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One Effect for Bodies with Double Porosity in the Case of Plane Deformation

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Abstract. We consider the basic two-dimensional differential equations of static equilibrium poroelastic materials with double porosity. We construct the general solution of this system of equations by means of three analytic functions of a complex variable and solution of the Helmholtz equation. On the basis of the constructed general solution we have defined the effect caused by pressures in a porous medium which is similar to temperature effect of Muskhelishvili.

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1. Introduction

The model of elastic balance of porous mediums with double porosity considered on this work was constructed in [1-3]. The theory grounded in these works unifies the earlier proposed models of Barenblatt for porous media with double porosity [4] and of Biot for porous media with single porosity [5]. The works [6-12] are dedicated to various questions of elastic balance of bodies with double porosity. In this work the static balance of the poroelastic materials with double porosity is considered in the case of plane deformation. The general solution of the relevant system of differential equations is constructed by means of three analytic functions of the complex variable and one arbitrary solution of Helmholtz equation. The received analogs of formulas of Kolosov-Muskhelishvili allow to construct the analytical solutions of a number of the boundary value problems of plane poroelasticity for the bodies consisting of materials with double porosity. On the basis of the constructed general solution we have defined the effect caused by pressures in a porous medium which is similar to temperature effect of Muskhelishvili [13].

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2. Main three-dimensional relations

Let the elastic body with double porosity occupy the domain $\overline{\Omega} \subset R^3$. We will denote the point of domain $\overline{\Omega}$ in the Cartesian system of coordinates through (x_1, x_2, x_3). The considered solid body with double porosity is characterized by a vector of displacement $\vec{u} = (u_1, u_2, u_3)$ and fluid pressures $p_1(x_1, x_2, x_3)$ and $p_2(x_1, x_2, x_3)$ in the pores and fissures which are available in the porous media.

Then the homogeneous system of equations of static equilibrium has the following form [10]

$$\partial_i \sigma_{ij} = 0, \quad j = 1, 2, 3, \quad in \;\; \Omega, \tag{1}$$

where σ_{ij} are the components of the total stresses tensor; $\partial_i \equiv \frac{\partial}{\partial x_i}$; the summation from 1 to 3 is supposed on the repeating index i.

Stresses σ_{ij} are connected with pressures p_1, p_2 and components of a tensor of deformation $e_{ij} = 0.5(\partial_i u_j + \partial_j u_i)$ by the following relations

$$\sigma_{ij} = (\lambda e_{kk} - \beta_1 p_1 - \beta_2 p_2)\delta_{ij} + 2\mu e_{ij} \;\; in \;\; \overline{\Omega}, \tag{2}$$

where λ and μ are the Lame constants; $e_{kk} = e_{11} + e_{22} + e_{33}$; β_1 and β_2 are the effective stress parameters; σ_{ij} is the Kroneckers delta.

The functions p_1 and p_2 in the stationary case satisfy the following system of equations [10]

$$\begin{aligned} &(k_1\Delta - \gamma)p_1 + (k_{12}\Delta + \gamma)p_2 = 0, \\ &(k_{21}\Delta + \gamma)p_1 + (k_2\Delta - \gamma)p_2 = 0 \end{aligned} \quad in \quad \Omega, \tag{3}$$

where $k_1 = \frac{\kappa_1}{\mu'}$, $k_2 = \frac{\kappa_2}{\mu'}, k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$; μ' is the fluid viscosity; $\kappa_1 > 0$ and $\kappa_2 > 0$ are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity; $\kappa_{12} \geq 0$ and $\kappa_{21} \geq 0$ are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; $\gamma > 0$ is the internal transport coefficient and corresponds to fluid transfer rate with respect to the intensity of flow between the pore and fissures; $\Delta \equiv \partial_{11} + \partial_{22} + \partial_{33}$ is the three-dimensional Laplace operator.

Remark 1 : Though we stated above that $\gamma > 0$, we can consider also the case when $\gamma = 0$, in fact, it means that pores of two types are isolated from each other, or we have a solid body with uniform porosity.

The three-dimensional system of equations (1), (2), (3) describes the static equilibrium of the poroelastic materials with double porosity. If we substitute relations (2) in (1), we obtain the equilibrium equations with respect to components of the displacement vector

$$\mu\Delta u_i + (\lambda + \mu)\partial_i e_{kk} - \partial_i(\beta_1 p_1 + \beta_2 p_2) = 0, \quad i = 1, 2, 3. \tag{4}$$

If we add the boundary conditions on the area boundary to the system of equilibrium equations we can consider the various classical boundary value problems. It is easy to prove the following lemma.

