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Antiplane Strain (Shear) of Isotropic Non-Homogeneous Prismatic Shell-Like Bodies

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Antiplane strain (shear) of an isotropic non-homogeneous prismatic shell-like body is considered when the shear modulus depending on the body projection (i.e., on a domain lying in the plane of interest) variables vanishes either on a part or on the entire boundary of the projection. The dependence of well-posedeness of boundary conditions on the character of vanishing of the shear modulus is studied. When the above-mentioned domain is either the half-plane or the half-disk and the shear modulus is a power function with respect to the variable along the perpendicular to the linear boundary, the basic boundary value problems are solved explicitly in quadratures.

Keywords: Antiplane Strain (shear) deformation, Degenerate elliptic equations.

AMS Subject Classification: 74K20, 35J70.

1. Introduction

For the convenience of the reader we remind that antiplane shear (strain) is a special state of strain in a body. This state is achived when the displacements in the body are zero in the plane of interest but nonzero in the direction perpendicular to the plane. If the plane Ox_1x_2 of the rectangular Cartesian frame $Ox_1x_2x_3$ is the plane of interest, then

$$u_\alpha(x_1,x_2,x_3) \equiv 0, \quad \alpha = 1,2; \quad u_3(x_1,x_2,x_3) = u_3(x_1,x_2), \qquad (x_1,x_2) \in \omega, \quad (1)$$

where u_i, $i = 1,2,3$, are the displacements, ω is a projection of the prismatic shell-like body Ω on the plane Ox_1x_2, correspondingly $\partial\omega$ is a projection of the lateral boundary S of Ω. The relations (1) mean that all the sections of the body parallel to the plane of interest Ox_1x_2 will be bent as its section by the plane Ox_1x_2. Ω may have either Lipschitz (see Figures 1-4) or non-Lipschitz (see Figures 5) boundary, ω has Lipschitz (see Figures 6-8) boundary.

For an isotropic linear elastic material the strain e_{ij} and stress X_{ij} tensors that

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result from a state of antiplane shear can be expressed as

$$e_{\alpha\beta} \equiv 0, \quad \alpha,\beta = 1,2; \quad e_{33} \equiv 0; \quad e_{\alpha 3} = \frac{1}{2}u_{3,\alpha}(x_1,x_2) \not\equiv 0, \quad \alpha = 1,2, \tag{2}$$

where the comma after the index means differentiation with respect to the variable corresponding to the index indicated after the comma, and

$$\begin{aligned} &X_{\alpha\beta} \equiv 0, \ \ \alpha,\beta = 1,2; \quad X_{33} \equiv 0; \\ &X_{3\alpha} = X_{\alpha 3} = \mu(x_1,x_2)u_{3,\alpha}(x_1,x_2), \ \ \alpha = 1,2, \end{aligned} \tag{3}$$

since for non-homogeneous body with the shear modulus $\mu(x_1,x_2)$ the Hook law looks like

$$X_{\alpha 3} = 2\mu e_{\alpha 3} = \mu(x_1,x_2)u_{3,\alpha}(x_1,x_2), \ \ \alpha = 1,2. \tag{4}$$

From (3), (4) it follows that at any point $x := (x_1,x_2,x_3)$ stress vector components

$$X_{n\alpha} = X_{j\alpha}n_j = X_{3\alpha}n_3 = \mu u_{3,\alpha}n_3, \quad \alpha = 1,2; \tag{5}$$

$$X_{n3} = X_{j3}n_j = X_{\alpha 3}n_\alpha = \mu u_{3,\alpha}n_\alpha = \mu\frac{\partial u_3}{\partial n}, \tag{6}$$

where Einstein's summation convention is used and n is the unit normal of a surface element passing through x.

The equilibrium equations reduce to

$$\Phi_\alpha \equiv 0, \quad \alpha = 1,2, \qquad X_{\alpha 3,\alpha} + \Phi_3 = 0, \tag{7}$$

where $\Phi_i, \ \ i = 1,2,3$, are the components of the volume force.

Let $u_3 \in C^2(\omega), \ \mu \in C^1(\omega)$, and $\Psi \in C(\overline{\omega})$. Substituting (3) into (7) we get only one governing equation

$$(\mu(x_1,x_2)u_{3,\alpha}(x_1,x_2))_{,\alpha} + \Phi_3(x_1,x_2) = 0, \quad (x_1,x_2) \in \omega. \tag{8}$$

In the dynamical case we will have

$$(\mu(x_1,x_2)u_{3,\alpha}(x_1,x_2,t))_{,\alpha} + \Phi_3(x_1,x_2,t) = \rho\ddot{u}_3(x_1,x_2,t), \ (x_1,x_2) \in \omega, \ t \geq t_0. \tag{9}$$

The case of antiplane shear when the shear modulus vanishes was not considered up to now. The aim of the present paper is to investigate boundary value problems (BVPs) for the symmetric prismatic shell-like body Ω (see [1,2]), in particular, of the constant thickness (which may also be infinite) when the shear modulus vanishes either on a part or on the entire boundary of the projection ω on the plane of interest Ox_1x_2 (see Figures 6-8). In the cases when ω is either the upper half-plane $x_2 \geq 0$ or a finite domain lying in the upper half-plane adjacent to x_1-axis and the shear modulus is a power function with respect to x_2 vanishing at boundary $x_2 = 0$, well-posedness of the basic BVPs are investigated. Moreover, in the case of the half-plane and the half-disk they are solved explicitly in quadratures.

