + \boldsymbol{n}\partial_{\bar{z}}\boldsymbol{U} + \partial_{\bar{z}}\boldsymbol{U}\partial_3\boldsymbol{U})(\bar{\boldsymbol{R}}^+ + 2\partial^{z}\boldsymbol{U})\right]\right\},
\end{aligned} \tag{11}$$

$$(\partial^3\boldsymbol{U} = \partial_3\boldsymbol{U}).$$

Here

$\boldsymbol{\tau}^+\boldsymbol{r}_+ = \left(\boldsymbol{\tau}^1 + i\boldsymbol{\tau}^2\right)(\boldsymbol{r}_1 + i\boldsymbol{r}_2)$, $\bar{\boldsymbol{\tau}}^+\boldsymbol{r}_+ = \left(\boldsymbol{\tau}^1 - i\boldsymbol{\tau}^2\right)(\boldsymbol{r}_1 + i\boldsymbol{r}_2)$, $\tau^{+\cdot}_{.3} = \boldsymbol{\tau}^+\boldsymbol{n}$,
$\tau^{3\cdot}_{.+} = \boldsymbol{\tau}^3\boldsymbol{r}_+$, $\tau_3^3 = \boldsymbol{\tau}^3\boldsymbol{n}$, $2\partial^{\bar{z}}\boldsymbol{U} = \left(\boldsymbol{R}^+\boldsymbol{R}^+\right)\partial_z\boldsymbol{U} + \left(\boldsymbol{R}^+\bar{\boldsymbol{R}}^+\right)\partial_{\bar{z}}\boldsymbol{U}$, $\tau^{\alpha\cdot}_{.\beta} = \boldsymbol{\tau}^\alpha\boldsymbol{r}_\beta$,

$\Theta = 2Re\left[\left(\boldsymbol{R}^+ + \partial^{\bar{z}}\boldsymbol{U}\right]\right)\partial_z\boldsymbol{U} + \partial_3\boldsymbol{U}_3 + 0.5\left(\partial_3\boldsymbol{U}\right)^2$, $\boldsymbol{R}^+ = \vartheta^{-1}\left[(1 - Hx_3)\,\boldsymbol{r}^+ +\right.$ $\left.+x_3Q\bar{\boldsymbol{r}}^+\right]$, $\boldsymbol{R}_+ = (1 - Hx_3)\boldsymbol{r}_+ - x_3Q\bar{\boldsymbol{r}}_+$, $\boldsymbol{R}_+ = \boldsymbol{R}_1 + i\boldsymbol{R}_2$, $\boldsymbol{R}^+ = \boldsymbol{R}^1 + i\boldsymbol{R}^2$,

$\boldsymbol{R}^+\boldsymbol{R}^+ = 4x_3(\Lambda\vartheta^2)^{-1}(1 - Hx_3)Q$, $\boldsymbol{R}^+\bar{\boldsymbol{R}}^+ = 2(\Lambda\vartheta^2)^{-1}(\vartheta + 2x_3^2Q\bar{Q})$,
$\boldsymbol{R}^+\boldsymbol{r}_+ = 2\vartheta^{-1}Qx_3$, $\bar{\boldsymbol{R}}^+\boldsymbol{r}_+ = 2\vartheta^{-1}(1 - Hx_3)$, $\boldsymbol{r}^+\bar{\boldsymbol{r}}^+ = 2\Lambda^{-1}$, $\boldsymbol{r}^+ = \boldsymbol{r}^1 + i\boldsymbol{r}^2$.

We have the formulas

$\boldsymbol{r}^+\partial_z\boldsymbol{U} = \Lambda^{-1}\partial_z U_+ - HU_3$, $\boldsymbol{r}^+\partial_{\bar{z}}\boldsymbol{U} = \partial_{\bar{z}}U^+ - QU_3$, $X_+ = X_1 + iX_2$,
$\boldsymbol{n}\partial_z\boldsymbol{U} = \partial_z U_3 + 0.5\left(\bar{Q}U_+ + H\bar{U}_+\right)$ $(U^+ = \boldsymbol{U}\boldsymbol{r}^+,\ U_+ = \boldsymbol{U}\boldsymbol{r}_+,\ U_3 = \boldsymbol{U}\boldsymbol{n})$.

4. In the present paper the three-dimensional problems of the theory of elasticity are reduced to the two-dimensional ones by the method suggested by I. Vekua. Since the system of Legendre polynomials $\left\{P_m\left(\frac{x_3}{h}\right)\right\}$ is complete in the interval [-h,h], for equation (8) we obtain the infinite system of two-dimensional equations

$$\int_{-h}^{h}\left[\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\vartheta\boldsymbol{\sigma}^{\alpha}}{\partial x^{\alpha}}+\frac{\partial\vartheta\boldsymbol{\sigma}^{3}}{\partial x^{3}}+\vartheta\boldsymbol{\Phi}\right]P_m\left(\tfrac{x_3}{h}\right)dx_3=0\quad(m=0,1,...)$$

or in a form

$$\nabla_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^{\alpha}-\frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^{3}+\overset{(m-3)}{\boldsymbol{\sigma}}{}^{3}+...\right)+\overset{(m)}{\boldsymbol{F}}=0, \tag{12}$$

where

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^{i},\overset{(m)}{\boldsymbol{\Phi}}\right)=\frac{2m+1}{2h}\int_{-h}^{h}\left(\vartheta\boldsymbol{\sigma}^{i},\vartheta\boldsymbol{\Phi}\right)P_m\left(\tfrac{x_3}{h}\right)dx_3=\frac{2m+1}{2h}\int_{-h}^{h}\left(\boldsymbol{\tau}^{i},\boldsymbol{X}\right)P_m dx_3,$$

$$\overset{(m)}{\boldsymbol{F}}=\overset{(m)}{\boldsymbol{\Phi}}+\frac{2m+1}{2h}\left(\overset{(+)}{\vartheta}\,\overset{(+)}{\boldsymbol{\sigma}}{}^{3}-(-1)^m\overset{(-)}{\vartheta}\,\overset{(-)}{\boldsymbol{\sigma}}{}^{3}\right)\quad\left(\overset{(\pm)}{\vartheta}=\vartheta(\pm h)\right),$$

∇_α are covariant derivatives on the midsurface S.

