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On the Static Problems of Beams and Plates

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The present work dedicated to the static problems of prismatic cusped beams and plates. The aim of the work is to study well-posedness of boundary value problem in case of (0,0) approximation of hierarchical models for beams and in case of $N = 0$ approximation of hierarchical models for plates.

Keywords: Cusped beams and plates, Vekua's hierarchical models, Partial differential equations with order degeneration.

AMS Subject Classification: 74K10, 74K20, 35J70, 35J75, 35L80, 35L81.

1. Introduction

In the fifties of XX century investigations of cusped elastic prismatic shells actually take origin, namely, in 1955 I.Vekua raised the problem of investigation of elastic cusped prismatic shells, whose thickness on the prismatic shell entire boundary or on its part vanishes (see [1-3]). In practice, such cusped prismatic shells, in particular, cusped plates, and cusped beams (i.e.,beams whose cross-sections area vanishes at least at one end of the beam) are often encountered in spatial structures with partly fixed edges, e.g., stadium ceilings, aircraft wings, submarine wings etc., in machine-tool design, as in cutting-machines, planning-machines, in astronautics, turbines, and in many other application fields of engineering. Investigation of elastic cusped prismatic shells, considered as 3D ones, may occupy 3D domains with non-Lipschitz boundaries, in general. The problem mathematically leads to the question of setting and solving boundary value problems for even order equations and systems of elliptic type with the order degeneration in the statical case and of initial boundary value problems for even order equations and systems of hyperbolic type with the order degeneration in the dynamical case (for corresponding investigations see the surveys [4,5] and also I. Vekua's comments in [3, p.86]). At the same time I.Vekua introduced a new mathematical model for elastic prismatic shells which was based on expansions of the three-dimensional displacement vector fields and the strain and stress tensors of linear elasticity into orthogonal Fourier-Legendre series with respect to the variable plate thickness. By taking only the first $N + 1$ terms of the expansions, he introduced the so called N-th approximation. Each of these approximations for $N = 0; 1; ...$ can be considered as an

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independent mathematical model of plates. In particular, the approximation for $N = 1$, actually, corresponds to the classical Kirchhoff plate model. In the sixties, I. Vekua developed the analogous mathematical model for thin shallow shells. All his results concerning plates and shells are collected in his monograph [3]. Works of I. Babuska, D. Gordeziani, V. Guliaev, I. Khoma, A. Khvoles, T. Meunargia, C. Schwab, T. Vashakmadze, V. Zhgenti, G. Jaiani, G. Tsikarishvili, M. and G. Avalishvili, W. Wendland, D. Natroshvili, S. Kharibegashvili, N. Chinchaladze, R. Gilbert, B.W. Schulze and others are devoted to further analysis of I. Vekua's models (rigorous estimation of the modeling error, numerical solutions, etc.) and their generalizations (see, e.g., [4-12]).

The analogues system in the case of (N_3, N_2) approximation of hierarchical models for cusped beams, in general, beams with variable rectangular cross-sections are derived by G. Jaiani ([13,14], see also [10]).

In [15] the hierarchical models for porous elastic and viscoelastic Kelvin-Voigt prismatic shells on the basis of linear theories are presented.

The present work is dedicated to the static problems of cusped prismatic beams and plates. In the (0,0) approximation of hierarchical models we consider a static problem for a beam whose length is L, width and thickness are given by the expressions (see Fig.1):

$$2h_2 = 2h_2^0 \quad \text{and} \quad 2h_3 = 2h_3^0 e^{-\frac{\kappa}{x_1}},$$

where

$$x_1 \in [0, L], \;\; x_2 \in \left[-h_2^0, h_2^0\right], \;\; h_2^0 = h_3^0 = const > 0, \; \kappa = const > 0, \; L = const > 0.$$

First we consider a weighted boundary condition on the cusped end of the beam and the Dirichlet boundary condition on the non-cusped end. We find the solution of the posed boundary value problem in an integral form. Then we investigate the static problem of a plate in the $N = 0$ approximation of hierarchical models when a half-thickness of the plate (prismatic shell) is given by the expression:

$$h(x_1, x_2) = h_0 e^{-\frac{\kappa}{x_1}},$$

where

$$(x_1, x_2) \in \omega, \quad h_0 = const > 0, \quad \kappa = const > 0,$$

the projection ω of the plate has the form given either in Fig. 2 or in Fig. 3.

