INVESTIGATION OF INTERIOR AND EXTERIOR DIRICHLET BVPs OF THERMO-ELECTRO-MAGNETO ELASTICITY THEORY AND REGULARITY RESULTS

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Abstract: We investigate the three-dimensional interior and exterior Dirichlet boundary value problems of statics of the thermo-electro-magneto-elasticity theory. We construct explicitly the fundamental matrix of the corresponding 6×6 non-self-adjoint matrix differential operator and study their properties near the origin and at infinity. We apply the potential method and reduce the corresponding BVPs to the equivalent system of boundary integral equations. We analyze the solvability of the resulting boundary pseudodifferential equations in the Hölder and Sobolev-Slobodetski spaces and prove the corresponding existence theorems.

1. Introduction. In the study of active material systems, there is significant interest in the coupling effects between elastic, electric, magnetic and thermal fields. The mathematical model of statics of the thermo-electro-magneto-elasticity theory is described by the non-self-adjoint 6×6 system of second order partial differential equations with the appropriate boundary conditions. The problem is to determine three components of the elastic displacement vector, the electric and magnetic scalar potential functions and the temperature distribution. Other field characteristics (e.g., mechanical stresses, electric and magnetic fields, electric displacement vector, magnetic induction vector, heat flux vector and entropy density) can be then determined by the gradient equations and the constitutive equations (for details see [2]-[5], [14], [21]).

For the equations of dynamics the uniqueness theorems of solutions for some mixed initial-boundary value problems are well studied. In particular, in the reference [14] the uniqueness theorem is proved without making restrictions on the positive definiteness on the elastic moduli. However, to the best of our knowledge, the uniqueness and existence of solutions of the boundary value problems of statics are not studied in the scientific literature.

Essential difficulties arise in the study of exterior BVPs for unbounded domains. The case is that one has to consider the problem in a class of vector functions which are bounded at infinity. This complicates proof of uniqueness theorems since Green's formulas do not hold for such vector functions. We have found efficient asymptotic conditions at infinity which guarantee uniqueness in the space of bounded vector functions and established uniqueness results (see [16]).

With the help of the potential method we reduce the three-dimensional interior and exterior Dirichlet boundary value problems of the thermo-electro-magneto-elasticity theory to the equivalent 6×6 system of pseudo-differential equations and analyze their solvability in the Hölder and Sobolev-Slobodetski spaces and prove the corresponding uniqueness and existence theorems.

2. Formulation of problems. Let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $S = \partial\Omega^+$ and $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\overline{\Omega^\pm}$ are filled with an anisotropic homogeneous material with thermo-electro-magneto-elastic properties.

Dirichlet problems $(D)^\pm$: Find a regular solution vector $U = (u, \varphi, \psi, \vartheta)^\top \in [C^1(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ (respectively $U \in [C^1(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6$, to the system of equations

$$A(\partial)\, U = \Phi \quad \text{in} \quad \Omega^\pm\,, \tag{1}$$

satisfying the Dirichlet type boundary conditions

$$\{\,U\}^\pm = f \quad \text{on} \quad S, \tag{2}$$

where $A(\partial)$ is a nonselfadjoint strongly elliptic matrix partial differential operator generated by the equations of statics of the theory of thermo-electro-magneto-elasticity,

$$A(\partial) := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} & [-\lambda_{rj}\,\partial_j]_{3\times 1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l & -p_j\,\partial_j \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l & -m_j\,\partial_j \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{6\times 6}, \tag{3}$$

the symbols $\{\cdot\}^\pm$ denote the one sided limits (the trace operators) on $\partial\Omega^\pm$ from $\Omega^\pm$, the summation over the repeated indices is meant from 1 to 3; ∂_j denotes partial differentiation with respect to x_j, $\partial_j := \partial/\partial x_j$. The components of the vector $U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top$ have the following physical sense: the first three components correspond to the elastic displacement vector $u = (u_1, u_2, u_3)$ in the theory of thermo-electro-magneto-elasticity, the forth and fifth ones, φ and ψ are respectively electric and magnetic potentials and the sixth component ϑ stands for the temperature distribution; c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{jk} are the thermal strain constants, η_{jk} are the heat conductivity coefficients. These constants satisfy specific symmetry conditions [4], [14], [18].

In our analysis we need special asymptotic conditions at infinity in the case of unbounded domains [16].

Definition 1 *We say that a continuous vector* $U = (u, \varphi, \psi, \vartheta)^\top \equiv (U_1, \cdots, U_6)^\top$ *in the domain* Ω^- *has the property* $\mathrm{Z}(\Omega^-)$*, if the following conditions are sat-*

isfied

$$\widetilde{U}(x) := (u(x), \varphi(x), \psi(x))^\top = \mathcal{O}(1),$$
$$U_6(x) = \vartheta(x) = \mathcal{O}(|x|^{-1}) \ \text{ as } \ |x| \to \infty, \tag{4}$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma_R} U_k(x)\, d\Sigma_R = 0, \quad k = \overline{1,5}, \tag{5}$$

where Σ_R *is a sphere centered at the origin and radius* R.