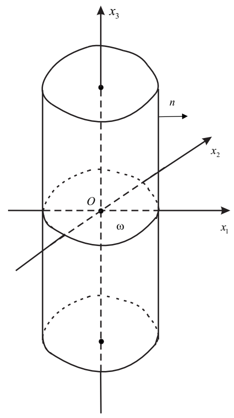

Figure 1. A non-homogeneous elastic cylinder

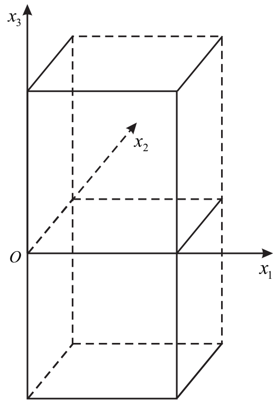

Figure 2. Ω with a Lipschitz boundary

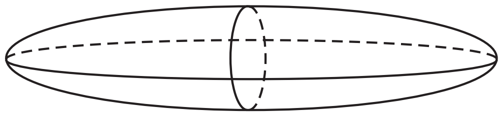

Figure 3. Ω with a smooth boundary

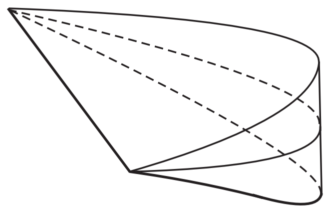

Figure 4. Ω with a Lipschitz boundary

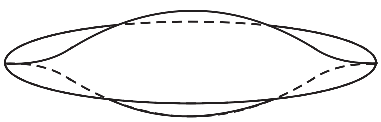

Figure 5. Ω with a non-Lipschitz boundary

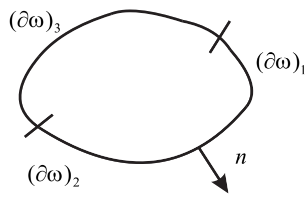

Figure 6. General case of ω

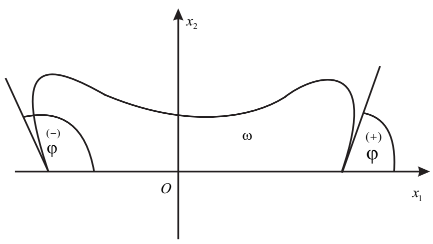

Figure 7. A finite ω

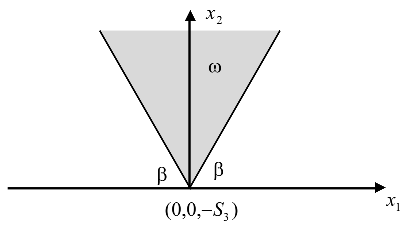

Figure 8. ω is an angle

2. Investigation of BVPs. General case

In this section we study the antiplane deformation of nonhomogeneous elastic cylinders and prismatic shell-like bodies (see Figures 1-5, for other examples see [2]).

Evidently (see (5), (6)), on the lateral boundary S of the cylinder Ω

$$X_{n\alpha} = 0, \quad \alpha = 1, 2, \qquad X_{n3} = \mu \frac{\partial u_3}{\partial n}, \tag{10}$$

where n is the outward normal to S.

If the cylinder is finite, on the upper and lower bases of the cylinder Ω (see (5), (6))

$$X_{\overset{(+)}{n}\alpha} = X_{3\alpha} = \mu u_{3,\alpha}, \quad \alpha = 1, 2, \quad X_{n3} = X_{33} = 0,$$

and

$$X_{\overset{(-)}{n}\alpha} = -X_{3\alpha} = -\mu u_{3,\alpha}, \quad \alpha = 1, 2; \quad X_{n3} = -X_{33} = 0,$$

respectively (in the case of prismatic shell-like bodies they are given by formulas (5), (6) on the face surfaces).