2. A static problem for beams

Governing equations in the (0,0) approximation have the following form (see [13, 14]):

$$\left(h_2 h_3 v_{j,1}(x_1, t)\right)_{,1} + \overset{0,0}{Y}_j = \Lambda_j^{-1} \rho h_2 h_3 \ddot{v}_j(x_1, t), \qquad j = 1, 2, 3, \tag{1}$$

where

$$\overset{0,\,0}{Y}_j := \frac{\overset{0,\,0}{X_1^0}}{\Lambda_1}, \qquad \overset{0,\,0}{Y}_i := \frac{\overset{0,\,0}{X_i^0}}{\Lambda_i}, \quad i=2,3,$$

$$\Lambda_j := \begin{cases} \lambda+2\mu, & j=1; \\ \mu, & j=2,3. \end{cases}$$

$$\overset{0,\,0}{X_j^0} = X_{j00}$$

$$+\int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \left[\sqrt{1+\left(\overset{(+)}{h}_{3,1}\right)^2} X_{\overset{(+)}{\nu_3}j}\left(x_1,\ x_2,\ \overset{(+)}{h}_3\right) + \sqrt{1+\left(\overset{(-)}{h}_{3,1}\right)^2} X_{\overset{(-)}{\nu_3}j}\left(x_1,\ x_2,\ \overset{(-)}{h}_3\right)\right] dx_2$$

$$+\int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} \left[\sqrt{1+\left(\overset{(+)}{h}_{2,1}\right)^2} X_{\overset{(+)}{\nu_2}j}\left(x_1,\ \overset{(+)}{h}_2,\ x_3\right) + \sqrt{1+\left(\overset{(-)}{h}_{2,1}\right)^2} X_{\overset{(-)}{\nu_2}j}\left(x_1,\ \overset{(-)}{h}_2,\ x_3\right)\right] dx_3,$$

$$j=1,2,3,$$

$$X_{\overset{(\pm)}{\nu_3}j}\left(x_1,\ x_2,\ \overset{(\pm)}{h}_3\right) = X_{kj}\ \overset{(\pm)}{\nu}_{3k}, \quad j=1,2,3,$$

$$X_{\overset{(\pm)}{\nu_2}j}\left(x_1,\ \overset{(\pm)}{h}_2,\ x_3\right) = X_{kj}\ \overset{(\pm)}{\nu}_{2k}, \quad j=1,2,3,$$

the last quantities are external forces acting on the face surfaces $x_i = \overset{(\pm)}{h}_i(x_1)$, $i=2,3$. $\overset{(\pm)}{\nu_i}$, $i=2,3$, are exterior normals. X_{j00} $(j=1,2,3)$ are double moments of volume forces in (0,0) approximation. v_j are weighted double moments in (0,0) approximation,

$$v_j(x_1,t) := \frac{u_{j00}(x_1,t)}{h_2(x_1)h_3(x_1)},$$

u_{j00} are double moments of displacements u_j in the (0,0) approximation, i.e.,

$$u_{j00} := \int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}} \int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}} u_j(x_1,x_2,x_3,t)\,dx_2 dx_3, \quad j=1,2,3,$$