In what follows we always assume that in the case of exterior boundary value problem a solution possesses $Z(\Omega^-)$ property.

3. Potentials and their properties. Denote by $\Gamma(x) = [\Gamma_{kj}(x)]_{6\times 6}$ the matrix of fundamental solutions of the operator $A(\partial)$,

$$A(\partial)\Gamma(x) = I_6\, \delta(x), \tag{6}$$

where $\delta(\cdot)$ is the Dirac's delta distribution and I_6 stands for the unit 6×6 matrix. Applying the generalized Fourier transform the fundamental matrix is constructed explicitly and its properties near the origin and at infinity are established. The entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in x and

$$\Gamma(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [0]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6}. \tag{7}$$

With the help of the fundamental matrix we construct the generalized single and double layer potentials,

$$V(h)(x) = \int_S \Gamma(x-y)\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{8}$$

$$W(h)(x) = \int_S [\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)]^\top\, h(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{9}$$

$$N_{\Omega^\pm}(g)(x) = \int_{\Omega^\pm} \Gamma(x-y)\, g(y)\, dy, \quad x \in \mathbb{R}^3, \tag{10}$$

where $S = \partial\Omega^\pm \in C^{m,\kappa}$withinteger $m \geq 1$and $0 < \kappa \leq 1$; $h = (h_1, \cdots, h_6)^\top$ and $g = (g_1, \cdots, g_6)^\top$ are density vector-functions defined respectively on S and in $\Omega^\pm$, and the so called *generalized stress operator* $\mathcal{P}(\partial, n)$ is given by the formula

$$\mathcal{P}(\partial,n) = [\ \mathcal{P}_{pq}(\partial,n)\ ]_{6\times 6}$$
$$= \begin{bmatrix} [c_{rjkl}\, n_j\, \partial_l]_{3\times 3} & [-e_{lrj}\, n_j\, \partial_l]_{3\times 1} & [-q_{lrj}\, n_j\, \partial_l]_{3\times 1} & [0]_{3\times 1} \\ [e_{jkl}\, n_j\, \partial_l]_{1\times 3} & \varkappa_{jl}\, n_j\, \partial_l & a_{jl}\, n_j\, \partial_l & 0 \\ [q_{jkl}\, n_j\, \partial_l]_{1\times 3} & a_{jl}\, n_j\, \partial_l & \mu_{jl}\, n_j\, \partial_l & 0 \\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\, n_j\, \partial_l \end{bmatrix} \tag{11}$$

$n=(n_1,n_2,n_3)$ is the outward unit normal vector with respect to Ω^+ at the point $x\in\partial\Omega^+$.

The qualitative and mapping properties of the layer potentials are described by the following theorems (cf. [11], [12], [17], [18]).

Theorem 2 *The generalized single and double layer potentials solve the homogeneous differential equation $A(\partial)U=0$ in $\mathbb{R}^3\setminus S$ and possess the property $\mathrm{Z}(\Omega^-)$.*

In what follows by L_p, W_p^r, H_p^s, and $B_{p,q}^s$ (with $r\ge 0$, $s\in\mathbb{R}$, $1<p<\infty$, $1\le q\le\infty$) we denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [22]). Recall that $H_2^r=W_2^r=B_{2,2}^r$, $H_2^s=B_{2,2}^s$, $W_p^t=B_{p,p}^t$, and $H_p^k=W_p^k$, for any $r\ge 0$, for any $s\in\mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Theorem 3 *Let $S=\partial\Omega^\pm\in C^{m,\kappa}$ with integers $m\ge 1$ and $k\le m-1$, and $0<\kappa'<\kappa\le 1$. Then the operators*

$$V\ :\ [C^{k,\,\kappa'}(S)]^6\to[C^{k+1,\kappa'}(\overline{\Omega^\pm})]^6,\quad W\ :\ [C^{k,\kappa'}(S)]^6\to[C^{k,\kappa'}(\overline{\Omega^\pm})]^6,\quad(12)$$

are continuous.

For any $g\in[C^{0,\kappa'}(S)]^6$, $h\in[C^{1,\kappa'}(S)]^6$, and any $x\in S$ we have the following jump relations:

$$\{\,V(g)(x)\,\}^\pm=V(g)(x)=\mathcal{H}\,g(x),\quad(13)$$

$$\{\,\mathcal{T}(\partial_x,n(x))\,V(g)(x)\,\}^\pm=[\,\mp 2^{-1}I_6+\mathcal{K}\,]\,g(x),\quad(14)$$

$$\{\,W(g)(x)\,\}^\pm=[\,\pm 2^{-1}I_6+\mathcal{N}\,]\,g(x),\quad(15)$$

$$\{\,\mathcal{T}(\partial_x,n(x))\,W(h)(x)\,\}^+=\{\,\mathcal{T}(\partial_x,n(x))\,W(h)(x)\,\}^-=\mathcal{L}\,h(x),\ m\ge 2,\quad(16)$$