The equation of state (9) may be written as

$$\begin{aligned}\overset{(m)}{\boldsymbol{\sigma}}{}^{i}=M^{i_1j_1p_1q_1}\sum_{m_1=0}^{\infty}\Bigg[&\left(\underset{(m_1)}{\overset{(m)}{A}}{}^{ip}_{i_1p_1}\boldsymbol{r}_{j_1}+\sum_{m_2=0}^{\infty}\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ijp}_{i_1j_1p_1}D_j\overset{(m_2)}{\boldsymbol{U}}\right)\left(\boldsymbol{r}_{q_1}\cdot D_p\overset{(m_1)}{\boldsymbol{U}}\right)\\+\frac{1}{2}\sum_{m_2=0}^{\infty}&\left(\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ipq}_{i_1p_1q_1}\boldsymbol{r}_{j_1}+\sum_{m_3=0}^{\infty}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ijpq}_{i_1j_1p_1q_1}D_j\overset{(m_2)}{\boldsymbol{U}}\right)\left(D_p\overset{(m_1)}{\boldsymbol{U}}D_q\overset{(m_3)}{\boldsymbol{U}}\right)\Bigg]\end{aligned} \tag{13}$$

where

$$D_i\overset{(m)}{\boldsymbol{U}}=\delta_i^\beta\partial_\beta\overset{(m)}{U}+\delta_i^3\overset{(m)}{U}{}';\quad \overset{(m)}{\boldsymbol{U}}{}'=\frac{2m+1}{h}\left(\overset{(m+1)}{\boldsymbol{U}}+\overset{(m+3)}{\boldsymbol{U}}+...\right), \tag{14}$$

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{ij}_{i_1j_1}=\frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1},A^j_{j_1},P_{m_1}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3$$

$$\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ijp}_{i_1j_1p_1}=\frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1},A^j_{j_1},A^p_{p_1}P_{m_1}P_{m_2}P_m dx_3, \tag{15}$$

$$\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ijpq}_{i_1j_1p_1q_1}=\frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1},A^j_{j_1},A^p_{p_1}A^q_{q_1}P_{m_1}P_{m_2}P_{m_3}P_m dx_3.$$

The boundary conditions on the lateral contour ∂S take the form:

a) for the stresses

$$\overset{(m)}{\boldsymbol{\sigma}}_{(l)} = \overset{(m)}{\sigma}_{(ll)}\boldsymbol{l} + \overset{(m)}{\sigma}_{(ls)}s + \overset{(m)}{\sigma}_{(ln)}\boldsymbol{n} = \frac{2m+1}{2h}\int\limits_{h}^{h} \boldsymbol{\sigma}_{(\boldsymbol{l})}\frac{d\hat{s}}{ds}P_m\left(\frac{x_3}{h}\right)dx_3, \quad (16)$$

b) for the displacements

$$\overset{m}{\boldsymbol{U}} = \frac{2m+1}{2h}\int\limits_{h}^{h} \boldsymbol{U}P_m\left(\frac{x_3}{h}\right)dx_3 = \overset{(m)}{U}_{(l)}\boldsymbol{l} + \overset{(m)}{U}_{(s)}(\boldsymbol{s}) + \overset{(m)}{U}_{3}\boldsymbol{n}. \quad (17)$$

Thus we have constructed an infinite system of two-dimensional equations of geometrically non-linear and non-shallow shells (12-17), which is consistent with the boundary conditions on the face surfaces, i.e. $\overset{(\pm)}{\boldsymbol{\sigma}}{}^3 = \boldsymbol{\sigma}^3(x^1, x^2, \pm h)$.

The passage to finite systems can be realized by various methods one of which consists in considering of a finite series, i.e.

$$(\vartheta\boldsymbol{\sigma}^i, \boldsymbol{U}, \vartheta\boldsymbol{\Phi}) = \sum_{m=0}^{N}\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\boldsymbol{U}}, \overset{(m)}{\Phi}\right)P_m\left(\frac{x_3}{h}\right) = (\boldsymbol{\tau}^i, \boldsymbol{U}, \boldsymbol{X}),$$

where N is a fixed nonnegative number. In other words, it is assumed that

$$\overset{(m)}{\boldsymbol{U}} = 0, \quad \overset{(m)}{\boldsymbol{\sigma}}{}^i = 0, \quad if \quad m > N.$$

Approximation of this type will be called approximation of order N.