Let the shear modules $\mu \in C^1(\overline{\omega}\backslash(\partial\omega)_2) \cap C(\overline{\omega})$ as already assumed be independent of x_3 and $\mu(x_1, x_2) > 0$ in $\omega \cup (\partial\omega)_3$ (see Figure 6), $\mu(x_1, x_2) = 0$ on $(\partial\omega)_0$, where the boundary $(\partial\omega)_0$ is divided in $(\partial\omega)_1$ and $(\partial\omega)_2$, i.e., $\partial\omega = \overline{(\partial\omega)}_1 \cup \overline{(\partial\omega)}_2 \cup \overline{(\partial\omega)}_3$ (correspondingly $S = \overline{S}_1 \cup \overline{S}_2 \cup \overline{S}_3$) with

$$\left.\frac{\partial \mu}{\partial n}\right|_{(\partial\omega)_2} = +\infty, \tag{11}$$

$$\left.\frac{\partial \mu}{\partial n}\right|_{(\partial\omega)_1} \geq 0, \tag{12}$$

then (compare with [3]): if $(\partial\omega)_1 \neq \varnothing$, a solution u_3 of equation (8) will be determined uniquely by its values prescribed only on $(\partial\omega)_2 \cup (\partial\omega)_3$ (Problem E: $u_3 \in C^2(\omega) \cap C(\bar{\omega} \setminus (\partial\omega))_1 \cap C^b(\omega)$, $C^b(\omega)$ means a class of bounded functions); if $(\partial\omega)_1 = \varnothing$, for unique solvability of the BVP the values of u_3 should be prescribed on the whole boundary $\partial\omega$ (Problem D: $u_3 \in C^2(\Omega) \cap C(\overline{\Omega})$).

The criteria (11) and (12) can be replaced by the equivalent criteria in the integral form (see [5], formulas (13), (14)).

If $X_{n3} = \varphi$ is prescribed on $\partial\omega$, then on $(\partial\omega)_0$ we have to consider with the weighted boundary condition (BC) (Problem W: $u_3 \in C^2(\Omega)$, $\mu\frac{\partial u_3}{\partial n} \in C(\overline{\Omega})$)

$$\lim_{(x_1,x_2)\to(\partial\omega)_0} \mu(x_1, x_2)\frac{\partial u_3}{\partial n} = \varphi. \tag{13}$$

The above mentioned problems are well-posed under some restrictions on classes of functions, where we are looking for solutions (for classical solutions see below; for $\mathcal{H}$-weak solutions [3,4] of Problems D and E see Appendix).

As we see in case of Problem D on the cylindrical boundary S deflections $u_3(x_1, x_2)$ are prescribed, while in case of Problem E deflections $u_3(x_1, x_2)$ should be prescribed only on $S_2 \cup S_3$. BC (13) means that at S shear stresses $X_{n\alpha}(x_1, x_2)$, $\alpha = 1, 2$, are applied. In all the cases at face surfaces should be applied stresses calculated by formulas (5), (6) in order to maintain the antiplane state in the body. Note that if the thickness of the prismatic shell-like body vanishes on a part of $\partial\omega$ or on the entire $\partial\omega$, then by the antiplane shear the character of the thickness vanishing does not affect on well-posedness of BVPs at cusped edges in contrast to cusped prismatic shells for which it is the case and depends (see [2]) on the sharpening geometry.

Let

$$\mu(x_1, x_2) = \mu_0 x_2^{\kappa}, \quad \mu_0 > 0, \quad \kappa \geq 0, \quad (x_1, x_2) \in \omega. \tag{14}$$

In this case equation (8) has the form

$$x_2(u_{3,11} + u_{3,22}) + \kappa u_{3,2} = -\frac{1}{\mu_0} x_2^{1-\kappa} \Phi_3(x_1, x_2). \tag{15}$$

The partial differential operator in the left-hand side of equation (15) has the order degeneration on the x_1-axis.

If ω is the upper unit half-disk using Weinstein's [6] fundamental solution

$$Z(x_1, x_2, \xi, \eta, \kappa) = \frac{1}{2\pi} \int\limits_0^{\pi} \left[(x_1 - \xi)^2 + (x_2 - \eta)^2 + 4x_2\eta sin^2\frac{\theta}{2} \right]^{-\kappa/2} sin^{\kappa-1}\theta d\theta$$

$$(-\infty < x_1,\ \xi < +\infty,\ x_2 \geq 0, \eta \geq 0)$$

of the degenerate PDE (15), by means of the method of potentials Problem D and Problem E are solved in the following explicit form [7], respectively:

$$u_3(x_1, x_2) = \frac{x_2^{1-\kappa}}{\mu_0} \int\limits_D\int \eta^{1-\kappa} \Phi_3(\xi, \eta) G(x_1, x_2, \xi, \eta, 2-\kappa) d\xi d\eta$$

$$+(1-\kappa) x_2^{1-\kappa} \int\limits_{-1}^{+1} G(x_1, x_2, \xi, 0, 2-\kappa) f(\xi, 0) d\xi$$

$$-x_2^{1-\kappa} \int\limits_0^{\pi} sin\varphi f(cos\varphi, sin\varphi) \frac{\partial G(x_1, x_2, cos\varphi, sin\varphi, 2-\kappa)}{\partial n} d\varphi, \quad \kappa < 1 \ \ ((\partial\omega)_1 = \varnothing)$$

