Bulletin of TICMI
Vol. 21, No. 2, 2017, 117–121

SHORT COMMUNICATIONS

On Asymptotic Behavior of Solutions of First Order Difference Equations with Several Delay

Roman Koplatadze*

I. Javakhishvili Tbilisi State University
I. Vekua Institute of Applied Mathematics & Faculty of Exact and Natural Sciences
2 University St., 0186, Tbilisi, Georgia
(*Received June 8, 2017; Revised October 20, 2017; Accepted November 20, 2017*)

Consider the first order difference equation with several retarded arguments

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\sigma_i(k)) = 0,$$

where $\Delta u(k) = u(k+1) - u(k)$, $p_i : N \to R_+$, $\sigma_i : N \to N$ and $\lim\limits_{k\to+\infty} \sigma_i(k) = +\infty$ $(i = 1, \dots, m)$. In the paper the oscillation of all solutions to this equation is reviewed and new sufficient conditions for the oscillation are established.

Keywords: Difference equations, Oscillation, Delay argument.

AMS Subject Classification: 34K11.

1. Introduction

Consider the difference equation with several retarded arguments

$$\Delta u(k) + \sum_{i=1}^{m} p_i(k)\, u(\sigma_i(k)) = 0, \tag{1.1}$$

where $m \in N$, $\Delta u(k) = u(k+1) - u(k)$, $p_i : N \to R_+$, $\sigma_i : N \to N$ and $\lim\limits_{k\to+\infty} \sigma_i(k) = +\infty$ $(i = 1, \dots, m)$. For each $n \in N$, denote $N_n = \{n, n+1, \dots\}$.

Definition 1.1: Let $n \in N$. We call a function $u : N \to R$ a proper solution of equation (1.1) on the set N_n, if it satisfies (1.1) on N_n and $\sup\{|u(i)| : i \geq k\} > 0$ for any $k \in N_n$.

*Email: r_koplatadze@yahoo.com

ISSN: 1512-0082 print

Definition 1.2: We say that a proper solution $u : N_n \to R$ of equation (1.1) is oscillatory if for any $k \in N_n$ there are $n_1, n_2 \in N_k$ such that $u(n_1)\, u(n_2) \le 0$. Otherwise the solution is called nonoscillatory.

In Sections 2 and 3 we give some well-known results for oscillation of solutions of difference equation(1.1) and the differential equation

$$u'(t) + \sum_{i=1}^{m} q_i(t)\, u(\tau_i(t)) = 0, \tag{1.2}$$

where $q_i \in L_{\text{loc}}(R_+; R_+)$, $\tau_i \in C(R_+; R)$ and $\lim\limits_{t\to+\infty} \tau_i(t) = +\infty$ $(i = 1, \dots, m)$.

For these two types of equations to give different and similarity between the oscillation of these criteria. In the final Section 4, we give new results for equation (1.1), where some results are new for $m = 1$.

In the special case where $m = 1$, equation (1.1) and (1.2) are reduced respectively to the equations

$$\Delta u(k) + p(k)\, u(\sigma(k)) = 0 \tag{1.3}$$

and

$$u'(t) + q(t)\, u(\tau(t)) = 0, \tag{1.4}$$

where $p : N \to R_+$, $\sigma : N \to N$, $\lim\limits_{k\to+\infty} \sigma(k) = +\infty$, $q \in L_{\text{loc}}(R_+; R_+)$, $\tau \in C(R_+; R)$ and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$.

2. Oscillation criteria for equations (1.3) and (1.4)

Theorem 2.1 : ([1]) *Let*

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} q(s)ds > \frac{1}{e}\,, \tag{2.1}$$

then all proper solutions of equation (1.4) *are oscillatory.*

Remark 1 : Condition (2.1) is optimal, that is, inequality (2.1) cannot be replaced by the condition

$$\liminf_{t\to+\infty} \int_{\tau(t)}^{t} q(s)ds \ge \frac{1}{e}\,.$$

Furthermore, if $\int\limits_{\tau(t)}^{t} q(s)ds$ is a nondecreasing function, then condition (2.1) is necessary and sufficient for oscillation of all solutions of equation (1.4).

Note that Theorem 2.1 was generalized in $[2-4]$ at different times.

Theorem 2.2: ([5]) *Let $n \in N$, $\sigma(k) = k - n$ and*

$$\liminf_{k\to+\infty} \sum_{i=k-n}^{k-1} p(i) > \Big(\frac{n}{n+1}\Big)^{n+1}. \tag{2.2}$$

Then all proper solutions of equation (1.3) *are oscillatory.*

Remark 2: Condition (2.2), for any $n \in N$, is optimal, that is condition (2.2) cannot be replaced by the condition

$$\liminf_{t\to+\infty} \sum_{i=k-n}^{k-1} p(i) \geq \Big(\frac{n}{n+1}\Big)^{n+1}.$$

It is obvious that $\big(\frac{n}{n+1}\big)^{n+1} < \frac{1}{e}$ and $\big(\frac{n}{n+1}\big)^{n+1} \uparrow \frac{1}{e}$ when $n \uparrow +\infty$.