The integrals of type (15) can be calculated [5], for example,

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{\alpha\beta}_{\alpha_1\beta_1} = \frac{2m+1}{2h}\int\limits_{-h}^{h}\vartheta^{-1}B^{\alpha}_{\alpha_1}(x_3)B^{\beta}_{\beta_1}(x_3)P_{m_1}\left(\frac{x_3}{h}\right)P_m\left(\frac{x_3}{h}\right)dx_3 =$$

$$\frac{2m+1}{2\sqrt{E}h}\left[B^{\alpha}_{\alpha_1}(hy)B^{\beta}_{\beta_1}(hy)\begin{pmatrix} P_{m_1}(y)Q_m(y), \ m_1 \le m \\ Q_{m_1}(y)P_m(y), \ m_1 \le m \end{pmatrix}\right]_{y_1}^{y_2} + \frac{L^{\alpha}_{\alpha_1}L^{\beta}_{\beta_1}}{K}\sigma^m_{m_1}, \quad (18)$$

if $E \neq 0$ $K \neq 0$ and $a^{\alpha}_{\alpha_1}a^{\beta}_{\beta_1}\delta^m_{m_1}$, if $E = H^2 - K = 0$; where $Q_m(y)$ is the Legendre function of the second kind, E is the Euler difference, $B^{\alpha}_{\beta}(x) = a^{\alpha}_{\beta} + xL^{\alpha}_{\beta}$, $L^{\alpha}_{\beta} = b^{\alpha}_{\beta} - 2Ha^{\alpha}_{\beta}$. Under the square brackets we mean the following:

$$[f(y)]^{y_2}_{y_1} = f(y_2) - f(y_1), \quad y_{1,2} = [(H \mp \sqrt{E})h]^{-1}.$$

For the integrals containing the product of three (four) Legendre polynomials we have

$$\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \frac{2m+1}{2n}\int\limits_{-h}^{h}\frac{B^{\alpha_1}_{\beta_1}B^{\alpha}_{\beta_2}B^{\alpha}_{\beta_3}}{1-2Hx_3+Kx_3}P_{m_1}P_{m_2}P_m dx_3 = \frac{2m+1}{K^2h^4}\times$$

$$\times \sum_{r=0}^{min(m_1,m_2)} \alpha_{m_1m_2r} \sum_{n=0}^{3} \overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} h^n \frac{\partial^2}{\partial y_1 \partial y_2} \left[\frac{y^n}{y_1 - y_2} \begin{pmatrix} P_s(y)Q_m(y), s \leq m \\ Q_s(y)P_m(y), s \geq m \end{pmatrix} \right]_{y_1}^{y_2},$$

where $s = m_1 + m_2 - 2r$,

$$a_{pqr} = \frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}} \frac{2(p+q)-4r+1}{2(p+q)-2r+1}, \quad A_p = \frac{1.3\cdots 2p-1}{p!},$$

$$B^{\alpha_1}_{\beta_1}(x)B^{\alpha_2}_{\beta_2}(x)B^{\alpha_3}_{\beta_3}(x) = \sum_{n=0}^{3} \overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} x^n,$$

5. Three-dimensional shell-type bodies are characterized by inequalities

$$|hb^\alpha_\beta| \leq q < 1 \quad (\alpha, \beta = 1, 2)$$

Therefore they can be represented as follows

$$|\varepsilon b^\alpha_\beta R| \leq q < 1,$$

where $\varepsilon = hR^{-1}$ is a small parameter.

Here h is semi-thickness of the shell, R is a certain characteristic radius of curvature of the midsurface S [4].

Now, following Signorini [3] we assume the validity of the expansions

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\mathbf{U}}, \overset{(m)}{\mathbf{F}} \right) = \sum_{n=1}^{\infty} \left(\overset{(m,n)}{\boldsymbol{\sigma}^i}, \overset{(m,n)}{\mathbf{U}}, \overset{(m,n)}{\mathbf{F}} \right) \varepsilon^n.$$

Substituting the above expansions into the (12,13) and (10,11) than equalizing the coefficients of expansions for ε^n we obtain the following 2-D finite system of equilibrium equations with respect to components of displacement vector in the isometric coordinates $a_{11} = a_{22} = \Lambda(x^1, x^2)$, which has the form:

$$\begin{aligned}
&4\mu\partial_{\bar z}\Big(\Lambda^{-1}\partial_z \overset{(m,n)}{u_+}\Big) + 2(\lambda+\mu)\partial_{\bar z} \overset{(m,n)}{\theta} + 2\lambda\partial_{\bar z} \overset{(m,n)}{u'_3} - (2m+1)\mu \\
&\Big[2\partial_{\bar z}\Big(\overset{(m-1,n)}{u_3} + \overset{(m-3,n)}{u_3} + \cdots\Big) + \overset{(m-1,n)}{u'_+} + \overset{(m-3,n)}{u'_+} + \cdots\Big] + \overset{(m,n)}{F_+} = 0, \quad (19)\\
&\mu\Big(\nabla^2 \overset{(m,n)}{u_3} + \overset{(m,n)}{\theta'}\Big) - (2m+1)\Big[\lambda\Big(\overset{(m-1,n)}{\theta} + \overset{(m-3,n)}{\theta} + \cdots\Big) + \\
&(\lambda+2\mu)\Big(\overset{(m-1,n)}{u'_3} + \overset{(m-3,n)}{u'_3} + \cdots\Big)\Big] + \overset{(m,n)}{F_3} = 0,
\end{aligned}$$

where $u_+ = u_1 + iu_2$, $\theta = \Lambda^{-1}\Big(\partial_z u_+ \partial_{\bar z} \overline{u}_+\Big)$, $z = x^1 + ix^2, 2\partial_z = \partial_1 - i\partial_2$, $\nabla^2 = \frac{4}{\Lambda}\frac{\partial^2}{\partial z \partial \bar z}$.

Obviously, in passing from the n-th step of approximation to the $(n+1)$-th step only the right-hand side of equations are changed. Below we will omit upper index n.

Consider now the cases: $N = 0, 1, 2, 3$ [6].