(double integral is continuous on $\overline{\omega}$ and vanishes on $\partial\omega$) and

$$u_3(x_1, x_2) = \frac{1}{\mu_0} \int\limits_D\int \Phi_3(\xi, \eta) G(x_1, x_2, \xi, \eta, \kappa) d\xi d\eta$$

$$-\int\limits_0^{\pi} sin^{\kappa}\varphi f(cos\varphi, sin\varphi) \frac{\partial G(x_1, x_2, cos\varphi, sin\varphi, \kappa)}{\partial n} d\varphi, \quad \kappa \geq 1 \ \ ((\partial\omega)_2 = \varnothing)$$

(double integral is continuous on $\omega \cup (\partial\omega)_3$, vanishes on the half-circle $(\partial\omega)_3$, and for its boundedness in ω by $\kappa \geq 2$ we assume additionally $\Psi_3(x_1, x_2) \equiv 0$ in some right neighborhood of $(\partial\omega)_2$)

$$G(x_1, x_2, \xi, \eta, \kappa) = Z(x_1, x_2, \xi, \eta, \kappa) - (\xi^2 + \eta^2)^{-\kappa/2} Z\Big(x_1, x_2, \frac{\xi}{\xi^2+\eta^2}, \frac{\eta}{\xi^2+\eta^2}, \kappa\Big).$$

If ω is a domain given in Figure 7, then Problem D, Problem E, and Problem W are uniquely solvable [8-10] in the indicated function classes.

Let us now consider a more general degenerate equation [10,11]

$$x_2(u_{,11} + u_{,22}) + au_{,1} + bu_{,2} = 0, \quad a, b = const. \tag{16}$$

Problem D and Problem E are uniquely solvable for equation (16) [8,10].

If in Problem D the prescribed function on a part of the boundary belonging to the x_1-axis has a finite number of points of discontinuity of the first kind, Problem D is uniquely solvable and, approaching point $(x_1, x_2) \in \omega$ to the point $(\xi_k, 0)$ of discontinuity of the boundary datum $f(x_1)$ along different ways, a unique solution $u(x_1, x_2)$ takes all the values between one-sided limits $\overset{(-)}{f}(\xi_k)$ and $\overset{(+)}{f}(\xi_k)$ of $f(x_1)$ along the x_1-axis.

Proof: A function

$$\overset{1}{u}_k(x_1, x_2) := \frac{h_k}{\alpha_k(b)} \int\limits_0^{arg(z-\xi_k)} e^{a\tau} sin^{-b}\tau d\tau, \tag{17}$$

where

$$z := x_1 + ix_2, \quad h_k := \overset{(+)}{f}(\xi_k) - \overset{(-)}{f}(\xi_k), \quad k = \overline{1, n},$$

$$\alpha_k(b) = -\Lambda(a, b), \quad k = \overline{2, n-1},$$

$$\alpha_1(b) = -\int\limits_0^{\overset{(-)}{\varphi}} e^{a\tau} sin^{-b}\tau d\tau, \quad \alpha_n(b) = -\int\limits_{\overset{(+)}{\varphi}}^{\pi} e^{a\tau} sin^{-b}\tau d\tau$$

(see Figure 7) is a solution of (16). Evidently

$$\overset{(1)}{u}_k(x_1, 0) = \begin{cases} \frac{h_k}{\alpha_k(b)} \Lambda(a, b), & x_1 \in]\xi_1, \xi_k[, \ k = \overline{2, n}; \\ 0, & x_1 \in]\xi_k, \xi_n[, \ k = \overline{1, n-1}. \end{cases} \tag{18}$$

If $(x_1, x_2) \to (\xi_k, 0)$ along the way with the tangent at the point $(\xi_k, 0)$ forming the angle φ with the x_1-axis (evidently, at the points $(\xi_k, 0)$, $k = \overline{2, n-1}$, the angle $\varphi \in [0,\pi]$; at the point $(\xi_1,0)$ the angle $\varphi \in [0, \overset{(-)}{\varphi}]$, at the point $(\xi_n,0)$ the angle $\varphi \in [\overset{(+)}{\varphi},\pi]$), then the function (17), i.e. $\overset{1}{u}_k$, tends to

$$\frac{h_k}{\alpha_k(b)}\int_0^{\varphi} e^{a\tau} sin^{-b}\tau d\tau.$$

By passing $(\xi_k,0)$ along $\partial\omega$ in the positive direction, $\overset{1}{u}_k$ undergoes jumps

$$0-(-h_k)=h_k$$

for $k=\overline{2,n-1}$;

$$0-\frac{h_1}{\alpha_1(b)}\int_0^{\overset{(-)}{\varphi}} e^{a\tau} sin^{-b}\tau d\tau = h_1$$

for $k=1$, and

$$\frac{h_n}{\alpha_n(b)}\Big[\int_0^{\overset{(+)}{\varphi}} e^{a\tau} sin^{-b}\tau d\tau - \int_0^{\pi} e^{a\pi} sin^{-b}\tau d\tau\Big] = h_n$$