Theorem 2.3: ([6]) *Let*

$$\liminf_{k\to+\infty} \sum_{i=\sigma(k)}^{k-1} p(i) > \frac{1}{e}. \tag{2.3}$$

Then all proper solutions of equation (1.3) *are oscillatory.*

Remark 3: Condition (2.3) is optimal. That is, we can construct the difference equation, where $k - \sigma(k) \to +\infty$ for $k \to +\infty$ and

$$\liminf_{k\to+\infty} \sum_{i=\sigma(k)}^{k-1} p(i) = \frac{1}{e},$$

but difference equation (1.3) has the positive solution.

3. Oscillation criteria for equations (1.1) and (1.2)

Theorem 3.1: ([7]) *Let the condition*

$$\sum_{s=1}^{+\infty} \big|p_i(s) - p_j(s)\big| < +\infty \quad (i, j = 1, \ldots, m),$$

be fulfilled and

$$\liminf_{k\to+\infty} \sum_{i=1}^{m} \Big(\sum_{s=\sigma_i(k)}^{k-1} p_i(s)\Big) > \frac{1}{e}.$$

Then all proper solutions of equation (1.1) *are oscillatory.*

Theorem 3.2 : ([8]) *Let* τ_i *be non-decreasing functions and*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \Big(\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} q_i(s)ds \Big)^{\frac{1}{m}} > \frac{1}{m}\,.$$

Then all proper solutions of equation (1.2) *are oscillatory.*

Theorem 3.3 : ([9]) *Let*

$$\limsup_{t\to+\infty} \prod_{j=1}^{m} \Big[\prod_{i=1}^{m} \int_{\tau_j(t)}^{t} q_i(s) \int_{\tau_i(s)}^{\tau_j(t)} \Big(\prod_{\ell=1} q_\ell(\xi) \Big)^{\frac{1}{m}} d\xi\, ds \Big]^{\frac{1}{m}} > 0$$

and

$$\liminf_{t\to+\infty} \sum_{i=1}^{m} \int_{\tau_i(t)}^{t} \Big(\prod_{\ell=1}^{m} q_\ell(s) \Big)^{\frac{1}{m}} ds > \frac{1}{e}\,.$$

Then all proper solutions of equation (1.2) *are oscillatory.*

4. New oscillation criteria for equation (1.1)

Theorem 4.1 : *Let*

$$\liminf_{k\to+\infty} p_i(k) > 0 \quad (i = 1, \dots, m)$$

and

$$\min \left\{ \frac{\prod\limits_{i=1}^{m} \liminf\limits_{k\to+\infty} \prod\limits_{j=\sigma_i(k)}^{k} \Big(1 + \alpha\, m\, p^{\frac{1}{m}}(j)\Big)}{\alpha^m} : \alpha \geq 1 \right\} > 1. \tag{4.1}$$

Then all proper solutions of equation (1.1) *are oscillatory, where* $p(j) = \prod\limits_{i=1}^{m} p_i(j)$.

Theorem 4.2 : *Let for large* $j \in N$, $p_i(j) \geq p_i > 0$ $(i = 1, \dots, m)$ *and*

$$p \liminf_{k\to+\infty} \frac{\Big(m + \sum\limits_{i=1}^{m} (k - \sigma_i(k))\Big)^{m + \sum\limits_{i=1}^{m} (k-\sigma_i(k))}}{\Big(\sum\limits_{i=1}^{m} (k - \sigma_i(k))\Big)^{\sum\limits_{i=1}^{m} (k-\sigma_i(k))}} > 1.$$

Then all proper solutions of equation (1.1) *are oscillatory, where* $p = \prod\limits_{i=1}^{m} p_i$.

Corollary 4.3: *Let* $n_i \in N$*, for large* $k \in N$*,* $\sigma_i(k) \le k - n_i$*,* $p_i(k) \ge p_i > 0$ $(i = 1, \dots, m)$ *and*

$$\frac{p\Big(m + \sum\limits_{i=1}^{m} n_i\Big)^{m+\sum\limits_{i=1}^{m} n_i}}{\Big(\sum\limits_{i=1}^{m} n_i\Big)^{\sum\limits_{i=1}^{m} n_i}} > 1. \tag{4.2}$$

Then all proper solutions of equation (1.1) *are oscillatory, where* $p = \prod\limits_{i=1}^{m} p_i$.

Remark 1 : In the case, observe that, when $m = 1$, condition (4.2) can be reduced to the (classical) condition (2.2). The following example illustrates the significance of our results.

Let $\sigma(k) = k - 1$, then condition (2.2) is reduced to the condition

$$\liminf_{k\to+\infty} p(k) > \frac{1}{4} \tag{4.3}$$

and condition (4.1) is reduced to the condition

$$\liminf_{k\to+\infty} \left(\sqrt{p(k-1)} + \sqrt{p(k)}\right)^2 > 1. \tag{4.4}$$

It is obvious that condition (4.3) implies Condition (4.4).