Case $N = 0$. From (3) we get

$$\begin{aligned} &4\mu\partial_{\bar z}\Big(\Lambda^{-1}\partial_z \overset{(0)}{u}_+\Big) + 2(\lambda+\mu)\partial_{\bar z}\overset{(0)}{\theta} = 0, \\ &\mu\nabla^2 \overset{(0)}{u}_3 = 0 \qquad \Big(\overset{(m)}{F}_+ = \overset{(m)}{F}_3 = 0,\ m = 0, 1, 2, 3\Big). \end{aligned} \tag{20}$$

The complex representation of general solutions has the form

$$\begin{aligned} &\overset{(0)}{u}_+ = -\frac{\lambda+3\mu}{\lambda+\mu}\,\frac{1}{\pi}\iint_S \frac{\varphi'(\zeta)dS}{\bar\zeta - \bar z} + \frac{1}{\pi}\iint_S \frac{\overline{\varphi'(\zeta)}dS}{\bar\zeta - \bar z} - \overline{\psi(z)}, \\ &\overset{(0)}{u}_3 = f(z) + \overline{f(z)}, \quad \Big(dS = \Lambda(\zeta,\bar\zeta)d\zeta\, d\bar\zeta,\ \ \zeta = \xi + i\eta\Big), \end{aligned} \tag{21}$$

where $f(z)$, $\varphi(z)$ and $\psi(z)$ are holomorphic functions of z. We note that for plane (i.e. $\Lambda = 1$) the expression of $\overset{(0)}{u}_+$ coincides with the well-known representation of Kolosov-Muskhelishvili.

Case $N = 1$. With respect to the components $(\overset{(0)}{u}_+, \overset{(1)}{u}_3)$ and $(\overset{(1)}{u}_1, \overset{(0)}{u}_3)$ we have two systems of equations:

$$\begin{aligned} &4\mu\partial_{\bar z}\Big(\Lambda^{-1}\partial_z \overset{(0)}{u}_+\Big) + 2(\lambda+\mu)\partial_{\bar z}\overset{(0)}{\theta} + 2\lambda\partial_{\bar z}\overset{(1)}{u}_3 = 0, \\ &\mu\nabla^2 \overset{(1)}{u}_3 + 3\Big[\lambda\overset{(0)}{\theta} + (\lambda+2\mu)\overset{(1)}{\theta}\Big] = 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} &4\mu\partial_{\bar z}\Big(\Lambda^{-1}\partial_z \overset{(1)}{u}_+\Big) + 2(\lambda+\mu)\partial_{\bar z}\overset{(1)}{\theta} - 3\mu\Big(2\partial_{\bar z}\overset{(0)}{u}_3 + \overset{(1)}{u}_+\Big) = 0, \\ &\nabla^2 \overset{(0)}{u}_3 + \overset{(1)}{\theta} = 0. \end{aligned} \tag{23}$$

The complex representation of general solutions has the form:

$$\begin{aligned} &\overset{(0)}{u}_+ = -\frac{5\lambda+6\mu}{3\lambda+2\mu}\,\frac{1}{\pi}\iint_S \frac{\varphi'(\zeta)dS}{\bar\zeta - \bar z} + \frac{1}{\pi}\iint_S \frac{\overline{\varphi'(\zeta)}dS}{\bar\zeta - \bar z} - \overline{\psi(z)} - \frac{\lambda}{6(\lambda+\mu)}\,\frac{\partial w}{\partial \bar z} \\ &\overset{(0)}{u}_3 = w - \frac{2\lambda}{\lambda+2\mu}(\varphi' + \overline{\varphi'}), \quad \Big(\nabla^2 w = \frac{12(\lambda+\mu)}{\lambda+2\mu}\,w\Big), \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\overset{(1)}{u}_+ = -\frac{1}{\pi}\iint_S \frac{\Phi'(\zeta) + \overline{\Phi'(\zeta)}}{\bar\zeta - \bar z}dS + \frac{4(\lambda+\mu)}{3\mu}\overline{\Phi''(z)} - 2\overline{\Psi'(z)} + i\frac{\partial\chi}{\partial\bar z}, \\ &\overset{(0)}{u}_3 = \Psi(z) + \overline{\Psi(z)} - \iint_S \Big(\Phi'(\zeta) + \overline{\Phi'(\zeta)}\Big)\ln|\zeta - z|dS \ \ (\nabla^2\chi = 3\chi), \end{aligned}$$

where $\Phi(z)$ and $\Psi(z)$ are holomorphic functions of z.

Note that the systems (10) and (11) coincide with I. Vekua's refined systems of equations for the stretch-strain and bending of plate, respectively.

Case $N = 2$. In this case with respect to the components $(\overset{(0)}{u}_1, \overset{(2)}{u}_+, \overset{(1)}{u}_3)$ and $(\overset{(1)}{u}_+, \overset{(0)}{u}_3, \overset{(2)}{u}_3)$ we have two systems of equations:

$$\begin{aligned}
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z \overset{(0)}{u}_+\Big) + 2(\lambda+\mu)\partial_{\overline{z}}\overset{(0)}{\theta} + 2\lambda\partial_{\overline{z}}\overset{(1)}{u}_3 = 0,\\
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z \overset{(2)}{u}_+\Big) + 2(\lambda+\mu)\partial_{\overline{z}}\overset{(2)}{\theta} - 5\mu\Big(2\partial_{\overline{z}}\overset{(1)}{u}_3 + 3\overset{(2)}{u}_+\Big) = 0,\\
&\mu\Big(\nabla^2\overset{(1)}{u}_3 + 3\overset{(2)}{\theta}\Big) - 3\Big[\lambda\overset{(0)}{\theta} + (\lambda+2\mu)\overset{(1)}{u}_3\Big] = 0
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z \overset{(1)}{u}\Big) + 2(\lambda+\mu)\partial_{\overline{z}}\overset{(1)}{\theta} - 3\mu\Big(2\partial_{\overline{z}}\overset{(0)}{u}_3 + \overset{(1)}{u}_+\Big) = 0,\\
&\nabla^2\overset{(0)}{u}_3 + \overset{(1)}{\theta} = 0,\\
&\mu\nabla^2\overset{(2)}{u}_3 - 5\Big[\lambda\overset{(1)}{\theta} + 3(\lambda+2\mu)\overset{(2)}{u}_3\Big] = 0.
\end{aligned} \tag{26}$$