for $k=n$. Therefore,

$$f(\zeta)-\sum_{k=1}^{n}\frac{h_k}{\alpha_k(b)}\int_0^{arg(\zeta-\xi_k)} e^{a\tau} sin^{-b}\tau d\tau = f(\xi)-\overset{1}{u}_k(\xi)-\sum_{\substack{j=i\\ j\neq k}}^{n}\overset{1}{u}_j(\xi), \quad \xi\in\partial\omega, \quad (19)$$

becomes continuous by passing through $(\xi_k,0)$, since $f(\xi)$ and $\overset{1}{u}_k(\xi)$ have the same jump h_k at the point ξ_k, while the sum

$$\sum_{\substack{j=1\\ j\neq k}}^{n}\overset{1}{u}_j(\zeta) \quad (20)$$

is a continuous function, at the point ξ_k. Let us denote by $U_1(x_1,x_2)$ a unique solution of Problem D for equation (16) with continuous datum (19) and consider the behavior of

$$\overset{1}{u}(x_1,x_2)=U_1(x_1,x_2)+\sum_{k=1}^{n}\overset{1}{u}_k(x_1,x_2) \quad (21)$$

which, evidently, is bounded, satisfies equation (18) and BC

$$\lim_{z\to\zeta\neq(\xi_k,\,0)} \overset{1}{u}(x_1,x_2)=f(\zeta).$$

Now, let $(x_1,x_2)\to(\xi_k,0)$ along the way with the tangent at the point $(\xi_k,0)$ forming the angle φ with the x_1-axis. Then from (21) we obtain the limit

$$\overset{1}{u}_{\varphi}(\xi_k)=\widetilde{U}_1(\xi_k)+\frac{h_k}{\alpha_k(b)}\int\limits_0^{\varphi} e^{a\tau}sin^{-b}\tau d\tau, \tag{22}$$

where $\widetilde{U}_1(\xi_k)$ is the limit of the sum of U_1 and (20) which does not depend on the way of approaching the point $(\xi_k,0)$. In particular, tending to $(\xi_k,0)$ along $\partial\omega$ in the negative direction, we get

$$\overset{(+)}{f}(\xi_k)=\widetilde{U}_1(\xi_k)+\overset{(+)}{\overset{1}{u}}_k(\xi_k), \tag{23}$$

where

$$\overset{(+)}{\overset{1}{u}}_k(\xi_k)=\begin{cases}0, & k=\overline{1,n-1};\\ \dfrac{h_n}{\alpha_n(b)}\displaystyle\int\limits_0^{\overset{(+)}{\varphi}} e^{a\tau}sin^{-b}\tau d\tau, & k=n.\end{cases}$$

Substituting calculated from (23) $\widetilde{U}_1(\xi_k)$ into (22), we have

$$\overset{1}{u}_{\varphi}(\xi_k)=\overset{(+)}{f}(\xi_k)+\begin{cases}\dfrac{h_k}{\alpha_k(b)}\displaystyle\int\limits_0^{\varphi} e^{a\tau}sin^{-b}\tau d\tau, & k=\overline{1,n-1};\\ \dfrac{h_n}{\alpha_n(b)}\displaystyle\int\limits_{\overset{(+)}{\varphi}}^{\varphi} e^{a\tau}sin^{-b}\tau d\tau, & k=n.\end{cases}$$

The right-hand side of the last equality is the continuous function of φ on the segments $[0,\pi]$, $[0,\overset{(-)}{\varphi}]$, and $[\overset{(+)}{\varphi},\pi]$ for $k=\overline{2,n-1}$, $k=1$, and $k=n$, respectively, and at the ends of the segments takes values $\overset{(+)}{f}_k(\xi_k)$ and $\overset{(-)}{f}_k(\xi_k)$. Whence, according to the Bolzano-Cauchy (intermediate value) theorem it takes at some points of the open intervals corresponding to above-mentioned segments all the values between $\overset{(+)}{f}(\xi_k)$ and $\overset{(-)}{f}(\xi_k)$.

This result for the Laplace equation is well-known (see, e.g., [11], p. 212).

If $a=0$, $b=\kappa$, then equation (16) coincides with (8) and the above result is valid for the antiplane shear in case (14). This result gives an interesting geometric interpretation of the surface of deflactions in the neighborhood of points of discontinuity of the first kind of deflections at the boundary $x_2=0$: it is the continuous

surface like a fan, in other words, like the graph of $x_3 = argz$ near $z = 0, \;\; x_2 \geq 0$.
□

3. Case of the half-space

Let us now consider antiplane shear of the nonhomogeneous elastic half-space with the half-plane $x_2 \geq 0$ as the plane of interest.

