ON THE ITERATIVE SOLUTION OF A SYSTEM OF DISCRETE TIMOSHENKO EQUATIONS

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Abstract. The variational and difference methods are used respectively for spatial and time variables to solve a nonlinear integro-differential Timoshenko dynamic beam equation. The resulting algebraic system of cubic equations is solved by the iterative method. The iteration process error is estimated.

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1. Statement of the problem

Let us consider the equation

$$u_{tt}(x,t)+u_{xxxx}(x,t)-hu_{xxtt}(x,t)-\frac{1}{2L}\Big(\lambda+\int_0^L u_x^2(x,t)\,dx\Big)u_{xx}(x,t)=0, \quad 0<x<L, \quad 0<t\le T, \tag{1}$$

with the initial boundary condition

$$\begin{aligned} &u(x,0)=u^0(x), \quad u_t(x,0)=u^1(x),\\ &u(0,t)=u(L,t)=0, \quad u_{xx}(0,t)=u_{xx}(L,t)=0,\\ &0\le x\le L, \quad 0\le t\le T, \end{aligned} \tag{2}$$

where $h>0$, $\lambda>0$, L, T and $u^0(t)$, $u^1(t)$ are the given constants and functions, $u(x,t)$ is the function we want to define.

The equation (1), which describes the oscillation of a beam by the Timoshenko theory, is considered in [1], [3] and [7]. In the present paper we introduce an approximate algorithm for the problem (1),(2) and study the accuracy of its iteration part. Note that the problem of construction of numerical methods through estimation of algorithm errors was investigated in [5] for other nonlinear Timoshenko beam equations.

2. Algorithm

a. Galerkin method

An approximate solution will be sought for in the form of a finite sum

$$u_n(x,t) = \sum_{i=1}^{n} u_{ni}(t) \sin \frac{i\pi}{L} x, \tag{3}$$

where, according to the Galerkin method, the coefficients $u_{ni}(t)$ are a solution of the system of ordinary differential equations

$$\left(1+h\left(\frac{i\pi}{L}\right)^2\right) u''_{ni}(t) + \left[\lambda+\left(\frac{i\pi}{L}\right)^2+\frac{1}{4}\sum_{j=1}^{n}\left(\frac{j\pi}{L}\right) u_{nj}^2(t)\right]\left(\frac{i\pi}{L}\right)^2 u_{ni}(t)=0, \quad i=1,2,\dots,n, \tag{4}$$

with the initial condition

$$u_{ni}(0) = a_i^0, \quad u'_{ni}(0) = a_i^1, \quad i = 1,2,\dots,n, \tag{5}$$

where

$$a_i^p = \frac{2}{L}\int_0^L u^p(x) \sin \frac{i\pi}{L} x\, dx, \quad i = 1,2,\dots,n, \quad p = 0,1.$$

b. Difference scheme

Let us introduce the functions

$$y_{ni}(t) = u'_{ni}(t), \quad z_{ni}(t) = \frac{i\pi}{L} u_{ni}(t), \quad i = 1,2,\dots,n, \tag{6}$$

and rewrite the system (4),(5) in the new notation as

$$\left(1+h\left(\frac{i\pi}{L}\right)^2\right) y'_{ni}(t) + \left[\lambda+\left(\frac{i\pi}{L}\right)^2+\frac{1}{4}\sum_{j=1}^{n} z_{nj}^2(t)\right]\frac{i\pi}{L} z_{ni}(t) = 0,$$
$$z'_{ni}(t) = \frac{i\pi}{L} y_{ni}(t), \tag{7}$$

$$y_{ni}(0) = a_i^1, \quad z_{ni}(0) = \frac{i\pi}{L} a_i^0, \quad i = 1,2,\dots,n. \tag{8}$$

The problem (7),(8) will be solved by the difference method. On a time interval $[0,T]$ we introduce a net with step $\tau = \frac{T}{M}$ and nodes $t_m = m\tau$, $m = 0,1,\dots,M$.

At the m-th layer, i.e., for $t = t_m$, the approximate values $y_{ni}(t)$ and $z_{ni}(t)$ are denoted by y_{ni}^m and z_{ni}^m.