In this case the complex representations of the general solutions of the equations (12) and (13) take the forms

$$\begin{aligned}
\overset{(0)}{u}_+ &= -\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{1}{\pi}\iint_S \frac{\varphi'(\zeta)dS}{\overline{\zeta}-z} + \frac{1}{\pi}\iint_S \frac{\overline{\varphi'(\zeta)}dS}{\overline{\zeta}-\overline{z}} - \overline{\psi(z)}\\
&\quad -\frac{2\lambda}{\lambda+2\mu}\sum_{k=1}^{2}\frac{1}{\alpha_k}\frac{\partial v_k}{\partial \overline{z}},\\
\overset{(2)}{u}_+ &= \frac{2}{3}\Big(i\frac{\partial\omega}{\partial\overline{z}} + \frac{2\lambda}{3\lambda+2\mu}\overline{\varphi''(z)} + \sum_{k=1}^{2}\frac{\alpha_{3-k}}{\alpha_k}\frac{\partial v_k}{\partial\overline{z}}\Big),\\
\overset{(1)}{u}_3 &= v_1 + v_2 - \frac{2\lambda}{\lambda+2\mu}\big(\varphi'(z) + \overline{\varphi'(z)}\big),
\end{aligned} \tag{27}$$

where

$$\nabla^2 v_k = \alpha_k v_k, \;\; \alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2} = 0 \;\; (k=1,2); \;\; \nabla^2\omega = 15\omega,$$

and

$$\begin{aligned}
\overset{(1)}{u}_+ = -\frac{1}{\pi}\iint_S \frac{\Phi'(\zeta)+\overline{\Phi'(\zeta)}}{\overline{\zeta}-\overline{z}}dS + \frac{16(\lambda+\mu)}{3(\lambda+2\mu)}\overline{\Phi''(z)} -\\
-2\overline{\Psi'(z)} + i\frac{\partial\chi}{\partial\overline{z}} - \frac{\lambda}{10(\lambda+\mu)}\frac{\partial w}{\partial\overline{z}},
\end{aligned} \tag{28}$$

$$\begin{aligned}
\overset{(0)}{u}_3 &= \Psi(z) + \overline{\Psi(z)} - \frac{1}{\pi}\iint_S \Big(\Phi'(\zeta) + \overline{\Phi'(\zeta)}\Big)\ln|\zeta - z|dS + \frac{\lambda}{20(\lambda+\mu)}w,\\
\overset{(2)}{u}_3 &= w - \frac{2\lambda}{3\lambda+2\mu}\Big(\Phi'(z) + \overline{\Phi'(z)}\Big),
\end{aligned}$$

where $\nabla^2 w = \frac{60(\lambda+\mu)}{\lambda+2\mu}\, w$, $\nabla^2\chi = 3\chi$.

Case $N = 3$. For this case we have

$$\overset{(0)}{u}_+ = -\frac{5\lambda+6\mu}{3\lambda+2\mu}\,\frac{1}{\pi}\iint_S \frac{\varphi' dS}{\overline{\zeta}-\overline{z}} + \frac{1}{\pi}\iint_S \frac{\overline{\varphi'} dS}{\overline{\zeta}-\overline{z}} - \overline{\psi} - \frac{2\lambda}{\lambda+2\mu}\sum_{k=1}^{3} \frac{1+\overset{(1)}{A}_k}{\alpha_k}\,\frac{\partial v_k}{\partial \overline{z}}$$

$$\begin{aligned}
\overset{(2)}{u}_+ &= \frac{2}{3}\left(i\frac{\partial\omega}{\partial\overline{z}} + \frac{2\lambda}{3\lambda+2\mu}\,\overline{\varphi''} + \sum_{k=1}^{3}\overset{(2)}{A}_k\frac{\partial v_k}{\partial\overline{z}}\right), \qquad (29)\\
\overset{(1)}{u}_3 &= \sum_{k=1}^{3}\overset{(1)}{A}_k v_k - \frac{2\lambda}{3\lambda+2\mu}(\varphi' + \overline{\varphi'}),\\
\overset{(3)}{u}_3 &= \sum_{k=1}^{3} v_k, \quad \Big(\nabla^2 v_k = \alpha_k v_k, \quad \nabla^2\omega = 15\omega\Big),
\end{aligned}$$

where

$$\alpha_k^3 - \frac{180(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k^2 + \frac{120(\lambda+\mu)(7\lambda+15\mu)}{(\lambda+2\mu)^2}\,\alpha_k + \frac{7\cdot 900(\lambda+\mu)}{\lambda+2\mu} = 0,$$

$$\overset{(1)}{A}_k = \left[\frac{3(9\lambda+4\mu)}{\lambda+2\mu}\,\alpha_k - \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\right]\left[\alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\right]^{-1},$$

$$\overset{(2)}{A}_k = -10\left[\frac{\lambda}{\lambda+2\mu}\,\alpha_k - \frac{12(\lambda+\mu)}{\lambda+2\mu}\right]\left[\alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\,\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\right]^{-1},$$