ρ is density, λ and μ are lamés constants, index after comma means differentiation with respect to x_1,

$$v_{j,1}(x_1,t) := \frac{\partial v_j(x_1,t)}{\partial x_1},$$

and $\dot{v}_j$ and $\ddot{v}_j$ mean the first and second order derivatives with respect to t, respectively

$$\dot{v}_j(x_1,t) := \frac{\partial v_j(x_1,t)}{\partial t}, \ \ \ddot{v}_j := \frac{\partial^2 v_j(x_1,t)}{\partial t^2}.$$

Let $\overline{V} \in R^3$ be occupied by an elastic beam,

$$V := \{(x_1,x_2,x_3) \in R^3 : 0 < x_1 < L, \ \ \overset{(-)}{h_i}(x_1) < x_i < \overset{(+)}{h_i}(x_1), \ \ i=2,3, \ \ L = const\},$$

$$2h_i(x_1) := \overset{(+)}{h_i}(x_1) - \overset{(-)}{h_i}(x_1) \geq 0, \ \ h_i \in C([0,L]) \cap C^1(]0,L[), \ \ i=2,3,$$

and let $2h_3$ and $2h_2$ be correspondingly the thickness and the width of the beam and their maxima be essentially less than the length L of the bar.

Let

$$\overset{(\pm)}{h_2}(x_1,x_3) := (\pm)h_2^0, \qquad h_2^0 = const > 0,$$

$$\overset{(\pm)}{h_3}(x_1,x_2) := \overset{(\pm)}{h_3^0}(x_1)e^{-\frac{\kappa}{x_1}}, \qquad \overset{(\pm)}{h_3^0} = const,$$

$$h_3^0 := \frac{\overset{(+)}{h_3^0} - \overset{(-)}{h_3^0}}{2}, \qquad \overset{(+)}{h_3^0} > \overset{(-)}{h_3^0},$$

then the profiles of the beam have three forms shown in Figure 1 a), b), c) in the cases $a)$ $\overset{(\pm)}{h_3^0} > 0$, $b)$ $\overset{(+)}{h_3^0} > 0$, $\overset{(-)}{h_3^0} < 0$, $c)$ $\overset{(\pm)}{h_3^0} < 0$, respectively.

In the case under consideration (1) has the following form

$$\left(e^{-\frac{\kappa}{x_1}} v_{j,1}(x_1,t)\right)_{,1} + \frac{\overset{0,0}{Y}_j}{h_2^0 h_3^0} = \Lambda_j^{-1}\rho e^{-\frac{\kappa}{x_1}}\ddot{v}_j(x_1,t), \qquad x_1 \in]0,L[. \tag{2}$$

(2) is a hyperbolic equation with the order degeneration at point $x_1 = 0$ since we may rewrite it in the following form

$$x_1^2 v_{j,11}(x_1,t) + æ v_{j,1}(x_1,t) + e^{\frac{\kappa}{x_1}}\frac{\overset{0,0}{Y}_j}{h_2^0 h_3^0}x_1^2 = \Lambda_j^{-1}\rho x_1^2 \ddot{v}_j(x_1,t)$$

Let us consider the static problem, then from (2) we have

$$\left(e^{-\frac{\kappa}{x_1}} v_{j,1}(x_1)\right)_{,1} = -\frac{\overset{0,0}{Y}_j}{h_2^0 h_3^0}, \qquad x_1 \in]0,L[, \ \ j=1,2,3. \tag{3}$$