where

$$\mathcal{H}\,g(x):=\int_S\Gamma(x-y)\,g(y)\,dS_y\,,\quad(17)$$

$$\mathcal{K}\,g(x):=\int_S[\mathcal{T}(\partial_x,n(x))\,\Gamma(x-y)]\ g(y)\,dS_y\,,\quad(18)$$

$$\mathcal{N}\,g(x):=\int_S[\,\mathcal{P}(\partial_y,n(y))\,\Gamma^\top(x-y)\,]^\top\,g(y)\,dS_y\,,\quad(19)$$

$$\mathcal{L}\,h(x):=\lim_{\Omega^\pm\ni z\to x\in S}\mathcal{T}(\partial_z,n(x))\int_S[\,\mathcal{P}(\partial_y,n(y))\,\Gamma^\top(z-y)\,]^\top\,h(y)\,dS_y\,,\quad(20)$$

and the matrix boundary operator $\mathcal{T}(\partial,n)$ reads as

$$\mathcal{T}(\partial,n):=\begin{bmatrix}[c_{rjkl}\,n_j\,\partial_l]_{3\times 3} & [e_{lrj}\,n_j\,\partial_l]_{3\times 1} & [q_{lrj}\,n_j\,\partial_l]_{3\times 1} & [-\lambda_{rj}\,n_j]_{3\times 1}\\ [-e_{jkl}\,n_j\,\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l & -p_j\,n_j\\ [-q_{jkl}\,n_j\,\partial_l]_{1\times 3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l & -m_j\,n_j\\ [0]_{1\times 3} & 0 & 0 & \eta_{jl}\,n_j\,\partial_l\end{bmatrix}_{6\times 6}\quad(21).$$

Theorem 4 *Let S be a Lipschitz surface. The operators V and W can be extended to the continuous mappings*

$$V : [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6, \quad W : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^1(\Omega^+)]^6.$$

The jump relations (13)-(16) *on S remain valid for the extended operators in the corresponding function spaces.*

Theorem 5 *Let S, m, κ, κ' and k be as in Theorem* 3. *Then the operators*

$$\begin{aligned}
\mathcal{H} \ &: \ [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6\,, \quad m \geq 1, && (22)\\
&: \ [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6\,, \quad m \geq 1, && (23)\\
\mathcal{K} \ &: \ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6\,, \quad m \geq 1, && (24)\\
&: \ [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6\,, \quad m \geq 1, && (25)\\
\mathcal{N} \ &: \ [C^{k,\kappa'}(S)]^6 \to [C^{k,\kappa'}(S)]^6\,, \quad m \geq 1, && (26)\\
&: \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6\,, \quad m \geq 1, && (27)\\
\mathcal{L} \ &: \ [C^{k,\kappa'}(S)]^6 \to [C^{k-1,\kappa'}(S)]^6\,, \quad m \geq 2,\ k \geq 1, && (28)\\
&: \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \quad m \geq 2, && (29)
\end{aligned}$$

are continuous. The operators (23), (25), (27), *and* (29) *are bounded if S is a Lipschitz surface.*

Proof. It is word for word of the proofs of the similar theorems in [6], [7], [13], [20]. □

The next assertion is a consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary (see, e.g., [1], [4], [8], [9], [10], [23], and the references therein).

Theorem 6 *Let V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{N}$ and $\mathcal{L}$ be as in Theorems 3 and let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $S \in C^\infty$. The layer potential operators* (12) *and the boundary integral* (*pseudodifferential*) *operators* (22)-(29) *can be extended to the following continuous operators*

$$\begin{aligned}
V \ &: \ [B_{p,p}^s(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6, & W \ &: \ [B_{p,p}^s(S)]^6 \to [H_p^{s+\frac{1}{p}}(\Omega^+)]^6,\\
\mathcal{H} \ &: \ [H_p^s(S)]^6 \to [H_p^{s+1}(S)]^6, & \mathcal{K} \ &: \ [H_p^s(S)]^6 \to [H_p^s(S)]^6,\\
\mathcal{N} \ &: \ [H_p^s(S)]^6 \to [H_p^s(S)]^6, & \mathcal{L} \ &: \ [H_p^{s+1}(S)]^6 \to [H_p^s(S)]^6.
\end{aligned}$$

The jump relations (13)-(16) *remain valid for arbitrary $g \in [B_{p,q}^s(S)]^6$ with $s \in \mathbb{R}$ if the limiting values* (*traces*) *on S are understood in the sense described in* [23].