Problem D: Find $u \in C^2(x_2 > 0) \cap C(x_2 \geq 0)$, satisfying (8) and the boundary condition

$$u(x_1, 0) = f(x_1), \quad b < 1.$$

Solution of this BVP has the form [8, 12, 13]

$$u(x_1, x_2) = \frac{x_2^{1-b}}{\Lambda(a,b)} \int\limits_{-\infty}^{+\infty} f(\xi) e^{a \cdot \, arcctg \frac{x_1 - \xi}{x_2}} \left[(x_1 - \xi)^2 + x_2^2\right]^{\frac{b}{2}-1} d\xi, \tag{24}$$

where

$$\Lambda(a,b) := \int\limits_0^\pi e^{a\theta} sin^{-b}\theta d\theta. \tag{25}$$

Problem W: Find $u \in C^2(x_2 > 0), \quad x_2^b u_{,2} \in C(x_2 \geq 0)$, satisfying (8) and BC

$$\lim_{x_2 \to 0+} x_2^b u_{,3} = f(x_1), \tag{26}$$

where either

$$a \neq 0, \quad b > 0 \quad or \quad a = 0, \quad b > -1 \quad but \quad b \neq 0. \tag{27}$$

Solution of this BVP has the form [8, 12, 13]

$$u := -\frac{a^2 + b^2}{b} \Lambda(a, -b) \int\limits_{-\infty}^{+\infty} f(\xi) e^{a \cdot \, arcctg \frac{x_1 - \xi}{x_2}} \left[(x_1 - \xi)^2 + x_2^2\right]^{-\frac{b}{2}} d\xi, \tag{28}$$

where

$$f(\xi) = O(|\xi|^{-\alpha}), \quad |\xi| \to +\infty, \quad \alpha > 1 - b. \tag{29}$$

4. Case of the angle

Let us consider the angle as ω (see Figure 8). In particular, if $\beta = 0$, we get the half-plane. Let $\Phi_3 \equiv 0$ and at the edge $r = 0$, $-\infty < x_3 < +\infty$, of the body (dihedron) the shear force $-S_3$ concentrated along the edge be applied. The governing equations (7), (3), (4), and the Saint-Venant condition in the cylindrical coordinates have the form

$$\frac{1}{r}\frac{\partial Z_\psi}{\partial \psi} + \frac{\partial Z_r}{\partial r} + \frac{Z_r}{r} = 0,$$

$$Z_r = 2\mu e_{3r}, \qquad Z_\psi = 2\mu e_{\psi 3},$$

$$e_{3r} = \frac{1}{2}\frac{\partial v_3}{\partial r}, \qquad e_{\psi 3} = \frac{1}{2r}\frac{\partial v_3}{\partial \psi},$$

$$\frac{\partial}{\partial r}\Big(\frac{rZ_\psi}{\mu}\Big) = \frac{\partial}{\partial \psi}\Big(\frac{Z_r}{\mu}\Big).$$

(14) will get the form

$$\mu = \mu_0 r^\kappa sin^\kappa \psi, \qquad \beta < \psi < \pi - \beta.$$

It is easily seen that the solution we are looking for has the form

$$Z_r = \frac{S_3}{\int\limits_\beta^{\pi-\beta} sin^\kappa \psi d\psi}\frac{sin^\kappa \psi}{r}, \qquad Z_\psi = 0, \quad \kappa \geq 0$$

$$u_3 = -\frac{1}{\kappa\mu_0 \int\limits_\beta^{\pi-\beta} sin^\kappa \psi d\psi}\frac{S_3}{r^\kappa}, \qquad \kappa > 0.$$

Appendix

Existence of $\mathcal{H}$-weak solutions

In order to apply existence results of G. Fichera [3] to equation (8) in the sense of $\mathcal{H}$-weak solutions in Hilbert spaces we need to introduce a smooting function χ [14-16, 5] with properties:

1. $\chi \in C^1(\overline{\omega})$;

2. $\chi > 0$ in $\overline{\omega}\backslash(\partial\omega)_0$;

3. $\chi\mu_{,\alpha} \in C(\overline{\omega})$;

4. $\mu\chi_{,\alpha} \in C^1(\overline{\omega})$;

5. $\mu\dfrac{\partial\chi}{\partial n}\Big|_P > 0$ if $\dfrac{\partial\mu}{\partial n}\Big|_P = +\infty, \quad P \in (\partial\omega)_0$;

6. $\mu\dfrac{\partial\chi}{\partial n}\Big|_P < 0$, if $\dfrac{\partial\mu}{\partial n}\Big|_P = 0, \quad P \in (\partial\omega)_0$;

7. $\chi_{,\alpha},\ \alpha = 1, 2$, are finite at points P, where $\dfrac{\partial\mu}{\partial n}\Big|_P \neq 0, +\infty$;

8. $\chi\mu\Big|_P = 0$ if $\dfrac{\partial\mu}{\partial n}\Big|_P = 0$

n is the inward normal to $\partial\omega$.