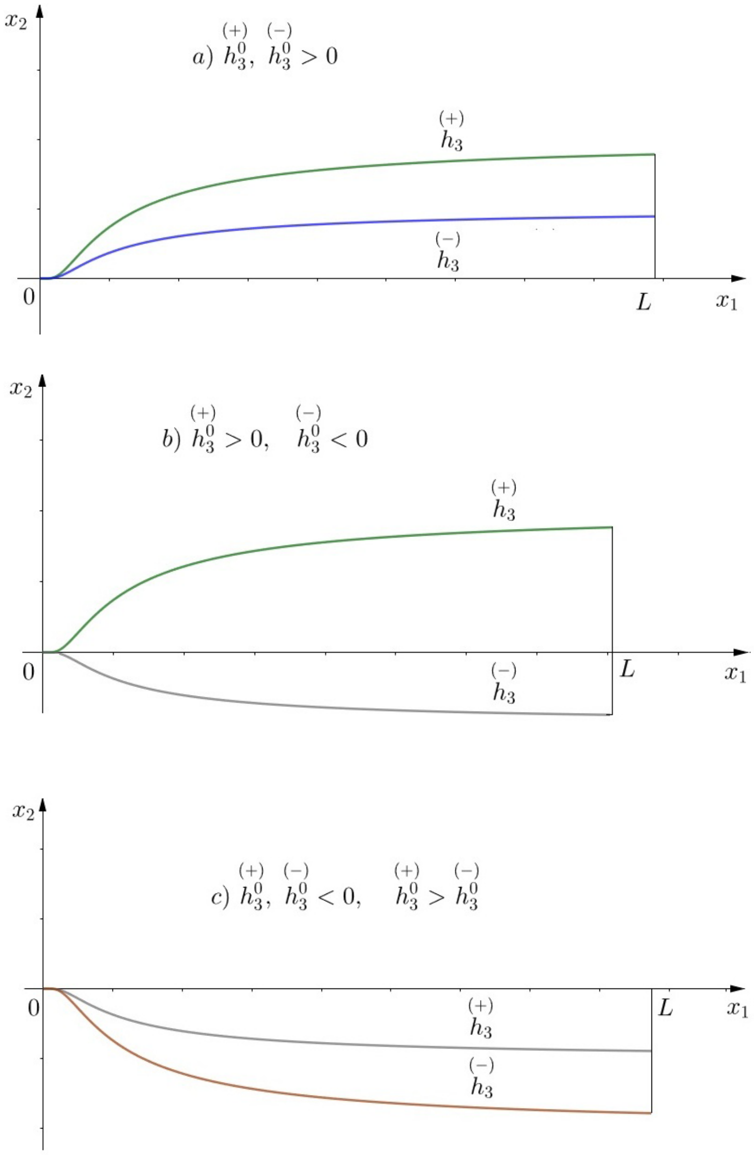

Figure 1. Profiles of the beam

For the sake of simplicity we assume $\overset{0,0}{Y}_j(x_1) \equiv 0$.
Problem 1. Find a solution

$$v_j(x_1) \in C^2(]0, L[) \cap C^1(]0, L]),$$

of equation (3) under the following boundary conditions:

$$v_j(L) = \varphi_j^L \tag{4}$$

and

$$X_{1j}(0) = \Lambda_j h_2^0 h_3^0 e^{-\frac{\kappa}{x_1}} v_{j,1}\Big|_{x_1=0} = \varphi_j^0 \tag{5}$$

where φ_j^L and φ_j^0 are given constants.

The general solution of the homogeneous equation (2) has the following form:

$$v_j(x_1) = c_{1j} \int\limits_L^{x_1} e^{\frac{\kappa}{\xi}} d\xi + c_{2j}.$$

Taking into account boundary conditions (4), (5), we easily find constants

$$c_{2j} = \varphi_j^L$$

and

$$c_{1j} = \frac{\varphi_j^0}{\Lambda_j h_2^0 h_3^0}.$$

So we arrive at the solution of Problem 1

$$v_j(x_1) = \frac{\varphi_j^0}{\Lambda_j h_2^0 h_3^0} \int\limits_L^{x_1} e^{\frac{\kappa}{\xi}} d\xi + \varphi_j^L. \tag{6}$$