Proof. It is word for word of the proofs of the similar theorems in [7]. □

Remark 7 *Let either $\Phi \in [L_p(\Omega^+)]^6$ or $\Phi \in [L_{p,\,comp}(\Omega^-)]^6$, $p>1$. Then the Newtonian volume potential $N_{\Omega^\pm}(\Phi)$ possesses the following properties (see, e.g.,* [15]) :

$$N_{\Omega^+}(\Phi) \in [W^2_p(\Omega^+)]^6, \quad N_{\Omega^-}(\Phi) \in [W^2_{p,\,loc}(\Omega^-)]^6,$$
$$A(\partial)N_{\Omega^\pm}(\Phi) = \Phi \ \text{almosteverywherein} \quad \Omega^\pm.$$

Therefore, without loss of generality, we can assume that in the formulation of the Dirichlet problems the right hand side function $\Phi(x) = 0$ in $\Omega^\pm$.

4. Investigation of the interior Dirichlet BVP: double layer approach. Let us consider the Dirichlet problem for the domain Ω^+:

$$A(\partial)U(x) = 0, \quad x \in \Omega^+, \tag{30}$$

$$\{U(x)\}^+ = f(x), \quad x \in S, \tag{31}$$

where $S \in C^{2,\kappa}$ with $0<\kappa\le 1$ and $f \in C^{1,\kappa'}$ with $0<\kappa'<\kappa\le 1$. We analyze this problem in the space of regular vector functions $[C^{1,\kappa'}(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$.

In [16] it is shown that this problem possesses at most one solution.

We look for a solution in the form of double layer potential,

$$U(x) \equiv W(h)(x) = \int_S [\mathcal{P}(\partial_y, n(y))\Gamma^\top(x-y)]^\top h(y)dS_y, \tag{32}$$

where $h = (h_1, ..., h_6)^\top \in [C^{1,\kappa'}(S)]^6$ is unknown density, while Γ and $\mathcal{P}$ are defined by (7) and (11) respectively. Using the boundary condition (31) and the jump relation for the double layer potential we arrive at the singular integral equation for the density vector h,

$$[2^{-1}I_6 + \mathcal{N}]\, h = f \quad on \ \ S. \tag{33}$$

By standard approach we show that the principal homogeneous symbol matrix of the operator $2^{-1}I_6 + \mathcal{N}$ is non-degenerate and the index equals to zero (cf. [18]). Therefore, for invertibility of the operator

$$2^{-1}I_6 + \mathcal{N} \ : \ [L_2(S)]^6 \to [L_2(S)]^6 \tag{34}$$

it suffices to show that the corresponding homogeneous equation

$$[2^{-1}I_6 + \mathcal{N}]\, h = 0 \text{ on } S \tag{35}$$

possesses only the trivial solution in the space $[L_2(S)]^6$.

Let $h^{(0)} \in [L_2(S)]^6$ be a solution of equation (35) and construct the double layer potential $W(h^{(0)})(x) \equiv U_0(x), \ \ x \in \mathbb{R}^3 \setminus S$. By the embedding theorems (see [13], Ch. 4), one can easily derive that actually $h^{(0)} \in [C^{1,\kappa'}(S)]^6$. Therefore $W(h^{(0)}) \in [C^{1,\kappa'}(\overline{\Omega^\pm})]^6$ due to Theorem 3.

It is evident that $A(\partial)\, U_0 = 0$ in $\Omega^{\pm}$ and $\{U_0\}^{+} = (2^{-1} I_6 + \mathcal{N})\, h^{(0)} = 0$ on S. Thus U_0 is a solution to the homogeneous interior Dirichlet problem and consequently $U_0 = 0$ in Ω^{+} by the uniqueness theorem ([16]).

With the help of the Liapunov-Tauber theorem (see (16)) we then conclude that $\{\mathcal{T} U_0\}^{+} = \{\mathcal{T} U_0\}^{-} = 0$, i.e., U_0 is a solution to the homogeneous exterior Neumann boundary value problem of the thermo-electro-magneto elasticity theory. Since $U_0 \in [C^{1,\kappa'}(\overline{\Omega^{-}})]^6 \cap Z(\Omega^{-})$, we can apply the uniqueness result obtained in [16] and conclude that $U_0 = W(h^{(0)}) = 0$ in Ω^{-}. Thus $U_0 = W(h^{(0)}) = 0$ in $\Omega^{+} \cup \Omega^{-}$. Whence, in view of the equality

$$\{U_0\}^{+} - \{U_0\}^{-} = \{W(h^{(0)})\}^{+} - \{W(h^{(0)})\}^{-} = h^{(0)} = 0 \text{ on } S, \tag{36}$$

which shows that the null space of the operator $2^{-1} I_6 + \mathcal{N}$ is trivial. Therefore the operators (34) and

$$2^{-1} I_6 + \mathcal{N} \;:\; [C^{1,\kappa'}(S)]^6 \to [C^{1,\kappa'}(S)]^6 \tag{37}$$

are invertible. Then it follows that the non-homogeneous equation (33) is uniquely solvable for arbitrary right hand side vector function f.

As a result we have the following existence theorem.