From these properties we easily conclude that:

(i) if

$$\frac{\partial\mu}{\partial n}\Big|_P = +\infty,$$

then χ is continuous at the point P and $\chi(P) = 0$;

(ii) if

$$\frac{\partial\mu}{\partial n}\Big|_P \neq 0, +\infty,$$

then χ is continuous at the point P;

(iii)

$$\chi\mu\Big|_{(\partial\Omega)_0} = 0.$$

We multiply equation (8) by χ and introduce a new unknown function w by the equality

$$u_3(x_1, x_2) = \psi(x_1, x_2)w(x_1, x_2),$$

where $\psi \in C^2(\overline{\omega})$ and $\phi > 0$ in $\overline{\omega}$.

So we arrive at the equation

$$\chi[\mu(\psi w)_{,\alpha}]_{,\alpha} + \chi\Phi_3 = 0,$$

i.e.,

$$Lw := \chi\mu\psi w_{,\alpha\alpha} + [2\chi\,\mu\psi_{,\alpha} + \chi\,\mu\psi\mu_{,\alpha}]w_{,\alpha} + \chi(\mu\psi_{,\alpha})_{,\alpha}w = -\chi\Phi_3 \qquad (30)$$

for the new unknown

$$w \in C^2(\omega) \cap C^1(\overline{\omega}\backslash(\partial\omega)_0) \cap C_b^0(\overline{\omega}\backslash(\partial\omega)_1).$$

To use Fichera's method we need to assume

$$\chi\mu\psi \in C_b^2(\omega), \quad [2\chi\mu\psi,_\alpha + \chi\psi\mu,_\alpha] \in C_b^1(\omega), \quad \chi(\mu\psi,_\alpha) \in C_b^0(\omega),$$

which will be guaranted by appropriate choice of χ and ψ; subscript "b" means boundedness of the indicated derivatives on ω.

Fichera's function [3,4]

$$F = [2\chi\mu\psi,_\alpha + \chi\psi\mu,_\alpha - (\chi\mu\psi),_\alpha]n_\alpha\Big|_{(\partial\omega)_0}$$
$$= (\chi\mu\psi,_\alpha - \chi,_\alpha \mu\psi)n_\alpha\Big|_{(\partial\omega)_0} = -\mu\psi\frac{\partial\chi}{\partial n}\Big|_{(\partial\omega)_0},$$

since

$$\chi\mu\Big|_{(\partial\Omega)_0} = 0.$$

Taking into account properties 5-7 of χ, we have

$$F(P) < 0 \quad for \quad P \in (\partial\omega)_2$$

$$F(P) \geq 0 \quad for \quad P \in (\partial\omega)_1.$$

Now, in order to prove the existence of a $\mathcal{H}$-weak solution of the problem

$$Lw = -\chi\Psi_3 \quad in \quad \omega, \quad w = 0 \quad on \quad (\partial\omega)_2 \cup (\partial\omega)_3, \tag{31}$$

following G. Fichera [3], we consider for w and a function $v \in C^1(\overline{\omega})$ the integral identity

$$\int\limits_\omega vLwd\omega = -\int\limits_\omega \Big(\chi\mu\psi w_{,\alpha} + w\mu(\chi\psi_{,\alpha} - \psi\chi_{,\alpha})v_{,\alpha}$$
$$+\big\{[\mu(\chi\psi_{,\alpha} - \psi\chi_{,\alpha})]_{,\alpha} - \chi(\mu\psi_{,\alpha})_{,\alpha}\big\}wv\Big)d\omega - \int\limits_{(\partial\omega)_3} v\chi\mu\psi w_{,\alpha}n_\alpha ds - \int\limits_{\partial\omega} wvds.$$

Let $\mathcal{W}$ be the class of functions belonging to $C^1(\overline{\omega})$ and vanishing on $(\partial\omega)_3$ (when $(\partial\omega)_3$ is not empty). If w vanishes a.e. on $(\partial\omega)_2 \cup (\partial\omega)_3$, then for any $v \in \mathcal{W}$ the identity

$$\int_{\omega} vLwd\omega = -\int_{\omega} \Big[\chi\mu\psi w_{,\alpha}v_{,\alpha} + w\mu(\chi\psi_{,\alpha} - \psi\chi_{,\alpha})v_{,\alpha}$$

$$+\big\{[\mu(\chi\psi_{,\alpha} - \psi\chi_{,\alpha})]_{,\alpha} - \chi(\mu\psi_{,\alpha})_{,\alpha}\big\}wv\Big]d\omega - \int_{(\partial\omega)_1} wvds$$

holds.

Let us introduce a scalar product in $\mathcal{W}$ in the following way:

$$(w,v) := \int_{\omega} (\chi\mu\psi w_{,\alpha}v_{,\alpha} + wv)d\omega + \int_{(\partial\omega)_1\cup(\partial\omega)_2} vw\mu\psi\Big|\frac{\partial\chi}{\partial n}\Big|ds.$$

The space $\mathcal{H}$ is the Hilbert space obtained by functional completion of $\mathcal{W}$ with the introduced scalar product.