We make use of the Crank-Nicolson scheme

$$\left(1 + h\left(\frac{i\pi}{L}\right)^2\right)\frac{y_{ni}^m - y_{ni}^{m-1}}{\tau} + \left[\lambda + \left(\frac{i\pi}{L}\right)^2 + \frac{1}{4}\sum_{j=1}^{n}\frac{(z_{nj}^m)^2 + (z_{nj}^{m-1})^2}{2}\right] \times\frac{i\pi}{L}\,\frac{z_{ni}^m + z_{ni}^{m-1}}{2} = 0, \tag{9}$$
$$\frac{z_{ni}^m - z_{ni}^{m-1}}{\tau} = \frac{i\pi}{L}\,\frac{y_{ni}^m + y_{ni}^{m-1}}{2}, \quad m = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,$$

with the condition

$$y_{ni}^0 = a_i^1, \quad z_{ni}^0 = \frac{i\pi}{L}a_i^0, \quad i = 1, 2, \ldots, n. \tag{10}$$

c. Iteration method

We will solve the system (9),(10) layer-by-layer.

Assuming that the solution has been obtained on the $(m-1)$th layer, to find it on the mth layer we will apply the Jacobi iteration method [4]. For the sake of simplicity we will neglect the error of the final iteration approximation on the $(m-1)$th layer. This means that for fixed m the counting will be carried out by the formulas

$$\left(1 + h\left(\frac{i\pi}{L}\right)^2\right)\frac{y_{ni,k+1}^m - y_{ni}^{m-1}}{\tau} + \left[\lambda + \left(\frac{i\pi}{L}\right)^2 + \frac{1}{4}\frac{(z_{ni,k+1}^m)^2 + (z_{ni}^{m-1})^2}{2} + \frac{1}{4}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(z_{nj,k}^m)^2 + (z_{nj}^{m-1})^2}{2}\right]\frac{i\pi}{L}\,\frac{z_{ni,k+1}^m + z_{ni}^{m-1}}{2} = 0, \tag{11.1}$$

$$\frac{z_{ni,k+1}^m - z_{ni}^{m-1}}{\tau} = \frac{i\pi}{L}\,\frac{y_{ni,k+1}^m + y_{ni}^{m-1}}{2}, \quad m = 1, 2, \ldots, M, \quad k = 0, 1, \ldots, \quad i = 1, 2, \ldots, n, \tag{11.2}$$

where y_{ni}^{m-1} and z_{ni}^{m-1} are the known values, $i = 1, 2, \ldots, n$, and

$$y_{ni}^0 = a_i^1, \quad z_{ni}^0 = \frac{i\pi}{L}\,a_i^0, \quad i = 1, 2, \ldots, n.$$

After expressing $y_{ni,k+1}^m$ in (11.2) through y_{ni}^{m-1}, z_{ni}^{m-1} and $z_{ni,k+1}^m$

$$y_{ni,k+1}^m = -y_{ni}^{m-1} + 2\,\frac{L}{i\pi}\,\frac{z_{ni,k+1}^m - z_{ni}^{m-1}}{\tau}, \tag{12}$$

and substituting (12) into (11.1), we come to the expression

$$\frac{2}{\tau}\left(\frac{L}{i\pi}+h\,\frac{i\pi}{L}\right)\frac{z^m_{ni,k+1}-z^{m-1}_{ni}}{\tau}+\left[\lambda+\left(\frac{i\pi}{L}\right)^2+\frac{1}{4}\,\frac{(z^m_{ni,k+1})^2+(z^{m-1}_{ni})^2}{2}\right.$$
$$\left.+\frac{1}{4}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(z^m_{nj,k})^2+(z^{m-1}_{nj})^2}{2}\right]\frac{i\pi}{L}\,\frac{z^m_{ni,k+1}+z^{m-1}_{ni}}{2}$$
$$-\frac{2}{\tau}\left(1+h\left(\frac{i\pi}{L}\right)^2\right)y^{m-1}_{ni}=0. \tag{13}$$

Hence it follows that for each k the iteration process means the realization of only one formula (13). After obtaining the final iteration approximation $z^m_{ni,k+1}$, we substitute this value into (12) to find an approximation for y^m_{ni}, $i=1,2,\dots,n$.