Lemma 2.1: *If $\gamma > 0, k_1k_2 - k_{12}k_{21} > 0$, then the system of equations (3) is equivalent to two independent equations: to the Laplace equation.*

$$\Delta[(k_1 + k_{21})p_1 + (k_2 + k_{12})p_2] = 0 \tag{5}$$

with respect to the combination $(k_1 + k_{21})p_1 + (k_2 + k_{12})p_2$ and the Helmholtz equation with respect to the difference $p_1 - p_2$

$$\Delta(p_1 - p_2) - \eta(p_1 - p_2) = 0, \tag{6}$$

where $\eta := \dfrac{\gamma(k_1 + k_2 + k_{12} + k_{21})}{k_1k_2 - k_{12}k_{21}} > 0.$

Proof: If we fold the first equation of system (3) with the second equation of this system, we will directly obtain equation (5).

The system of equations (3) may be written in a matrix form

$$\begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} \begin{pmatrix} \Delta p_1 \\ \Delta p_2 \end{pmatrix} - \gamma \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By the condition the determinant of a matrix $\det \begin{pmatrix} k_1 & k_{12} \\ k_{21} & k_2 \end{pmatrix} = k_1k_2 - k_{12}k_{21} > 0$ is positive. Multiplying all members of the last equation by the inverse matrix on the left, we will obtain the system

$$\begin{cases} \Delta p_1 - \dfrac{\gamma(k_2 + k_{12})}{k_1k_2 - k_{12}k_{21}}(p_1 - p_2) = 0, \\ \Delta p_2 + \dfrac{\gamma(k_1 + k_{21})}{k_1k_2 - k_{12}k_{21}}(p_1 - p_2) = 0. \end{cases}$$

If we subtract the second equation of this system from the first equation of the last system, we will obtain equation (6).

Since

$$\det \begin{pmatrix} k_1 + k_{21} & k_2 + k_{12} \\ 1 & -1 \end{pmatrix} = -(k_1 + k_2 + k_{12} + k_{21}) < 0,$$

so the lemma is proved. □

Corollary 2.2: *If on the boundary $\partial\Omega$ of domain Ω $p_1 = p_2$ then $p_1 = p_2$ in the whole body $\overline{\Omega}$.*

This corollary follows from the fact that the homogeneous Helmholtz equation with zero boundary conditions has only the trivial solution.

3. The case of plane deformation

We have obtained the basic equation for the case of plane deformation from the main three-dimensional equations. Let Ω be a rather long cylindrical body with the generatrix parallel to the axis Ox_3. Let us denote by ω the cross section of this cylindrical body, thus $\omega \subset R^2$. In the case of plane deformation $u_3 = 0$ and functions u_1, u_2, p_1 and p_2 do not depend on the coordinate x_3.

Let's continue with the following notations $x := x_1,\ y := x_2$. We will obtain also

$$u := u_1,\ \ \nu := u_2,\ \ \sigma_{xx} := \sigma_{11},\ \ \sigma_{yy} := \sigma_{22},\ \ \sigma_{xy} := \sigma_{12},\ \ \sigma_{yx} := \sigma_{21}.$$

In case of plane deformation, as it appears from formulas (2) $\sigma_{13} = \sigma_{31} = 0$ and the system of equilibrium equations (1) will have the form

$$\begin{cases} \partial_x \sigma_{xx} + \partial_y \sigma_{yx} = 0, \\ \partial_x \sigma_{xy} + \partial_y \sigma_{yy} = 0 \end{cases} \quad in \quad \omega. \tag{7}$$

Relations (2) will be written as follows

$$\begin{aligned} &\sigma_{xx} = -\beta_1 p_1 - \beta_2 p_2 + \lambda\theta + 2\mu\partial_x u, \\ &\sigma_{yy} = -\beta_1 p_1 - \beta_2 p_2 + \lambda\theta + 2\mu\partial_y \nu, \\ &\sigma_{xy} = \sigma_{yx} = \mu(\partial_x \nu + \partial_y u), \\ &\sigma_{33} = -\beta_1 p_1 - \beta_2 p_2 + \lambda\theta, \end{aligned} \tag{8}$$

where $\theta = \partial_x u + \partial_y \nu$.

Equations (5) and (6) will have the form

$$\Delta_2[(k_1 + k_{21})p_1 + (k_2 + k_{12})p_2] = 0 \quad in \quad \omega, \tag{9}$$

$$\Delta_2(p_1 - p_2) - \eta(p_1 - p_2) = 0 \quad in \quad \omega, \tag{10}$$

where $\Delta_2 := \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplace operator.

If we substitute relations (8) in system (7), we obtain a system of equilibrium equations for the components u and v of the displacement vector

$$\begin{cases} \mu\Delta_2 u + (\lambda + \mu)\partial_x\theta - \partial_x(\beta_1 p_1 + \beta_2 p_2) = 0, \\ \mu\Delta_2 \nu + (\lambda + \mu)\partial_y\theta - \partial_y(\beta_1 p_1 + \beta_2 p_2) = 0 \end{cases} \quad in \quad \omega. \tag{11}$$

We introduce the complex variable $z = x + iy = re^{i\alpha}$ $(i^2 = -1)$ and the following operators $\partial_z = 0.5(\partial_x - i\partial_y),\ \ \partial_{\bar{z}} = 0.5(\partial_x + i\partial_y),\ \ \bar{z} = x - iy$, on the plane Oxy.