Theorem 8 *Let $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$ with $0 < \kappa' < \kappa \le 1$. Then the interior Dirichlet BVP* (30)-(31) *is uniquely solvable in the space of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^{+}})]^6 \cap [C^{2}(\Omega^{+})]^6$ for arbitrary f and the solution is representable by the double layer potential, $U = W(h)$, where the density vector function $h \in [C^{1,\kappa'}(S)]^6$ is defined by the uniquely solvable singular integral equation* (33).

5. Investigation of the interior and exterior Dirichlet BVPs: single layer approach. First, we consider the interior Dirichlet problem and look for a solution to the problem (30)-(31) in the form of single layer potential

$$U(x) \equiv V(h)(x) = \int_S \Gamma(x-y)\, h(y)\, dS_y, \tag{38}$$

where Γ is given by equation (7) and $h = (h_1, ..., h_6)^{\top}$ is an unknown density. By Theorem 3 and in view of the boundary condition (31), we get the following integral equation for the density vector function h

$$\mathcal{H}\, h = f \quad \texttt{on} \quad S. \tag{39}$$

Here $\mathcal{H}$ is a weakly singular integral operator defined by (17) and is a strongly elliptic pseudodifferential operator of order -1 with index zero. Now, we show that the null space of the operator

$$\mathcal{H} \;:\; [H_2^{-\frac{1}{2}}(S)]^6 \to [H_2^{\frac{1}{2}}(S)]^6 \tag{40}$$

is trivial. To this end, we have to show that the corresponding homogeneous equation

$$\mathcal{H}\, h = 0 \quad \text{on} \;\; S \tag{41}$$

has only the trivial solution. Let $h^{(0)} \in [H_2^{-\frac{1}{2}}(S)]^6$ be a solution of equation (41), i.e. $\mathcal{H}\, h^{(0)} = 0$ on S, and construct the single layer potential $U_0(x) \equiv V(h^{(0)})(x)$, $x \in \Omega^{\pm}$.

By the embedding theorems (see [13], [15]), one can easily derive that actually $h^{(0)} \in [C^{0,\kappa'}(S)]^6$ and consequently $U_0 \in [C^{1,\kappa'}(S)]^6$. It is clear that $A(\partial)\, U_0 = 0$ in $\Omega^{\pm}$. Further, $\{U_0\}^+ = \mathcal{H}\, h^{(0)} = 0$ on S. Hence U_0 is a solution to the homogeneous interior Dirichlet BVP. Therefore, by the uniqueness theorem for the interior Dirichlet BVP we conclude that $U_0 = 0$ in Ω^+. Due to the continuity of the single layer potential $\{U_0\}^- = \{U_0\}^+ = \mathcal{H} h^{(0)} = 0$ on S (see Theorem 3). Then it follows that the vector function $U_0 = V(h^{(0)})$ is a solution of the exterior homogeneous Dirichlet BVP. Since the single layer potential possesses the property $Z(\Omega^-)$, we can apply the uniqueness theorem for the exterior Dirichlet BVP and we arrive at the equality $U_0 = V(h^{(0)}) = 0$ in Ω^-. Thus $U_0 = V(h^{(0)}) = 0$ in $\Omega^{\pm}$. Using the jump relation (14) we get

$$\{\mathcal{T} U_0\}^+ - \{\mathcal{T} U_0\}^- = \{\mathcal{T} V(h^{(0)})\}^+ - \{\mathcal{T} V(h^{(0)})\}^- = -h^{(0)} \;= 0 \quad \text{on} \;\; S. \tag{42}$$

Therefore, the homogeneous equation (41) has only the trivial solution. Since the operator (40) has zero index it then follows that it is invertible. Therefore the operator

$$\mathcal{H} \;\; : \;\; [C^{k,\kappa'}(S)]^6 \to [C^{k+1,\kappa'}(S)]^6 \tag{43}$$

is also invertible. Finally we conclude that the integral equation (39) is uniquely solvable in the space $[C^{k,\kappa'}(S)]^6$ for an arbitrary right hand side $f \in [C^{k+1,\kappa'}(S)]^6$. In particular, if $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$, than the integral equation (39) has a unique solution $h \in [C^{0,\kappa'}(S)]^6$. Clearly, then $V(h) \in [C^{1,\kappa'}(\overline{\Omega^{\pm}})]^6 \cap [C^{\infty}(\Omega^{\pm})]^6$.
Thus we have the following existence theorem.

Theorem 9 *Let $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$ with $0 < \kappa' < \kappa \leq 1$. Then the interior Dirichlet BVP* (30)-(31) *is uniquely solvable in the space of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ and the solution is representable in the form of the single layer potential, $U = V(h)$, where the density vector function $h \in [C^{0,\kappa'}(S)]^6$ is defined by the uniquely solvable integral equation* (39)*.*

The proof of the following theorem is word for word.