By the expression (13) we conclude that we have to solve a cubic equation with respect to $z^m_{ni,k+1}$ at the $(k+1)$th iteration step for each i. This equation is written in the form

$$(z^m_{ni,k+1})^3+a_i(z^m_{ni,k+1})^2+b_i z^m_{ni,k+1}+c_i=0, \tag{14}$$

where

$$\begin{aligned} &a_i=z^{m-1}_{ni},\quad b_i=d_i+(z^{m-1}_{ni})^2+e_i,\\ &c_i=\left(-d_i+(z^{m-1}_{ni})^2+e_i\right)z^{m-1}_{ni}-\tau d_i\,\tfrac{i\pi}{L}\,y^{m-1}_{ni}, \end{aligned} \tag{15}$$

and

$$d_i=\frac{32}{\tau^2}\left(h+\left(\frac{L}{i\pi}\right)^2\right),\quad e_i=8\left(\lambda+\left(\frac{i\pi}{L}\right)^2\right)+\sum_{\substack{j=1\\ j\neq i}}^{n}\left((z^m_{nj,k})^2+(z^{m-1}_{nj})^2\right). \tag{16}$$

Let us apply Cardano's formula [2] to the equation (14) and the relations (15),(16). Recall that the a priori real root of the equation

$$w^3+aw^2+bw+c=0$$

is equal to

$$w=-\frac{a}{3}+\sum_{p=1}^{2}\left[(-1)^p\frac{s}{2}+\left(\frac{s^2}{4}+\frac{r}{27}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}},$$

where

$$r=-\frac{a^2}{3}+b,\quad s=\frac{2a^3}{27}-\frac{ab}{3}+c.$$

Thus we get

$$z^m_{ni,k+1}=-\frac{z^{m-1}_{ni}}{3}+\sum_{p=1}^{2}(-1)^{p+1}\sigma_{i,p}\,, \tag{17}$$
$$k=0,1,\dots,\quad i=1,2,\dots,n,$$

where

$$\sigma_{i,p} = \left[(-1)^p \frac{s_i}{2} + \left(\frac{s_i^2}{4} + \frac{r_i^3}{27} \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} \tag{18}$$

and

$$\begin{aligned} r_i &= d_i + \tfrac{2}{3}(z_{ni}^{m-1})^2 + e_i, \\ s_i &= \tfrac{2}{3}\, z_{ni}^{m-1} \left(-2d_i + \tfrac{10}{9}(z_{ni}^{m-1})^2 + e_i\right) - \tau d_i \tfrac{i\pi}{L}\, y_{ni}^{m-1}. \end{aligned} \tag{19}$$

The considered algorithm of solution of the problem (1), (2) should be understood as the counting carried out by the formula (17). Having $z_{ni,k}^m$ and taking (6) and (3) into consideration, we construct the approximated value of the function $u(x,t)$ for $t = t_m$ as the sum

$$\sum_{i=1}^{n} \frac{L}{i\pi}\, z_{ni,k}^m \sin \frac{i\pi}{L}\, x. \tag{20}$$

3. Iteration method error

Under the iteration method error $\Delta u_{n,k}^m$ we will understand the difference between (20) and the sum

$$\sum_{i=1}^{n} \frac{L}{i\pi}\, z_{ni}^m \sin \frac{i\pi}{L}\, x,$$

which would give an approximate value of the function $u(x,t)$ for $t = t_m$ if the difference scheme (9), (10) were solved exactly. Thus

$$\Delta u_{n,k}^m(x) = \sum_{i=1}^{n} \frac{L}{i\pi}(z_{ni,k}^m - z_{ni}^m) \sin \frac{i\pi}{L}\, x. \tag{21}$$

Our aim consists in estimating the error $\Delta u_{n,k}^m(x)$.

Let us represent the system (17) as

$$z_{ni,k+1}^m = \varphi_i(z_{n1,k}^m, z_{n2,k}^m, \ldots, z_{nn,k}^m) \tag{22}$$

and consider the Jacobi matrix

$$J = \left(\frac{\partial \varphi_i}{\partial z_{nj,k}^m} \right)_{i,j=1}^{n}. \tag{23}$$

By virtue of (17)–(19) and (22) the diagonal terms of the matrix J are equal to zero, while for the nondiagonal terms we have

$$\frac{\partial \varphi_i}{\partial z_{nj,k}^m} = -\frac{z_{nj,k}^m}{9} \sum_{p=1}^{2} \frac{1}{\sigma_{i,p}^2} \left[2z_{ni}^{m-1} + (-1)^p \left(s_i z_{ni}^{m-1} + \frac{1}{3}\, r_i^2 \right) \left(\frac{s_i^2}{4} + \frac{r_i^3}{27} \right)^{-\frac{1}{2}} \right]. \tag{24}$$