We will write equations (9) and (10) as follows

$$\partial_{\bar{z}}\partial_z[(k_1 + k_{21})p_1 + (k_2 + k_{12})p_2] = 0 \quad in \quad \omega, \tag{12}$$

$$4\partial_{\bar{z}}\partial_z(p_1-p_2)-\eta(p_1-p_2)=0 \quad in \quad \omega. \tag{13}$$

We will write the system of equations (11) in a complex form

$$2\mu\partial_{\bar{z}}\partial_z u_+ + (\lambda+\mu)\partial_{\bar{z}}\theta - \partial_{\bar{z}}(\beta_1 p_1+\beta_2 p_2)=0 \quad in \quad \omega, \tag{14}$$

where $u_+ := u+i\nu$; $\theta=\partial_z u_+ + \partial_{\bar{z}}\bar{u}_+$.

4. General solution of equations (12)-(14)

In this paragraph we will construct the formulas of Kolosov-Muskhelishvili [13] for system (12)-(14).

From equations (12) and (13) it follows that

$$\begin{cases} (k_1+k_{21})p_1+(k_2+k_{12})p_2=k_0[f'(z)+\overline{f'(z)}], \\ p_1-p_2=k_0\chi(z,\bar{z}), \end{cases} \tag{15}$$

where $k_0=k_1+k_2+k_{12}+k_{21}$; $f(z)$ is an arbitrary analytic function of a complex variable z in domain ω and $\chi(z,\bar{z})$ is an arbitrary solution of the Helmholtz equation

$$4\partial_{\bar{z}}\partial_z\chi-\eta\chi=0.$$

From system (15) we will easily obtain expressions for pressures p_1 and p_2

$$\begin{aligned} p_1&=f'(z)+\overline{f'(z)}+(k_2+k_{12})\chi(z,\bar{z}), \\ p_2&=f'(z)+\overline{f'(z)}-(k_1+k_{21})\chi(z,\bar{z}). \end{aligned} \tag{16}$$

Theorem 4.1: *The general solution of equation (14) is as follows:*

1) when $\gamma>0$

$$\begin{aligned} 2\mu u_+ &= \kappa\varphi(z)-\overline{z\varphi'(z)}-\overline{\psi(z)} \\ &+\frac{\mu}{\lambda+2\mu}\left[(\beta_1+\beta_2)(f(z)+z\overline{f'(z)})+\frac{4\delta}{\eta}\partial_{\bar{z}}\chi(z,\bar{z})\right], \end{aligned} \tag{17}$$

where $\delta:=(k_2+k_{12})\beta_1-(k_1+k_{21})\beta_2$;

2)when $\gamma=0$

$$2\mu u_+ = \kappa\varphi(z)-\overline{z\varphi'(z)}-\overline{\psi(z)}+\frac{\mu}{\lambda+2\mu}\left[F(z)+z\overline{F'(z)}\right], \tag{18}$$

where $\kappa=\frac{\lambda+3\mu}{\lambda+\mu}$; $\varphi(z)$ and $\psi(z)$ are arbitrary analytic functions of a complex variable z in the domain ω and $F(z)$ is also an analytic function, connected with the values p_1 and p_2 by means of the relation

$$F'(z)+\overline{F'(z)}=\beta_1 p_1+\beta_2 p_2.$$

Proof: We will prove the point a), i.e. the formula (17). Proof of the formula (18) is absolutely similar.

Let's take out the operator $\partial_{\bar{z}}$ from the brackets in the left part of the equation (14)

$$\partial_{\bar{z}}(2\mu\partial_z u_+ + (\lambda+\mu)\theta - (\beta_1 p_1 + \beta_2 p_2)) = 0. \tag{19}$$

(19) is therefore a system of Cauchy-Riemann equations

$$2\mu\partial_z u_+ + (\lambda+\mu)\theta = (\kappa+1)\varphi'(z) + \beta_1 p_1 + \beta_2 p_2, \tag{20}$$

where $\varphi(z)$ are arbitrary analytic functions of a complex variable z.