and

$$\begin{aligned}
\overset{(1)}{u}_+ &= -\frac{1}{\pi}\iint_S \frac{\Phi' + \overline{\Phi'}}{\overline{\zeta}-\overline{z}}\,dS + \frac{4}{15}\,\frac{23\lambda+24\mu}{\lambda+2\mu}\overline{\Phi''} - 2\overline{\Psi} +\\
&\quad + \sum_{k=1}^{2}\left(i\frac{\varkappa_k - 3}{3}\,\frac{\partial\chi_k}{\partial\overline{z}} - \frac{6\lambda}{\lambda+2\mu}\,\frac{\partial w_k}{\partial\overline{z}}\right),\\
\overset{(3)}{u}_+ &= \sum_{k=1}^{2}\left(i\frac{\partial\chi_k}{\partial\overline{z}} + \frac{2}{5}\,\frac{\gamma_{3-k}}{\gamma_k}\,\frac{\partial w_k}{\partial\overline{z}}\right) - \frac{4}{15}\,\frac{3\lambda+2\mu}{\lambda+2\mu}\,\overline{\Phi''}(z), \qquad (30)\\
&\qquad (\nabla^2 w_k = \gamma_k w_k, \quad \nabla^2\chi_k = \varkappa_k\chi_k,),\\
\overset{(0)}{u}_3 &= \sum_{k=1}^{2}\left(\frac{3\lambda}{\lambda+2\mu} - \frac{\gamma_{3-k}}{5}\right)\frac{1}{\gamma_k}\,w_k - \iint_S (\Phi' + \overline{\Phi'})\ln|\zeta - z|dS + \Psi + \overline{\Psi},\\
\overset{(2)}{u}_3 &= \sum_{k=1}^{2} w_k - \frac{2\lambda}{3(\lambda+2\mu)}(\Phi' + \overline{\Phi'})\\
&\quad \left(\gamma_k^2 - \frac{60(\lambda+\mu)}{3(\lambda+2\mu)}\,\gamma_k + 60\frac{35\mu(\lambda+\mu)}{(\lambda+2\mu)^2} = 0, \quad \varkappa_k^2 - 45\varkappa_k + 105 = 0\right).
\end{aligned}$$

The general solution of the homogeneous system (19) we can find the form

$$\overset{(m)}{u}_+ = \partial_{\overline{z}} \overset{(m)}{V}_+ + \Big(\frac{1}{\pi}\iint\limits_S \frac{\overline{\varphi_0'(\zeta)} - æ_1\varphi_0'(\zeta)dS_\zeta}{\overline{\zeta}-\overline{z}} - \overline{\psi_0'(z)}\Big)\delta_{0m} -$$

$$\Big(\frac{1}{\pi}\iint\limits_S \frac{\varphi_1'(\zeta) + \overline{\varphi_1'(\zeta)}dS_\zeta}{\overline{\zeta}-\overline{z}} + \eta_1\overline{\varphi_1''(z)} - 2\overline{\psi_1'(z)}\Big)\delta_{1m}$$

$$+æ_2\overline{\varphi_0''(z)}\delta_{2m} + \eta_2\overline{\varphi_1''(z)}\delta_{3m}, \tag{31}$$

$$\overset{(m)}{u}_3 = \overset{(m)}{V}_3 - \Big(\frac{1}{\pi}\iint\limits_S (\varphi_1'(\zeta) + \overline{\varphi_1'(\zeta)}) ln|\zeta - z| dS_\zeta - \psi_1(z) - \overline{\psi_1(z)}\Big)\delta_{0m}$$

$$-\frac{3}{2}æ_2\Big[(\varphi_0'(z) + \overline{\varphi_0'(z)})\delta_{1m} - (\varphi_1'(z) + \overline{\varphi_1'(z)})\delta_{2m}\Big] \qquad (m = 0, 1, ..., N)$$

$$\overset{(0)}{V}_1 = \overset{(0)}{V}_2 = 0, \quad \overset{(0)}{u}_3 = \psi_1(z) + \overline{\psi_1(z)}, \quad if\ N = 0$$

$$(dS_\zeta = \Lambda(\zeta,\overline{\zeta})d\zeta d\overline{\zeta},\ \zeta = \xi + i\eta).$$

where $\varphi_0'(z)$,$\varphi_1'(z)$,$\psi_0'(z)$,$\psi_1'(z)$ are holomorphic functions of z and express the biharmonic solution of the system (19). Then $æ_1$, $æ_2$, η_1, η_2 are known constants.

Substituting expressions (31) into (19) the matrix equations for $\overset{(m)}{V_i}$ are obtained

$$\nabla^2 V - AV = X, \quad \nabla^2\Omega - B\Omega = Y, \tag{32}$$

where V and Ω are column-matrices of the form

$$V = \big(\overset{(0)}{V}_1, \overset{(1)}{V}_1, ..., \overset{(N)}{V}_1, \overset{(0)}{V}_3, \overset{(1)}{V}_3, ..., \overset{(N)}{V}_3\big)^T, \qquad \Omega = \big(\overset{(0)}{V}_2, \overset{(1)}{V}_2, ..., \overset{(N)}{V}_2\big)^T,$$

and A and B are block-matrices $2N+2\times 2N+2$ and $N+1\times N+1$ respectively.

Using now the formulae Vekua-Bitsadze for the homogenous matrix equations (32) we obtain the following complex representation of the general solutions

$$V = 2Re\{\varphi(z) + \frac{A}{4}\int\limits_{z_0}^{z}\int\limits_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})R(z,\overline{z},t,\overline{t})\varphi(t)dtd\overline{t}\},$$

$$\Omega = 2Re\{f(z) + \frac{B}{4}\int\limits_{z_0}^{z}\int\limits_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})r(z,\overline{z},t,\overline{t})f(t)dtd\overline{t}\},$$

or

$$V(z,\overline{z}) = 2Re\Big[\alpha R(z,\overline{z},z_0,\overline{z}_0)\varphi(z) + \int\limits_{z_0}^{z}\Phi(t)R(z,\overline{z},t,\overline{z}_0)dt\Big],$$

$$\Omega(z,\overline{z}) = 2Re\Big[\beta r(z,\overline{z},z_0,\overline{z}_0) + \int_{z_0}^{z} \Psi(t) r(z,\overline{z},t,\overline{z}_0) f(t)dt\Big],$$