By (18) and (19)

$$\sigma_{i,1}\sigma_{i,2} = \frac{r_i}{3}\,, \quad \sigma_{i,2}^3 - \sigma_{i,1}^3 = s_i, \quad \left(\frac{s_i^2}{4} + \frac{r_i^3}{27}\right)^{\frac{1}{2}} = \frac{\sigma_{i,1}^3 + \sigma_{i,2}^3}{2}\,. \tag{25}$$

From (24) and (25) follows

$$\frac{\partial \varphi_i}{\partial z_{nj,k}^m} = -\frac{4}{9}\, z_{nj,k}^m z_{ni}^{m-1} \left(\sigma_{i,1}^2 - \frac{r_i}{3} + \sigma_{i,2}^2\right)^{-1}$$
$$+\frac{2}{3}\, z_{nj,k}^m s_i \left(\sigma_{i,1}^4 + \frac{r_i^2}{9} + \sigma_{i,2}^4\right)^{-1}, \quad i \neq j. \tag{26}$$

Now we apply the first relation of (25) to the estimate $\sigma_{i,1}^{2p} + \sigma_{i,2}^{2p} \geq 2(\sigma_{i,1}\sigma_{i,2})^p$, $p = 1, 2$. We get

$$\sigma_{i,1}^{2p} + \sigma_{i,2}^{2p} \geq 2\left(\frac{r_i}{3}\right)^p.$$

Using this inequality, (19) and (16), from (26) we obtain

$$\left|\frac{\partial \varphi_i}{\partial z_{nj,k}^m}\right| \leq \left(\frac{4}{3r_i}\,|z_{ni}^{m-1}| + \frac{2}{r_i^2}\,|s_i|\right) |z_{nj,k}^m|$$
$$\leq \left[\frac{1}{24}\,\tau^2 \left(h + \left(\frac{L}{i\pi}\right)^2\right)^{-1} |z_{ni}^{m-1}| + 2\,\frac{1}{r_i^2}\sum_{p=1}^{3} |s_i^{(p)}|\right] |z_{nj,k}^m|, \tag{27}$$

where

$$|s_i^{(1)}| = \tfrac{2}{3}\left[\tfrac{64}{\tau^2}\left(h + \left(\tfrac{L}{i\pi}\right)^2\right) + \tfrac{4}{9}\,(z_{ni}^{m-1})^2\right] |z_{ni}^{m-1}|,$$
$$|s_i^{(2)}| = \tfrac{2}{3}\left[8\left(\lambda + \left(\tfrac{i\pi}{L}\right)^2\right) + \tfrac{2}{3}\,(z_{ni}^{m-1})^2 + \sum_{\substack{j=1\\ j\neq i}}^n \left((z_{nj,k}^m)^2 + (z_{nj}^{m-1})^2\right)\right] |z_{ni}^{m-1}|,$$
$$|s_i^{(3)}| = 32\,\tfrac{1}{\tau}\left(h + \left(\tfrac{L}{i\pi}\right)^2\right) \tfrac{i\pi}{L}\,|y_{ni}^{m-1}|.$$

We again apply (19) and (16) and also the inequalities $\frac{\xi}{(a+\xi)^2} \leq \frac{1}{4a}$ for $\xi > 0$, $a > 0$ and $4ab \leq (a+b)^2$ for arbitrary a and b. The results is as follows

$$\frac{|s_i^{(1)}|}{r_i^2} \leq \tfrac{1}{24}\,\tau^2\left(h + \left(\tfrac{L}{i\pi}\right)^2\right)^{-1} |z_{ni}^{m-1}|,$$
$$\frac{|s_i^{(2)}|}{r_i^2} \leq \tfrac{1}{192}\,\tau^2\left(h + \left(\tfrac{L}{i\pi}\right)^2\right)^{-1} |z_{ni}^{m-1}|,$$
$$\frac{|s_i^{(3)}|}{r_i^2} \leq \tfrac{\tau}{32}\left(\lambda + \left(\tfrac{i\pi}{L}\right)^2\right)^{-1} \tfrac{i\pi}{L}\,|y_{ni}^{m-1}|.$$