If we add the formula (23) with the conjugated expression and consider the formula

$$\theta = \partial_z u_+ + \partial_{\bar{z}}\bar{u}_+,$$

we will obtain

$$\theta = \frac{1}{\lambda+\mu}(\varphi'(z) + \overline{\varphi'(z)}) + \frac{1}{\lambda+2\mu}(\beta_1 p_1 + \beta_2 p_2). \tag{21}$$

If we substitute the formula (21) in (20), we will obtain

$$2\mu\partial_z u_+ = \kappa\varphi'(z) - \overline{\varphi'(z)} + \frac{\mu}{\lambda+2\mu}(\beta_1 p_1 + \beta_2 p_2). \tag{22}$$

From the formulas (16) obtained above we'll find the following expression for the combination $\beta_1 p_1 + \beta_2 p_2$

$$\beta_1 p_1 + \beta_2 p_2 = (\beta_1+\beta_2)(f'(z) + \overline{f'(z)}) + \delta\chi(z,\bar{z}).$$

If we substitute the last formula in (22) and integrate on z and we consider also that

$$\chi = \frac{4}{\eta}\partial_{\bar{z}}\partial_z\chi,$$

we will obtain the formula (17) to be proved. Thus if the solution of the equation (14) is rather smooth, it is presented in the form (17). On the contrary, if we substitute the expression (17) in (14), it will satisfy this equation.

As it was noted above, the formula (18) is proved absolutely similarly.

Substituting the expression (17) into formula (8), for the combinations of the components of the stress tensor we obtain the following formulas

$$\sigma_{xx} + \sigma_{yy} = 2[\varphi'(z) + \overline{\varphi'(z)}] - \frac{2\mu}{\lambda+2\mu}[(\beta_1+\beta_2)(f'(z) + \overline{f'(z)}) + \delta\chi(z,\bar{z})], \tag{23}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = 2[-z\overline{\varphi''(z)} - \overline{\psi'(z)}]$$

$$+\frac{2\mu}{\lambda+2\mu}\left[(\beta_1+\beta_2)z\overline{f''(z)}+\frac{4\delta}{\eta}\partial_{\bar z}\partial_{\bar z}\chi(z,\bar z)\right], \tag{24}$$

$$\sigma_{33}=\frac{\lambda}{\lambda+\mu}[\varphi'(z)+\overline{\varphi'(z)}]-\frac{2\mu}{\lambda+2\mu}[(\beta_1+\beta_2)(f'(z)+\overline{f'(z)})+\delta\chi(z,\bar z)]. \tag{25}$$

Thus, the general solution of the two-dimensional system of the differential equations describing static equilibrium of the poroelastic environment with double porosity is represented by means of three analytic functions of the complex variable and solution of the Helmholtz equation. The appropriate selection of these functions can satisfy four independent classical boundary conditions. □

5. The case of a simply-connected body

We will consider at first a cylindrical body with simply-connected cross-section ω with unit outward normal $\vec{l}$. The vector $\vec{l}$ forms the angle α with the positive direction of the x-axis. The unit vector $\vec{s}$ is perpendicular to $\vec{l}$ and

$$\vec{l}\times\vec{s}=\vec{e}_3,$$

where $\vec{e}_3$, the unit vector along the axis x_3 (Fig. 1).

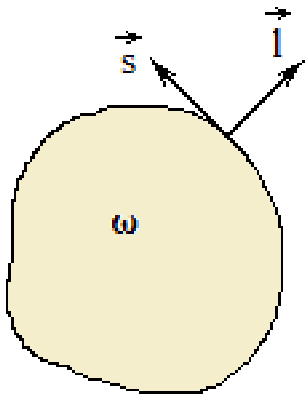

Fig. 1. The simply-connected domain ω

For a combination of stresses $\sigma_{ll}+i\sigma_{ls}$ the formula is valid

$$\sigma_{ll}+i\sigma_{ls}=0.5[\sigma_{xx}+\sigma_{yy}-(\sigma_{xx}-\sigma_{yy}+2i\sigma_{xy})e^{-2i\alpha}].$$

Let the side surface of the considered body be free from the external stresses, i.e.

$$\sigma_{ll}+i\sigma_{ls}=0 \quad on \quad \partial\omega, \tag{26}$$

and let boundary conditions for pressures p_1 and p_2 be also arbitrary.

Theorem 5.1: *Let us have a rather long cylindrical body with double porosity with simply-connected cross section ω. If the side surface of this body is free from* *stresses and the following condition is satisfied*

$$(k_2 + k_{12})\beta_1 - (k_1 + k_{21})\beta_2 = 0, \tag{27}$$

in a static case, despite pressures $p_1(x,y)$ *and* $p_2(x,y)$ *in pores, the stresses* σ_{xx}, σ_{yy} *and* σ_{xy} *in a body is absent, i.e. on the whole body we have*

$$\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = 0. \tag{28}$$

On the contrary, if for any $p_1(x,y)$ *and* $p_2(x,y)$ *satisfying the equations (9) and (10), on the whole body there are no stresses* σ_{xx}, σ_{yy} *and* σ_{xy} *condition (27) is satisfied.*