Let $\varepsilon \in (0, \frac{1}{4}]$, $p(2k) = \varepsilon$ and $p(2k+1) = 1 - \varepsilon$ $(k = 1, 2, \dots)$. It is obvious that condition (4.4) is fulfilled but condition (4.3) is not fulfilled.

References

[1] R.G. Koplatadze, T.A. Chanturiya, *Oscillating and monotone solutions of first-order differential equations with deviating argument* (Russian), Differentsialnye Uravneniya, **18**, 8 (1982), 1463-1465, 1472

[2] R. Koplatadze, G. Kvinikadze, *On the oscillation of solutions of first-order delay differential inequalities and equations*, *Georgian Math. J.*, **1**, 6 (1994), 675-685

[3] M.K. Grammatikopoulos, R. Koplatadze, I.P. Stavroulakis, *On the oscillation of solutions of first order differential equations with retarded arguments*, *Georgian Math. J.*, **10**, 1 (2003), 63-76

[4] G.K. Berikelashvili, O.M. Dzhokhadze, R.G. Koplatadze, *On an approach to the investigation of the asymptotic properties of solutions of first-order ordinary differential equations with delay* (Russian), translated from Differ. Uravn., **44**, 1 (2008), 19-38, 141, *Differ. Equ.*, **44**, 1 (2008), 19-39

[5] G. Ladas, CH.G Philos, Y.G. Sficas, *Sharp conditions for the oscillation of delay difference equations*, J. Appl. Math. Simulation, **2**, 2 (1989), 101-111

[6] G.E. Chatzarakis, R. Koplatadze, I.P. Stavroulakis, *Optimal oscillation criteria for first order difference equations with delay argument*, Pacific J. Math., **235**, 1 (2008), 15-33

[7] R. Koplatadze, S. Pinelas, *Oscillation criteria for first-order linear difference equation with several delay arguments*, translated from Nelnn Koliv., **17**, 2 (2014), 248-267, J. Math. Sci. (N.Y.), **208**, 5 (2015), 571-592

[8] G. Infante, R. Koplatadze, I.P. Stavroulakis, *Oscillation criteria for differential equations with several retarded arguments*, Funkcial. Ekvac., **58**, 3 (2015), 347-364

[9] R. Koplatadze, *Specific properties of solutions of first order differential equations with several delay arguments*, translated from Izv. Nats. Akad. Nauk Armenii Mat., **50**, 5 (2015), 24-33, J. Contemp. Math. Anal., **50**, 5 (2015), 229-235

R E P O R T

o f

T I C M I   A C T I V I T I E S

January - December 2017

TICMI was organizer of

- the Workshop in Discrete Mathematics II (Tbilisi, Georgia, November 27-28, 2017). Information about the conference can be found on the following web-site:

http://www.viam.science.tsu.ge/ticmi/programs/program/2017/;

BULLETIN of TICMI, vol. 21, No. 1 and No. 2, 2017 have been published. The electronic versions can be found on the following web-sites:

http://www.viam.science.tsu.ge/ticmi/bulletin/bulletin
http://www.emis.de/journals/TICMI/.

LECTURE NOTES of TICMI, vol. 18, 2017 have been published. The electronic versions can be found on the following web-sites:

http://www.viam.science.tsu.ge/others/ticmi/lnt/lecturen.htm
http://www.emis.de/journals/TICMI/.

Chairman of *ISC* of *TICMI*

Prof. G. Jaiani

CONTENTS

Information for the Authors

1. Papers written in English should be submitted as tex and PDF files using online submitted form (http://www.viam.science.tsu.ge/admin)
2. Papers should be prepared in any standard version of TeX (i.e. plain, LaTeX, AMS-(La)Tex)
3. Letter's size and intervals between lines should be 12 pt, with a printed text area of 140mm x 225mm on a page.
4. Manuscripts for short Communications and Main Articles should be preferably not longer than 4 and 15 pages, respectively.
5. Manuscripts should be compiled in the following order: title page; abstract; keywords; main text; acknowledgments; appendices (as appropriate); references; table(s) with caption(s) (on individual pages); figure caption(s) (as a list).
6. Abstracts of 100-150 words are required for all papers submitted.
7. Each paper should have 3-6 keywords. In addition to keywords, authors are encouraged to provide 2-6 AMS 2010 Mathematics Subject Classification codes.
8. Section headings should be concise and numbered sequentially, using a decimal system for subsections.
9. All the authors of a paper should include their full names, affiliations, postal addresses, telephone and fax numbers and email addresses on the cover page of the manuscript. One author should be identified as the Corresponding Author.
10. The decision on the acceptance is taken after a peer-reviewing procedure.
11. Authors submit their papers on the condition that they have not been published previously and are not under consideration for publication elsewhere.

Published by TSU Press

1 Ilia Chavchavadze Avenue, Tbilisi 0179
Tel 995 (32) 225 14 35, 995 (32) 225 27 36
www.press.tsu.edu.ge