Using these relations in (27) we come to a conclusion that

$$\left|\frac{\partial\varphi_i}{\partial z^m_{nj,k}}\right| \le \left(\frac{13}{96}\,\tau^2 \left(h+\left(\frac{L}{i\pi}\right)^2\right)^{-1} |z^{m-1}_{ni}| \right.$$
$$\left. +\frac{1}{16}\,\tau \left(\lambda+\left(\frac{i\pi}{L}\right)^2\right)^{-1} \frac{i\pi}{L}\,|y^{m-1}_{ni}|\right)|z^m_{nj,k}|. \tag{28}$$

We will need a vector norm equal to $\|v\|_1 = \sum\limits_{i=1}^{n} |v_i|$ and the corresponding norm for the matrices $\|U\|_1 = \max\limits_{1\le j\le m} \sum\limits_{i=1}^{n} |u_{ij}|$, where $v = (v_i)^n_{i=1}$, $U = (u_{ij})^n_{i,j=1}$. By (23) and (27) we get

$$\|J_1\| \le \Big(\max_{1\le j\le n} |z^m_{nj,k}|\Big)\left(\frac{13}{96}\,\tau^2 \sum_{i=1}^{n}\left(h+\left(\frac{L}{i\pi}\right)^2\right)^{-1} |z^{m-1}_{ni}| \right.$$
$$\left. +\frac{1}{16}\,\tau \sum_{i=1}^{n}\left(\lambda+\left(\frac{i\pi}{L}\right)^2\right)^{-1} \frac{i\pi}{L}\,|y^{m-1}_{ni}|\right). \tag{29}$$

By the principle of compressed mapping let us assume that the condition $\|J\|_1 \le q$ is fulfilled for $0 < q < 1$ and $(z^m_{ni,k})^n_{i=1}$, $k = 0, 1, \ldots$, belonging to the domain

$$\Big\{(v_i)^n_{i=1} \in R^n:\ \sum_{i=1}^{n} |v_i - z^m_{ni,0}| \le \frac{1}{1-q}\sum_{i=1}^{n} |z^m_{ni,1} - z^m_{ni,0}|\Big\}.$$

As seen from (28), for this it suffices that the restriction

$$\tau \le \frac{1}{2\alpha}\,\Big[-\beta + (\beta^2 + 4\alpha\gamma)^{\frac{1}{2}}\Big] \tag{30}$$

be fulfilled for the step τ. Here

$$\alpha = \sum_{i=1}^{n} \frac{1}{h+(\frac{L}{i\pi})^2}\,|z^{m-1}_{ni}|, \quad \beta = \frac{6}{13}\sum_{i=1}^{n} \frac{1}{\lambda+(\frac{i\pi}{L})^2}\,\frac{i\pi}{L}\,|y^{m-1}_{ni}|,$$
$$\gamma = \frac{96}{13}\,q\Big[\sum_{i=1}^{n}\Big(|z^m_{ni,0}| + \frac{1}{1-q}\,|z^m_{ni,1} - z^m_{ni,0}|\Big)\Big]^{-1}.$$

If this restriction is fulfilled, then the system (9),(10) has a unique solution y^m_{ni}, z^m_{ni}, the iteration process (17) converges, $\lim\limits_{k\to\infty} z^m_{ni,k} = z^m_{ni}$, $i = 1, 2, \ldots, n$, and the convergence rate is defined by the inequality

$$\sum_{i=1}^{n} |z^m_{ni,k} - z^m_{ni}| \le \frac{q^k}{1-q}\sum_{i=1}^{n} |z^m_{ni,1} - z^m_{ni,0}|.$$

Using this relation in (21), we come to a conclusion that if the condition (30) is fulfilled, then for the $L^2(0,L)$-norm of the iteration method error we have the estimate

$$\Big\| \frac{d^p}{dx^p}\, \Delta u^m_{n,k}(x) \Big\|_{L^2(0,L)} \le \left(\frac{L}{\pi}\right)^{1-p} \sqrt{\frac{L}{2}}\, \frac{q^k}{1-q} \sum_{i=1}^{n} |z^m_{ni,1} - z^m_{ni,0}|,$$
$$p = 0,1, \quad m = 1,2,\ldots,M, \quad k = 1,2,\ldots\ .$$

The question of accuracy of the Galerkin method for the problem (1),(2) is studied in [6].

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