3. Static problems for plates

Let us consider the $N = 0$ approximation of Vekua's hierarchical models (for details see, e.g. [3,16]). The system of governing equations for the weighted zero moments of the displacement vector components u_i has the following forms (below Einstein's summation convention is used, α, β, γ take values 1, 2)

$$\mu\left[(hv_{\alpha 0,\beta})_{,\beta} + (hv_{\beta 0,\alpha})_{,\beta}\right] + \lambda\delta_{\alpha\beta}(hv_{\gamma 0,\gamma})_{,\beta} + Q_{\overset{(+)}{\nu}\alpha}\sqrt{\left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2 + 1}$$
$$+Q_{\overset{(-)}{\nu}\alpha}\sqrt{\left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2 + 1} + \Phi_{\alpha 0} = \rho h\ddot{v}_{\alpha 0}, \quad \alpha = 1, 2, \tag{7}$$

$$\mu(hv_{30,\beta})_{,\beta} + Q_{\overset{(+)}{\nu}3}\sqrt{\left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2 + 1}$$
$$+Q_{\overset{(-)}{\nu}3}\sqrt{\left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2 + 1} + \Phi_{30} = \rho h\ddot{v}_{30}, \tag{8}$$

where Φ_{i0} is the zero moment of the volume force component Φ_i, $Q_{\overset{(+)}{\nu}i}$, $Q_{\overset{(-)}{\nu}i}$ are stresses given on the upper $x_3 = \overset{(+)}{h}(x_1, x_2)$ and lower $x_3 = \overset{(+)}{h}(x_1, x_2)$ surfaces of the prismatic shell,

$$v_{i0} = \frac{1}{h} u_{i0},$$

$$u_{i0}(x_1, x_2, t) := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} u_i(x_1, x_2, x_3, t)dx_3,$$

$$\Phi_{i0}(x_1, x_2, t) := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \Phi_i(x_1, x_2, x_3, t)dx_3,$$

$$\overset{(\pm)}{\nu}_\beta = \frac{\mp \overset{(\pm)}{h}_{,\beta}}{\sqrt{\left(\overset{(\pm)}{h}_{,1}\right)^2 + \left(\overset{(\pm)}{h}_{,2}\right)^2 + 1}}, \quad \overset{(\pm)}{\nu}_3 = \pm \frac{1}{\sqrt{\left(\overset{(\pm)}{h}_{,1}\right)^2 + \left(\overset{(\pm)}{h}_{,2}\right)^2 + 1}},$$

$$X_{i\beta}\left(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2), t\right)\overset{(\pm)}{\nu}_\beta + X_{i3}\left(x_1, x_2, \overset{(\pm)}{h}(x_1, x_2), t\right)\overset{(\pm)}{\nu}_3 = Q_{\overset{(\pm)}{\nu}i}(x_1, x_2, t).$$

Let (see Fig.2)

$$\begin{gathered} \overset{(\pm)}{h}(x_1, x_2) = \overset{(\pm)}{h}_0 e^{-\frac{\kappa}{x_1}}, \quad (x_1, x_2) \in \omega, \\ \overset{(\pm)}{h}_0 = const, \quad \overset{(+)}{h}_0 - \overset{(-)}{h}_0 > 0, \quad \kappa = const > 0, \end{gathered} \tag{9}$$

then the thickness

$$2h(x_1, x_2) = 2h_0 e^{-\frac{\kappa}{x_1}},$$

$$h_0 = \frac{\overset{(+)}{h}_0 - \overset{(-)}{h}_0}{2} = const > 0.$$

The profiles of the plate look like the above-considered beams (see Fig.1).

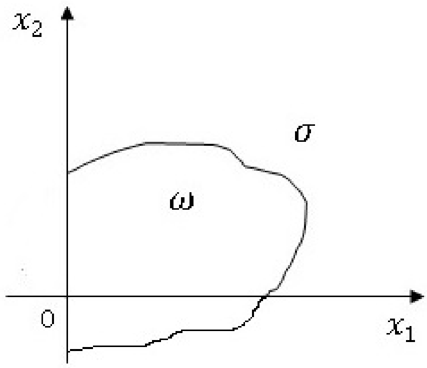

Figure 2. Projection of the plate

Let us first consider cylindrical deformation, i.e., all the quantities depend only on x_1, when

$$\omega :=]0, L]\times]-\infty, +\infty[.$$

Using (9) from the system (7), (8) in the static case we arrive at the system of independent three equations of type (3) for $v_{j0}(x_1),\ x_1 \in]0, L]$ (see Fig. 3):