$$\Big(\ \varphi(z) = \frac{\alpha}{2}\int_{z_0}^{z} \Phi(t)dt, \quad V(z_0,\overline{z}_0) = \alpha, \quad \Phi(t) = \frac{\partial V(z_0,\overline{z}_0)}{\partial z}\Big),$$

$$\Big(\ f(z) = \frac{\beta}{2}\int_{z_0}^{z} \Psi(t)dt, \quad \Omega(z_0,\overline{z}_0) = \beta, \quad \Psi(t) = \frac{\partial \Omega(z_0,\overline{z}_0)}{\partial z}\Big),$$

where R and r are the Riemann's matrix functions of the equations (32), $\varphi(z)$ and $f(z)$ are holomorphic column-matrices:

$$\varphi(z) = (\varphi_0(z),\cdot\cdot\cdot,\varphi_N(z),\varphi_{N+1}(z),\cdot\cdot\cdot\varphi_{2N}(z))^T, \quad f(z) = (f_0(z),\cdot\cdot\cdot,f_N(z))^T.$$

Then particular solutions of the matrix equations (32) have the form

$$\hat{V}(z,\overline{z}) = \frac{1}{4}\int_{z_0}^{z}\int_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t}) R(z,\overline{z},t,\overline{t}) X(t,\overline{t}) dt d\overline{t},$$

$$\overset{\wedge}{\Omega}(z,\overline{z}) = \frac{1}{4}\int_{z_0}^{z}\int_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t}) r(z,\overline{z},t,\overline{t}) Y(t,\overline{t}) dt d\overline{t},$$

where

$$R(z,\overline{z},t,\overline{t}) = E + \frac{A}{4}\int_{t}^{z}\int_{\overline{t}}^{\overline{z}} \Lambda(t_1,\overline{t}_1) dt_1 d\overline{t}_1 +$$

$$\Big(\frac{A}{4}\Big)^2 \int_{t}^{z}\int_{\overline{t}}^{\overline{z}} \Lambda(t_1,\overline{t}_1)\Big(\int_{t}^{t_1}\int_{\overline{t}}^{\overline{t}_1} \Lambda(t_2,\overline{t}_2) dt_2 d\overline{t}_2\Big) dt_1 d\overline{t}_1 \cdots$$

$$r(z,\overline{z},t,\overline{t}) = E + \frac{B}{4}\int_{t}^{z}\int_{\overline{t}}^{\overline{z}} \Lambda(t_1,\overline{t}_) dt_1 d\overline{t}_1 +$$

$$\Big(\frac{B}{4}\Big)^2 \int_{t}^{z}\int_{\overline{t}}^{\overline{z}} \Lambda(t_1,\overline{t}_1)\Big(\int_{t}^{t_1}\int_{\overline{t}}^{\overline{t}_1} \Lambda(t_2,\overline{t}_2) dt_2 d\overline{t}_2\Big) dt_1 d\overline{t}_1 + \cdots$$

For the first boundary condition (in stress) we have

$$(\lambda+\mu)\overset{(m)}{\theta} - 2\mu\Lambda\frac{\partial \overset{(m)}{u}_+}{\partial \overline{z}}\Big(\frac{d\overline{z}}{ds}\Big)^2 = \overset{(m)}{a_1} + i\,\overset{(m)}{b_1}, \quad Im\Big(\frac{\partial \overset{(m)}{u_3}}{\partial \overline{z}}\frac{d\overline{z}}{ds}\Big) = \overset{(m)}{c_1} \quad (on\ \partial S)$$

The second boundary condition (in displacements) for any m takes the form

$$\overset{(m)}{u}_+ \frac{d\bar{z}}{ds} = \overset{(m)}{a}_2 + i\,\overset{(m)}{b}_2, \quad \overset{(m)}{u}_3 = \overset{(m)}{c}_2 \quad (on\ \partial S).$$

The basic boundary conditions ($N = 0$) for any n have the form:
a) for the first boundary problem (in displacements)

$$\overset{(0,n)}{U}_{(\ell)} + i\,\overset{(0,n)}{U}_{(s)} = i\,\overset{(0,n)}{U}_+ \frac{d\bar{z}}{ds} = \overset{(n)}{d}_+, \quad \overset{(0,n)}{U}_3 = \overset{(n)}{d}_3 \ \text{ on } \partial D \tag{33}$$

b) for the second boundary problem (in stresses)

$$\begin{aligned} &\overset{(0,n)}{\sigma}_{(\ell\ell)} + i\,\overset{(0,n)}{\sigma}_{(\ell s)} = \frac{1}{2}\left[\overset{(0,n)}{\bar{\boldsymbol{\sigma}}}{}^{+}\boldsymbol{r}_+ - \left(\overset{(0,n)}{\boldsymbol{\sigma}}{}_{+}\boldsymbol{r}_+\right)\frac{dz}{d\bar{z}}\right] = \overset{(n)}{e}_+, \\ &\overset{(0,n)}{\sigma}_{(\ell n)} = -Im\left[\left(\overset{(0,n)}{\boldsymbol{\sigma}}{}_{+}\boldsymbol{n}\right)\frac{d\bar{z}}{ds}\right] = \overset{(n)}{e}_3 \ \text{ on } \partial D. \end{aligned} \tag{34}$$

Here we present a general scheme of solution of boundary problems when the domain D is a circle of radius r_0.