Theorem 10 *Let $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$ with $0 < \kappa' < \kappa \leq 1$. Then the exterior Dirichlet boundary value problem*

$$A(\partial)\, U(x) = 0, \quad x \in \Omega^-, \tag{44}$$

$$\{\, U(x)\}^- = f, \quad x \in S, \tag{45}$$

is uniquely solvable in the space of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$ *and the solution is representable in the form of the single layer potential,* $U = V(h)$*, where the density vector function* $h \in [C^{0,\kappa'}(S)]^6$ *is defined by the uniquely solvable integral equation* (39)*.*

Corollary 11 *Denote by* $\mathcal{H}^{-1}$ *the operator inverse to* $\mathcal{H}$*,*

$$\mathcal{H}^{-1} \ : \ [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6, \quad \mathcal{H}^{-1} \ : \ [C^{1,\kappa'}(S)]^6 \to [C^{0,\kappa'}(S)]^6. \tag{46}$$

An arbitrary solution $U \in [C^{1,\kappa'}(\overline{\Omega^+})]^6 \cap [C^2(\Omega^+)]^6$ *in* Ω^+ *or* $U \in [C^{1,\kappa'}(\overline{\Omega^-})]^6 \cap [C^2(\Omega^-)]^6 \cap Z(\Omega^-)$ *in* Ω^- *of the equation* $A(\partial)\,U = 0$ *is representable in the form*

$$U(x) = V(\mathcal{H}^{-1}\,\{U\}^{\pm})(x), \ \ x \in \Omega^{\pm}. \tag{47}$$

6. Investigation of the exterior Dirichlet BVP: an alternative approach. We assume that $S \in C^{2,\kappa}$ and $f \in [C^{1,\kappa'}(S)]^6$ with $0 < \kappa' < \kappa \le 1$ and look for a solution to the exterior Dirichlet BVP (44)-(45) in the form of linear combination of the single and double layer potentials

$$U(x) = W(h)(x) + \alpha V(h)(x), \tag{48}$$

where $h = (h_1, ..., h_6)^\top \in [C^{1,\kappa'}(S)]^6$ is an unknown density vector and α is a positive constant. By Theorem 3 and in view of the boundary condition (45), we get the following equation for the density vector function h

$$-\frac{1}{2}\,h + \mathcal{N}\,h + \alpha\mathcal{H}\,h = f, \tag{49}$$

where the operators $\mathcal{H}$ and $\mathcal{N}$ are defined by (17) and (19) respectively. The operator $-\frac{1}{2}\,I_6 + \mathcal{N}$ is a singular integral operator of normal type with index zero and $\mathcal{H}$ is a compact operator. Denote

$$\mathcal{M} := -\frac{1}{2}\,I_6 + \mathcal{N} + \alpha\mathcal{H}\,. \tag{50}$$

This operator possesses the following mapping properties

$$\mathcal{M} \quad : \quad [L_2(S)]^6 \to [L_2(S)]^6, \tag{51}$$

$$\quad : \quad [C^{1,\kappa'}(S)]^6 \to [C^{1,\kappa'}(S)]^6. \tag{52}$$

Moreover, $\mathcal{M}$ is a singular integral operator of normal type with zero index. Now we show that the null space of the operator $\mathcal{M}$ is trivial. Let $h^{(0)} \in [L_2(S)]^6$ be a solution of the homogeneous integral equation

$$\mathcal{M}\,h^{(0)} \equiv -\frac{1}{2}\,h^{(0)} + \mathcal{N}\,h^{(0)} + \alpha\,\mathcal{H}\,h^{(0)} = 0 \ \ \texttt{on} \ \ S. \tag{53}$$

From the embedding theorems it follows that $h^{(0)} \in [C^{1,\kappa'}(S)]^6$. Further we construct the vector function

$$U_0(x) = W(h^{(0)})(x) + \alpha\, V(h^{(0)})(x), \quad x \in \Omega^{\pm}. \tag{54}$$

Clearly, $U_0 \in [C^1(\overline{\Omega^{\pm}})]^6 \cap [C^{\infty}(\Omega^{\pm})]^6$ and it has the property $Z(\Omega^-)$ (see Theorem 2). In view of (53)-(54) we have

$$\mathcal{M}\, h^{(0)} = \{U_0\}^- = 0 \;\; \texttt{on} \;\; S. \tag{55}$$

Therefor, U_0 is a solution to the exterior homogeneous Dirichlet BVP and by the uniqueness theorem $U_0 = 0$ in Ω^-. Applying the jump relations we can write the following equalities

$$\{U_0\}^+ - \{U_0\}^- = h^{(0)}, \tag{56}$$

$$\{\mathcal{T}\, U_0\}^+ - \{\mathcal{T}\, U_0\}^- = -\alpha\, h^{(0)}, \tag{57}$$

on S, i.e., $\{U_0\}^+ = h^{(0)}$ and $\{\mathcal{T}\, U_0\}^+ = -\alpha\, h^{(0)}$ on S. Whence it follows that the function U_0 solves the interior Robin type BVP:

$$A(\partial)\, U_0 = 0 \;\; \texttt{in} \;\; \Omega^+, \tag{58}$$

$$\{\mathcal{T}\, U_0\}^+ + \alpha\, \{U_0\}^+ = 0 \;\; \texttt{on} \;\; S. \tag{59}$$

Now we prove that the BVP (58)-(59) admits only the zero solution.