Displacements in this case are set by means of formulas

$$u = \frac{\beta_1 + \beta_2}{\lambda + \mu} \operatorname{Re} f(z), \qquad \nu = \frac{\beta_1 + \beta_2}{\lambda + \mu} \operatorname{Im} f(z), \tag{29}$$

and stress σ_{33} *will have the form*

$$\sigma_{33} = -\frac{\mu}{\lambda + \mu}(\beta_1 + \beta_2)(f'(z) + \overline{f'(z)}). \tag{30}$$

Proof: Let the condition (27) be satisfied, then formula (17) will have the form

$$2\mu u_+ = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)} + \frac{\mu}{\lambda + \mu}(\beta_1 + \beta_2)(f'(z) + \overline{f'(z)}). \tag{31}$$

and formulas (23), (24) and (25) can be rewritten as follows

$$\sigma_{xx} + \sigma_{yy} = 2[\varphi'(z) + \overline{\varphi'(z)}] - \frac{2\mu}{\lambda + 2\mu}(\beta_1 + \beta_2)(f'(z) + \overline{f'(z)}), \tag{32}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = 2[-z\overline{\varphi''(z)} - \overline{\psi'(z)}] + \frac{\mu}{\lambda + 2\mu}(\beta_1 + \beta_2)z\overline{f''(z)}, \tag{33}$$

$$\sigma_{33} = \frac{\lambda}{\lambda + \mu}[\varphi'(z) + \overline{\varphi'(z)}] - \frac{2\mu}{\lambda + 2\mu}(\beta_1 + \beta_2)(f'(z) + \overline{f'(z)}). \tag{34}$$

We introduce the following notation

$$\varphi_0(z) := \varphi(z) - \frac{\mu}{\lambda + 2\mu}(\beta_1 + \beta_2)f(z). \tag{35}$$

Of course $\varphi_0(z)$ is an arbitrary analytic function.

From the last formula we have

$$\varphi'(z) = \varphi_0'(z) + \frac{\mu}{\lambda + 2\mu}(\beta_1 + \beta_2)f'(z), \tag{36}$$

$$-z\overline{\varphi''(z)} = -z\overline{\varphi_0''(z)} - \frac{\mu}{\lambda+2\mu}(\beta_1+\beta_2)z\overline{f''(z)}. \tag{37}$$

Substituting formulas (36) and (37) in relations (32) and (33), we will obtain

$$\sigma_{xx}+\sigma_{yy} = 2[\varphi_0'(z)+\overline{\varphi_0'(z)}], \tag{38}$$

$$\sigma_{xx}-\sigma_{yy}+2i\sigma_{xy} = 2[-z\overline{\varphi_0''(z)}-\overline{\psi'(z)}], \tag{39}$$

Formulas (38) and (39) coincide with formulas of Kolosov-Muskhelishvili for a solid isotropic body [13]. Therefore since the side surface of the body is free from stresses, and rigid displacement doesn't interest us, from formulas (38) and (39) we will obtain

$$\varphi_0(z)=\psi(z)=0.$$

Therefore, from the same formulas (38) and (39) we obtain that on the whole body

$$\sigma_{xx}=\sigma_{yy}=\sigma_{xy}=0.$$

Suppose now that, for any pressures $p_1(x,y)$ and $p_2(x,y)$, stresses σ_{xx},σ_{yy} and σ_{xy} in the body are absent. We will show that condition (27) is necessarily satisfied. Indeed, from formula (23), with the notation (35) we have

$$\sigma_{xx}+\sigma_{yy} = 2[\varphi_0'(z)+\overline{\varphi_0'(z)}] - \frac{2\mu\delta}{\lambda+2\mu}\chi(z,\bar z).$$

But the stresses σ_{xx},σ_{yy} are absent, therefore from the last equality we will obtain

$$\frac{\mu\delta}{\lambda+2\mu}\chi(z,\bar z) = [\varphi_0'(z)+\overline{\varphi_0'(z)}].$$

If $(k_2+k_{12})\beta_1-(k_1+k_{21})\beta_2 \neq 0$ then it turns out that the arbitrary solution of the Helmholtz equation $4\partial_{\bar z}\partial_z\chi-\eta\chi=0$ is a harmonic function therefore condition (27) has to be satisfied.

Expressing $\varphi(z)$ by function $\varphi_0(z)$ from (35) and substituting this expression into formula (31) and (34) we will have

$$2\mu u_+ = \kappa\varphi_0(z) - z\overline{\varphi_0'(z)} - \overline{\psi(z)} + \frac{2\mu}{\lambda+\mu}(\beta_1+\beta_2)f(z),$$

$$\sigma_{33} = \frac{\lambda}{\lambda+\mu}[\varphi_0'(z)+\overline{\varphi_0'(z)}] - \frac{\mu}{\lambda+\mu}(\beta_1+\beta_2)(f'(z)+\overline{f'(z)}).$$

On the basis of these last formulas we obtain the formula $\varphi_0(z)=\psi(z)=0$ we'll receive formulas (29) and (30). Thus, Theorem 5.1 is proved. □

The effect formulated in Theorem 5.1 is the analogue of temperature effect of Muskhelishvili, but in our case the pressures in pores affect the body instead of the temperature field.