Let us consider for $w, v \in \mathcal{W}$ the bilinear form

$$B(w,v) = -\int_{\omega} \Big[\chi\mu\psi w_{,\alpha}v_{,\alpha} + w\mu(\chi\psi_{,\alpha} - \psi\chi_{,\alpha})v_{,\alpha}$$

$$+\big\{[\mu(\chi\psi_{,\alpha} - \psi\chi_{,\alpha})]_{,\alpha} - \chi(\mu\psi_{,\alpha})_{,\alpha}\big\}wv\Big]d\omega - \int_{(\partial\omega)_1} wvds.$$

It is easily seen that

$$|B(w,v)| \le M\Big[\int_{\omega} (v_{,\alpha}\, v_{,\alpha} + v^2)d\omega + \int_{(\partial\omega)_1} |v|^2 ds\Big]^{\frac{1}{2}} ||w||_{L^2(\omega)},$$

where M is a constant depending on the coefficients of L. Hence, according to the Riesz theorem, for any fixed $v \in \mathcal{W}$, $B(w,v)$ can be considered as a linear bounded functional of w defined on $\mathcal{H}$.

For given $-\chi\Psi_3 \in L_2(\omega)$ we define as an $\mathcal{H}$-weak solution of the problem under consideration a function w in $\mathcal{H}$ satisfying the equation

$$-\int_{\omega} v\chi\Psi_3 d\omega = B(w,v) \qquad \forall v \in \mathcal{W}.$$

For the representation theorem of linear functionals in a Hilbert space, we have for $w \in \mathcal{H}$, $v \in \mathcal{W}$:

$$B(w,v) = (w, T(v)).$$

$T(v)$ is a linear transformation defined in $\mathcal{W}$ and with range in $\mathcal{H}$.

Let Fichera's condition [3,4] for equation (30)

$$\frac{1}{2}\big(2\chi\mu\psi_{,\alpha}+\chi\psi\mu_{,\alpha}\big)_{,\alpha}-\frac{1}{2}(\chi\mu\psi)_{,\alpha\alpha}-\chi(\mu\psi_{,\alpha})_{,\alpha}$$
$$=-\frac{1}{2}\Big[\chi\mu_{,\alpha}\psi_{,\alpha}+(\chi_{,\alpha}\mu)_{,\alpha}\psi+\chi\mu\psi_{,\alpha\alpha}\Big]>c_0=const>0 \quad in\ \overline{\omega}, \qquad (32)$$

be fulfilled (it should be achieved by the appropriate choice of ψ). Then for $v\in\mathcal{W}$:

$$|B(v,v)|\geq\lambda_0||v||^2_{\mathcal{H}}, \qquad \lambda_0=const>0.$$

It follows that

$$||v||^2_{\mathcal{H}}\leq\frac{1}{\lambda_0}|B(v,v)|\leq\frac{1}{\lambda_0}|(v,T(v))|\leq\frac{1}{\lambda_0}||v||_{\mathcal{H}}||T(v)||_{\mathcal{H}},$$

whence,

$$||v||_{\mathcal{H}}\leq\frac{1}{\lambda_0}||T(v)||_{\mathcal{H}},$$

which means that the mapping of $\mathcal{W}$ on $\mathcal{H}$ is one-to-one. We denote by $\mathcal{H}'$ the closure of the set $T(v)$, $v\in\mathcal{W}$, by the norm $\mathcal{H}$.

Since

$$\Big|\int_\omega\chi\Psi_3d\omega\Big|\leq\frac{1}{\lambda_0}||\chi\Psi_3||_{L_2(\omega)}||T(v)||_{\mathcal{H}},$$

$-\int_\omega\chi\Psi_3d\omega$ can be considered as a linear continuous functional over $\mathcal{H}'$. Hence, according to the Riesz theorem,

$$-\int_\omega\chi\Psi_3d\omega=\Big(w,T(v)\Big)_{\mathcal{H}}\equiv B(w,v), \ \ \forall v\in\mathcal{W}.$$

Thus, if condition (32) is satisfied, for any $-\chi\Psi_3\in L_2(\Omega)$ $\mathcal{H}$-weak solution w of problem (31) exists.

Uniqueness of classical solution can be easily verified. For uniqueness of $\mathcal{H}$-weak solution see [3,4].

If

$$\mu=\mu_0x_2^\kappa, \quad \mu_0,\kappa=const>0,$$

one can choose χ and ψ as follows

$$\chi=x_2^{1-\kappa}, \ \ \psi=e^{-ax_2}, \ \ a\in]0,d[, \qquad (33)$$

where $d := \operatorname{diam}\overline{\omega}$. Indeed, by virtue of (33),

$$-\frac{1}{2}\Big[\chi\mu,_\alpha\psi,_\alpha+(\chi,_\alpha\mu),_\alpha\psi+\chi\mu\psi,_{\alpha\alpha}\Big]$$
$$=\frac{1}{2}\mu_0 a e^{-ax_2}(\kappa-ax_2)\geq\frac{1}{2}\mu a e^{-ad}(\kappa-ad)=:c_0>0.$$

if $a\in]0,\frac{\kappa}{d}[$.

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