Lemma 12 *In the space of regular vector-functions the homogeneous Robin type BVP* (58)-(59) *with $\alpha > 0$ has only the trivial solution.*

Proof. Let $U = (u, \varphi, \psi, \vartheta)^{\top}$ be a solution to the problem (58)-(59). Then ϑ is a solution to the following BVP:

$$\eta_{jl}\, \partial_j\, \partial_l \vartheta = 0 \;\; \texttt{in} \;\; \Omega^+, \tag{60}$$

$$\{\eta_{jl}\, n_j\, \partial_l \vartheta\}^+ + \alpha\, \{\vartheta\}^+ = 0 \;\; \texttt{on} \;\; S. \tag{61}$$

Using Green's formula

$$\int_{\Omega^+} \eta_{jl}\, \partial_j\, \partial_l \vartheta \cdot \vartheta\, dx = -\int_{\Omega^+} \eta_{jl}\, \partial_l \vartheta\, \partial_j \vartheta\, dx + \int_S \{\eta_{jl}\, n_j\, \partial_l \vartheta\}^+ \cdot \{\vartheta\}^+\, dS \tag{62}$$

and with the help of (60)-(61), we derive

$$\int_{\Omega^+} \eta_{jl}\, \partial_l \vartheta \cdot \partial_j \vartheta\, dx = -\alpha \int_S |\{\vartheta\}^+|^2\, dS. \tag{63}$$

Note that the matrix $[\eta_{jl}]_{3\times 3}$ is positive definite ([21], [19], [18]). Taking into account that $\alpha > 0$, from (63) we easily conclude $\vartheta = const$ in Ω^+ and $\{\vartheta\}^+ = 0$

on S, whence $\vartheta = 0$ in Ω^+ follows. Now, from (58)-(59) it follows that the vector function $\widetilde{U} = (u, \varphi, \psi)^\top$ is a solution of the homogeneous BVP

$$\widetilde{A}(\partial)\,\widetilde{U}(x) = 0, \quad x \in \Omega^+, \tag{64}$$

$$\{T(\partial, n)\,\widetilde{U}(x)\}^+ + \alpha\,\{\widetilde{U}(x)\}^+ = 0, \quad x \in S, \tag{65}$$

where $\widetilde{A}^{(0)}$ is the 5×5 matrix differential operator generated by the equations of statics of thermo-electro-magneto elasticity theory (without taking into account thermal effects, see [4], [5])

$$\widetilde{A}^{(0)}(\partial) := \begin{bmatrix} [c_{rjkl}\,\partial_j\,\partial_l]_{3\times 3} & [e_{lrj}\,\partial_j\partial_l]_{3\times 1} & [q_{lrj}\,\partial_j\partial_l]_{3\times 1} \\ [-e_{jkl}\,\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\,\partial_j\,\partial_l & a_{jl}\,\partial_j\,\partial_l \\ [-q_{jkl}\,\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\,\partial_l & \mu_{jl}\,\partial_j\,\partial_l \end{bmatrix}_{5\times 5}, \tag{66}$$

and $T(\partial, n)$ is the corresponding 5×5 generalized stress operator

$$T(\partial, n) = \begin{bmatrix} [c_{rjkl}\,n_j\,\partial_l]_{3\times 3} & [e_{lrj}\,n_j\,\partial_l]_{3\times 1} & [q_{lrj}\,n_j\,\partial_l]_{3\times 1} \\ [-e_{jkl}\,n_j\,\partial_l]_{1\times 3} & \varkappa_{jl}\,n_j\,\partial_l & a_{jl}\,n_j\,\partial_l \\ [-q_{jkl}\,n_j\,\partial_l]_{1\times 3} & a_{jl}\,n_j\,\partial_l & \mu_{jl}\,n_j\,\partial_l \end{bmatrix}_{5\times 5}. \tag{67}$$

For arbitrary regular solution $\widetilde{U}$ of equation (64) there holds Green's identity [4],

$$\int_{\Omega^+} \mathcal{E}(\widetilde{U}, \widetilde{U})\,dx = \int_S \{T\,\widetilde{U}\}^+ \cdot \{\widetilde{U}\}^+\,dS, \tag{68}$$

where

$$\mathcal{E}(\widetilde{U}, \widetilde{U}) = c_{rjkl}\,\partial_l u_k\,\partial_j u_r + \varkappa_{jl}\,\partial_l\varphi\,\partial_j\varphi + 2a_{jl}\,\partial_l\varphi\,\partial_j\psi + \mu_{jl}\,\partial_l\psi\,\partial_j\psi \quad \texttt{in}\ \Omega^+. \tag{69}$$