Remark 1 : In the case when on a boundary $\partial\omega$ besides condition (26) condition $p_1 = p_2$ is also satisfied (according to Corollary 1 from Lemma 1, in this case $\chi(z,\bar{z}) = 0$), relations (28)-(30) will be valid without obligatory performance of condition (27).

Remark 2 : In the case when $\gamma = 0$, relation (28) will be valid without obligatory performance of condition (27) and instead of formulas (29) and (30) we have

$$u = \frac{1}{\lambda+\mu}\operatorname{Re} F(z), \quad \nu = \frac{1}{\lambda+\mu}\operatorname{Im} F(z), \quad \sigma_{33} = -\frac{\mu}{\lambda+\mu}(\beta_1 p_1 + \beta_2 p_2).$$

6. The case of a multi-connected body

Let now ω be a multi-connected domain which is limited by several simple closed contours $L_1, L_2, \ldots, L_m, L_{m+1}$ from which the last one covers the others (Fig. 2)

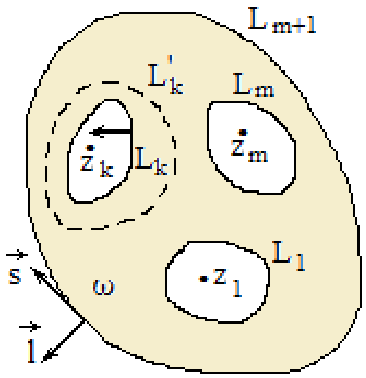

Fig. 2. The considered multi-connected domain ω

In this domain the analytic functions φ_0 (see formula (35)) and $\psi(z)$ will have the form

$$\varphi_0(z) = z\sum_{k=1}^{m} A_k \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k \ln(z - z_k) + \varphi^*(z), \tag{40}$$

$$\psi(z) = \sum_{k=1}^{m} \gamma'_k \ln(z - z_k) + \psi^*(z), \tag{41}$$

where $\varphi^*(z)$ and $\psi^*(z)$ is a holomorphic function in the domain ω; A_k are real constants, γ_k and γ'_k are complex constants.

Similarly

$$f'(z) = \sum_{k=1}^{m} B_k \ln(z - z_k) + f^*(z), \tag{42}$$

where $f^*(z)$ is a holomorphic function in the domain ω and B_k are real constants.
From (42) we obtain the following expression for the analytic function $f(z)$

$$f(z) = z \sum_{k=1}^{m} B_k \ln(z - z_k) + \sum_{k=1}^{m} (a_k^* + i\beta_k^*) \ln(z - z_k) + \tilde{f}(z), \tag{43}$$

where $\widetilde{f}(x)$ is a holomorphic function in the domain ω.
Substituting formulas (40)-(43) in general representation (17) we establish that at single round counterclockwise of a contour L_k' (see Fig. 2), function $2\mu\, u_+$ will obtain an increment

$$2\mu[u + i\nu]_{L_k'} = 2\pi i \left\{ \left[(\kappa + 1) A_k + \frac{2\mu}{\lambda + \mu}(\beta_1 + \beta_2) B_k \right] z + \kappa \gamma_k + \overline{\gamma_k'} \right.$$

$$\left. + \frac{2\mu}{\lambda + 2\mu}(\beta_1 + \beta_2)(a_k^* + i\beta_k^*) \right\}. \tag{44}$$

Taking into account formula (44) using proof of Theorem 4.1, the next theorem can be proved.

Theorem 6.1: *Let us have a rather long cylindrical body with double porosity with multi-connected cross section ω. If the side surface of this body is free from stresses and the following condition (27) is satisfied, then, in a static case, stresses σ_{xx}, σ_{yy} and σ_{xy}, caused by pressures $p_1(x, y)$ and $p_2(x, y)$ in pores is the same as in case when a body is not affected by pressures but is subjected to dislocation with characteristics*

$$\varepsilon_k = -\frac{2\pi}{\lambda + \mu}(\beta_1 + \beta_2) B_k,$$

$$\alpha_k^0 = \frac{2\pi}{\lambda + \mu}(\beta_1 + \beta_2)\beta_k^*, \quad \beta_k^0 = -\frac{2\pi}{\lambda + \mu}(\beta_1 + \beta_2)\alpha_k^*,$$

Remark 1: When on the boundary of a multi-connected domain the condition $p_1 = p_2$ is also satisfied (in this case $\chi(z, \bar{z}) = 0$), Theorem 6.1 is valid even if condition (27) does not hold.