The first boundary problem for any n takes the form (on $|z| = r_0$),

$$\overset{(0,n)}{U}_+ = -\frac{æ}{\pi}\iint_D \frac{\Lambda\varphi'(\zeta)d\xi d\eta}{\bar{\zeta} - \bar{z}} + \left(\frac{1}{\pi}\iint_D \frac{\Lambda d\xi d\eta}{\bar{\zeta} - \bar{z}}\right)\overline{\varphi'(z)} - \overline{\psi(z)} = \overset{(n)}{G}_+ \tag{35}$$

$$\overset{(0,n)}{U}_3 = f(z) + \overline{f(z)} = \overset{(n)}{G}_3 \quad (z = re^{i\varphi}, \zeta = \rho e^{i\psi}), \tag{36}$$

where $\overset{(n)}{G}_+$ and $\overset{(n)}{G}_3$ are the known values containing solutions $\overset{(0,1)}{U}_i, \cdots, \overset{(0,n-1)}{U}_i$ of the previous approximations.

Let $\Lambda(z, \bar{z})$ depend only on $r = |z|$, next $\varphi'(z)$, $\psi(z)$ and $\overset{(n)}{G}_+$ are expanded in power series of the type

$$\varphi'(z) = \sum_{k=0}^{\infty} a_k z^k, \quad \Psi(z) = \sum_{k=0}^{\infty} b_k z^k, \quad \overset{(n)}{G}_+ = \sum_{k=-\infty}^{\infty} A_k e^{ik\vartheta}.$$

Substituting these expansions into (24), we obtain

$$a_0 = \frac{r_0}{\alpha_0}\frac{æA_1 + \bar{A}_1}{æ^2 - 1}, \ a_k = \frac{r_0^{k+1}A_{k+1}}{æ\alpha_k} \ (k \geq 1),$$

$$b_k = -\frac{\bar{A}_k}{r_0^k} - \frac{\alpha_0 r_0^{k+2}}{æ\alpha_{k+1}}A_{k+2}, (k \geq 0), \quad \alpha_k = 2\int_0^{r_0} \rho^{2k+1}\Lambda(\rho)d\rho.$$

$\overset{(0,n)}{U}_3$ is representable in the form of the Poisson integral,

$$\overset{(0,n)}{U}_3(r,\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} \overset{(n)}{G}_3(\psi)\frac{r_0^2 - r^2}{r^2 - 2r_0 r\cos(\psi-\vartheta) + r_0^2}d\psi. \tag{37}$$

Thus for any n we can construct formal solutions of the problem (22), when $N = 0$.

From the second boundary condition (23), we obtain (on ∂D)

$$\overset{(0,n)}{\sigma}_{(\ell\ell)} + i\overset{(0,n)}{\sigma}_{(\ell s)} = \overset{(n)}{\ell}_+ \Rightarrow (\lambda+\mu)\overset{(0,n)}{\Theta} - 2\mu\left(\frac{1}{\Lambda}\overset{(0,n)}{U}_+\right)\frac{d\bar z}{dz} = \overset{(n)}{P}_+, \tag{38}$$

$$\overset{(0,n)}{\sigma}_{(\ell n)} = \overset{(n)}{\ell}_3 \Rightarrow \mathrm{Im}\left(\frac{\partial \overset{(0,n)}{U}_3}{\partial z}\frac{d\bar z}{ds}\right) = \overset{(n)}{P}_3, \tag{39}$$

$$\left(\overset{(0,n)}{\Theta} = \frac{1}{\Lambda}\left(\frac{\partial \overset{(0,n)}{U}_+}{\partial z} + \frac{\partial \overset{(0,n)}{\bar U}_+}{\partial \bar z}\right)\right).$$

Consider the case of a spherical shell, whose midsurface is a spherical segment of radius $R_0 \sin\vartheta$, where R_0 is the radius of a sphere. Isometric coordinates on the sphere can be represented in the form

$$z = x^1 + ix^2 = re^{i\varphi},\ \ r = \mathrm{tg}\frac{\vartheta}{2},\ \ \Lambda = 4R^2(1+z\bar z)^{-2} \quad (0 \le \vartheta \le \vartheta_0).$$

Let the expressions

$$(\varphi'(z), \Psi'(z), f(z)) = \sum_{k=0}^{\infty}(a_k, b_k, c_k)z^k,\ \ \left(\overset{(n)}{g}_+, \overset{(n)}{g}_3\right) = \sum_{k=-\infty}^{\infty}(A_k, B_k)e^{ik\varphi},$$

be valid, where $\overset{(n)}{g}_+$ and $\overset{(n)}{g}_3$ are known values expressed by $\overset{(0,1)}{U}_i, \cdots, \overset{(0,n-1)}{U}_i$ of the previous approximations. Substituting these expansions into (27), (28) and taking into account that principal vector and moment of stresses are zero, we obtain

$$a_k = \frac{A_k}{2\mu r_0^k}\frac{1}{1+2æ(1+r_0^2)\beta_k},$$

$$b_k = \frac{-1}{2\mu r_0^{k-1}}\frac{(1+r_0^2)^{-1}}{k+2r_0^2}\left[\frac{((1+r_0^2)k + 2r_0^2)A_{k+1}}{1+2æ(1+r_0^2)\beta_{k+1}r_0^2} + \bar A_{k-1}\right] \quad (k \ge 0).$$

$$c_k = \frac{2}{\mu}\frac{R_0}{1+r_0^2}\frac{B_k}{kr_0^{k-1}} \ \ (k \ge 1),\ \ B_0 = 0,\ \beta_k(z) = \frac{1}{z^{k+2}}\int_0^z \frac{(z-t)t^k dt}{(1+t\bar z)^3}.$$

From here we obtain the well-known Dini's formula

$$\overset{(0,n)}{U}_3(r_0,\varphi) = -\frac{r_0}{\pi}\int_0^{2\pi} \overset{(n)}{P}_3(r_0,\varphi)\ln|\sigma - z|d\vartheta + \text{const} \quad (\sigma = r_0 e^{i\vartheta}).$$

R e f e r e n c e s

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