Recall that (see [4], [5], [18], [21])

$$\mathcal{E}(\widetilde{U}, \widetilde{U}) \geq \delta_0\,[\,e_{kj}(u)e_{kj}(u) + |\nabla\varphi|^2 + |\nabla\psi|^2\,] \tag{70}$$

with some positive constant δ_0, where $e_{kj}(u) = 2^{-1}(\partial_k u_j + \partial_j u_k)$ denotes the mechanical strain tensor. By virtue of the condition (65) we easily derive that

$$\int_{\Omega^+} \mathcal{E}(\widetilde{U}, \widetilde{U})\,dx = -\alpha \int_S |\{\widetilde{U}\}^+|^2\,dS. \tag{71}$$

Due to the condition $\alpha > 0$, from (70) and (71) we conclude

$$\mathcal{E}(\widetilde{U}, \widetilde{U}) = 0 \quad \texttt{in}\ \Omega^+, \quad \{\widetilde{U}\}^+ = 0 \quad \texttt{on}\ S. \tag{72}$$

The first equation in (72) implies that

$$u(x) = [a \times x] + b, \quad \varphi(x) = b_4, \quad \psi(x) = b_5, \quad \texttt{in}\ \ \Omega^+, \tag{73}$$

where $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ are arbitrary constant vectors, and b_4 and b_5 are arbitrary constants. By the second equation in (72) it then follows that $a=b=0$, $b_4=b_5=0$ and, consequently, $\widetilde{U}=0$ on Ω^+ which completes the proof. □

Applying Lemma 12 we conclude that the vector U_0 defined by (54), which is a solution of the homogeneous Robin type BVP (58)-(59), vanishes in Ω^+ as well. Thus $U_0=0$ in $\Omega^+\cup\Omega^-$. Then from relations (56)-(57) it follows that $h^{(0)}=0$ on S. Therefore the null spaces of the operators (51)-(52) are trivial and they are invertible. Consequently the integral equation (49) is uniquely solvable for an arbitrary right hand side vector f. Thus, we have the following existence result.

Theorem 13 *Let $S\in C^{2,\kappa}$ and $f\in[C^{1,\kappa'}(S)]^6$ with $0<\kappa'<\kappa\le 1$. Then the exterior Dirichlet boundary value problem* (44)-(45) *is uniquely solvable in the space of regular vector-functions $[C^{1,\kappa'}(\overline{\Omega^-})]^6\cap[C^2(\Omega^-)]^6\cap \mathrm{Z}(\Omega^-)$, and the solution is representable in the form of linear combination of the single and double layer potentials*

$$U(x)=W(h)(x)+\alpha\, V(h)(x), \tag{74}$$

where α is an arbitrary positive constant and the density vector function $h=(h_1,...,h_6)^\top\in[C^{1,\kappa'}(S)]^6$ is defined by the uniquely solvable singular integral equation (49).

The counter parts of the above formulated existence theorems hold also in the Sobolev-Slobodetski and Bessel potential spaces. We formulate here the existence theorem for the interior Dirichlet problem.

Theorem 14 *Let $S\in C^\infty$ and $f\in[B_{p,p}^{1-\frac{1}{p}}(S)]^6$ with $p>1$. Then the pseudodifferential operator*

$$2^{-1}I_6+\mathcal{N}\ :\ [B_{p,p}^{1-\frac{1}{p}}(S)]^6\to[B_{p,p}^{1-\frac{1}{p}}(S)]^6 \tag{75}$$

is invertible and the interior Dirichlet BVP (30)-(31) is uniquely solvable in the space $[H_p^1(\Omega^+)]^6=[W_p^1(\Omega^+)]^6$, and the solution is representable in the form of double layer potential $U=W(h)$ with the density vector function $h\in[B_{p,p}^{1-\frac{1}{p}}(S)]^6$ being a unique solution of the equation

$$[\,2^{-1}I_6+\mathcal{N}\,]\,h=f\ \text{ on }\ S. \tag{76}$$

If $f\in[B_{p,p}^{k-\frac{1}{p}}(S)]^6$ with $p>1$ and integer $k\ge 1$, then $h\in[B_{p,p}^{k-\frac{1}{p}}(S)]^6$ and the corresponding solution $U=W(h)\in[H_p^k(\Omega^+)]^6$.

There holds also the following generalization of Corollary 11 (cf. [18]).

Corollary 15 *Let $k \geq 1$ be integer and $U \in [H_p^k(\Omega^+)]^6$ be a solution to the homogeneous equation $A(\partial)U = 0$ in Ω^+. Then U is uniquely representable in the form*

$$U(x) = V(\mathcal{H}^{-1}\{U\}^+)(x), \quad x \in \Omega^+, \tag{77}$$

where $[U]^+ \in [B_{p,p}^{k-\frac{1}{p}}(S)]^6$ is the traces of U on S from Ω^+ and $\mathcal{H}^{-1}$ stands for the operator inverse to the operator

$$\mathcal{H} \,:\, [B_{p,p}^{k-1-\frac{1}{p}}(S)]^6 \to [B_{p,p}^{k-\frac{1}{p}}(S)]^6.$$

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