Remark 2: In case when $\gamma = 0$, Theorem 6.1 is satisfied even if condition (27) does not hold, only characteristics of a dislocation will have the form

$$\varepsilon_k = -\frac{2\pi}{\lambda + \mu} C_k,$$

$$\alpha_k^0 = \frac{2\pi}{\lambda+\mu}\eta_k^*, \quad \beta_k^0 = -\frac{2\pi}{\lambda+\mu}\delta_k^*,$$

where constants $C_k, \eta_k^*, \delta_k^*$ are taken from the following expression of an analytic function $F(z)$ ($F^*(z)$ is a holomorphic function in the domain ω)

$$F(z) = z\sum_{k=1}^{m} C_k \ln(z-z_k) + \sum_{k=1}^{m}(\delta_k^* + i\eta_k^*)\ln(z-z_k) + F^*(z).$$

7. Problem for a concentric circular ring

Finally, we will solve a simple problem for a concentric circular ring.

Let the poroelastic body with double porosity occupy the domain ω, which is limited to concentric circles L_1 and L_2 with radiuses R_1 and $R_2 (R_1 < R_2)$ (Fig. 3).

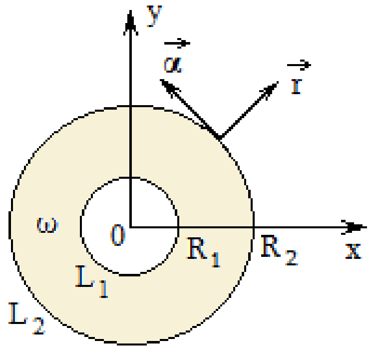

Fig. 3. The considered circular ring

We consider the following boundary value problem

$$\sigma_{rr} - i\sigma_{ra} = \begin{cases} 0, & r = R_1, \\ 0, & r = R_2, \end{cases} \tag{45}$$

$$(k_1 + k_{21})p_1 + (k_2 + k_{12})p_2 = \begin{cases} p_+^{(1)}, & r = R_1, \\ p_+^{(2)}, & r = R_2, \end{cases} \tag{46}$$

$$p_1 - p_2 = \begin{cases} p_-^{(1)}, & r = R_1, \\ p_-^{(2)}, & r = R_2, \end{cases} \tag{47}$$

where $p_+^{(1)}, p_+^{(2)}, p_-^{(1)} p_-^{(1)}$ are the set constants.

As $k_1 p_1 + k_2 p_2$ satisfies to Laplace's equation (see formula (9)), having satisfied boundary conditions (46), we will have

$$(k_1 + k_{21})p_1 + (k_2 + k_{12})p_2 =$$

$$\frac{p_+^{(2)} - p_+^{(1)}}{\ln R_2 - \ln R_1} ln\, r + \frac{p_+^{(1)} \ln R_2 - p_+^{(2)} \ln R_1}{\ln R_2 - \ln R_1}. \tag{48}$$

And the analytic function $f'(z)$ will have the form

$$f'(z) = \frac{1}{2k_0} \left(\frac{p_+^{(2)} - p_+^{(1)}}{\ln R_2 - \ln R_1} ln\, z + \frac{p_+^{(1)} \ln R_2 - p_+^{(2)} \ln R_1}{\ln R_2 - \ln R_1} \right). \tag{49}$$

We present the function $\chi(z, \bar{z})$ in the form of the series

$$\chi(z, \bar{z}) = \frac{1}{k_0} \sum_{-\infty}^{+\infty} (\alpha_n I_n(\sqrt{\eta} r) + \beta_n K_n(\sqrt{\eta} r)) e^{in\alpha},$$

where $I_n(\sqrt{\eta} r), K_n(\sqrt{\eta} r)$ are modified Bessel functions of n order.
Considering the boundary conditions (47), we obtain

$$\alpha_n = \beta_n = 0, \ when\ n \neq 0,$$

and for coefficients of α_0 and β_0 we will obtain the following system of the equations

$$\begin{cases} \alpha_0 I_0(\sqrt{\eta} R_1) + \beta_0 K_0(\sqrt{\eta} R_1) = p_-^{(1)}, \\ \alpha_0 I_0(\sqrt{\eta} R_2) + \beta_0 K_0(\sqrt{\eta} R_2) = p_-^{(2)}, \end{cases}$$

whence

$$\alpha_0 = \frac{p_-^{(1)} K_0(\sqrt{\eta} R_2) - p_-^{(2)} K_0(\sqrt{\eta} R_1)}{I_0(\sqrt{\eta} R_1) K_0(\sqrt{\eta} R_2) - I_0(\sqrt{\eta} R_2) K_0(\sqrt{\eta} R_1)}, \tag{50}$$

$$\beta_0 = \frac{p_-^{(2)} K_0(\sqrt{\eta} R_1) - p_-^{(1)} K_0(\sqrt{\eta} R_2)}{I_0(\sqrt{\eta} R_1) K_0(\sqrt{\eta} R_2) - I_0(\sqrt{\eta} R_2) K_0(\sqrt{\eta} R_1)}. \tag{51}$$

Thus

$$\chi(z, \bar{z}) = \frac{\alpha_0}{k_0} I_0(\sqrt{\eta} r) + \frac{\beta_0}{k_0} K_0(\sqrt{\eta} r). \tag{52}$$

and where the coefficients α_0 and β_0 are calculated according to the formula (50) and (51).

We will satisfy now the boundary conditions (45). According to the general representations, we have .

$$\sigma_{rr} - i\sigma_{r\alpha} = 0.5[\sigma_{xx} + \sigma_{yy} - (\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy})e^{2i\alpha}]$$

$$= \varphi_0'(z) + \overline{\varphi_0'(z)} - e^{2i\alpha}(\bar{z}\varphi_0''(z) + \psi'(z))$$

$$-\frac{\mu\delta}{\lambda + 2\mu}\left(\chi(z,\bar{z}) + \frac{4}{\eta}\partial_z\partial_z\chi(z,\bar{z})e^{2i\alpha}\right) = \begin{cases} 0, & r = R_1, \\ 0, & r = R_2, \end{cases} \tag{53}$$

From (52) we have

$$\partial_z\partial_z\chi(z,\bar{z}) = \frac{\eta}{2k_0}(\alpha_0 I_2(\sqrt{\eta}r) + \beta_0 K_2(\sqrt{\eta}r))e^{-2i\alpha}, \tag{54}$$

and analytical functions φ_0' and $\psi'(z)$ will be presented as the series

$$\varphi_0'(z) = A\ln z + \sum_{-\infty}^{+\infty} a_n r^n e^{in\alpha}, \quad \psi_0'(z) = \sum_{-\infty}^{+\infty} b_n r^n e^{in\alpha}. \tag{55}$$

If we substitute the representations (52), (54) and (55) in boundary conditions (53), we will have

$$\left[2A\ln rA - A + \sum_{-\infty}^{+\infty}(1-n)a_n r^n e^{in\alpha} + \sum_{-\infty}^{+\infty}\bar{a}_n r^n e^{-in\alpha} - \sum_{-\infty}^{+\infty} b_{n-2} r^{n-2} e^{-in\alpha}\right]_{r=R_j}$$

$$= A_j, \quad j = 1,2 \tag{56}$$

where the notation is introduced

$$A_j = \frac{2\mu}{\lambda + 2\mu}\frac{\delta}{k_0\sqrt{\eta}R_j}[\alpha_0 I_1(\sqrt{\eta}R_j) - \beta_0 K_1(\sqrt{\eta}R_j)], \quad j = 1,2.$$

We assume that $a_0 = \bar{a}_0$(see [13]) and if we equate free members in both parts of equality (56), we will obtain the system

$$\begin{cases} 2A\ln R_1 - A + 2a_0 - b_{-2}R_1^{-2} = A_1, \\ 2A\ln R_2 - A + 2a_0 - b_{-2}R_2^{-2} = A_2 \end{cases} \tag{57}$$

The coefficient A is determined from the first condition of uniqueness displacements (formulas (35), (49) and (55) are here taken into account)

$$A = -\frac{\mu}{2(\lambda + 2\mu)}\frac{\beta_1 + \beta_2}{k_0}\frac{p_+^{(2)} - p_+^{(1)}}{\ln R_2 - \ln R_1}.$$

After the coefficient A is found, from system (57) we define the coefficients a_0 and b_{-2}

$$a_0 = \frac{R_1^2(A_1 + A - 2A\ln R_1) - R_2^2(A_2 + A - 2A\ln R_2)}{2(R_1^2 + R_2^2)}$$

$$b_{-2} = \frac{R_1^2 R_2^2[A_1 - A_2 + 2A(\ln R_2 - \ln R_1)]}{R_1^2 + R_2^2}$$

The second condition of uniqueness displacements have the form

$$\kappa a_{-1} + \bar{b}_{-1} = 0,$$

but from (56) we get

$$a_{-1} + \bar{b}_{-1} = 0,$$

therefore $a_{-1} = b_{-1} = 0$. It is easy to see that all other factors are equal to zero. Thus, the problem is solved.

8. Conclusion

In the paper static poroelastic equilibrium the of materials with double porosity is considered in case of plane deformation. The general solution of the relevant system of differential equations is constructed by means of three analytic functions of the complex variable and solution of the Helmholtz equation. The constructed general solution allows to construct the analytical solution of quite a wide class of boundary value problems. In the paper the boundary problem for a circular ring in an explicit form is solved. It is also shown that when performing a certain condition the analog of temperature effect of Muskhelishvili occurs, but with the difference that in the considered case the pressure in pores an effects the body instead of a temperature field.

In our opinion, it will be interesting to consider the case of porothermoelasticity in future, when the influence of change of temperature is taken into account also in addition to the effect of pressures in pores.

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