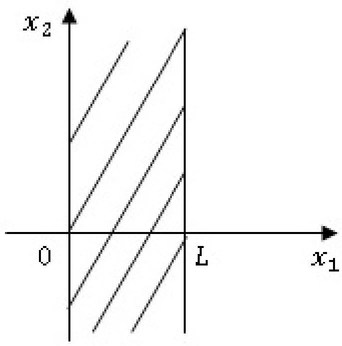

Figure 3. Projection a strip

$$\begin{gathered}
x_1^2(\lambda+2\mu)v_{10,11} + (\lambda+2\mu)\kappa v_{10,1} + F_{10} = 0, \\
x_1^2\mu v_{20,11} + \mu\kappa v_{20,1} + F_{20} = 0, \\
x_1^2 v_{30,11} + \kappa v_{30,1} + \tfrac{F_{30}}{\mu} = 0,
\end{gathered}$$

i.e.,

$$x_1^2 v_{j0,11} + \kappa v_{j0,1} + \Lambda^{-1}F_{j0} = 0, \quad j = 1, 2, 3,$$

where

$$F_{j0} = \frac{e^{\frac{\kappa}{x_1}}}{h_0}\left[Q_{\overset{(+)}{\nu}3}\sqrt{\left(\overset{(+)}{h_0}\,\frac{\kappa}{x_1^2}e^{-\frac{\kappa}{x_1}}\right)^2+1}+Q_{\overset{(-)}{\nu}3}\sqrt{\left(\overset{(-)}{h_0}\,\frac{\kappa}{x_1^2}e^{-\frac{\kappa}{x_1}}\right)^2+1}+\Phi_{30}\right],$$

$$j = 1, 2, 3.$$

Whence a problem with boundary conditions (4) (5) for each equation of the system (10) will be solved in the explicit form.

Now let's go back to the general case. For the sake of simplicity let $Q_{\overset{(\pm)}{\nu}3} \equiv 0$, $\Phi_{(30)} \equiv 0$, then in the static case from (8) we obtain

$$x_1^2(v_{30,11} + v_{30,22}) + \kappa v_{30,1} = 0 \tag{10}$$

Problem 2. Find the solution v_{30} of equation (10)

$$v_{30}(x_1, x_2) \in C^2(\omega)$$

under the following boundary conditions

$$v_{30}|_\sigma = \varphi$$

and

$$v_{j0}(x_1, x_2) = O(1), \quad j = 1, 2, 3, \quad x_1 \to 0,$$

where $\varphi \in C(\sigma)$ is a given function.

To investigate this problem we use the following theorem (see [17])

Theorem 3.1 : *If the coefficients a_α, $\alpha = 1, 2$, and c of the equation*

$$x_1^{\kappa_\alpha} u_{,\alpha\alpha} + a_\alpha(x_1, x_2)u_{,\alpha} + c(x_1, x_2)u = 0, \quad c \leq 0, \quad \kappa_\alpha = const \geq 0, \quad \alpha = 1, 2, \tag{11}$$

are analytic in $\overline{\omega}$, then

(i) if either $\kappa_1 < 1$, or $\kappa_1 \geq 1$,

$$a_1(x_1, x_2) < x_1^{\kappa_2 - 1}$$

in $\overline{\omega_\delta}$ for some $\delta = const > 0$, where

$$\omega_\delta := \{(x_1, x_2) \in \omega : \ 0 < x_1 < \delta\},$$

the Dirichlet problem is well-posed;

(ii) if $\kappa_1 \geq 1$,

$$a_1(x_1, x_2) \geq x_1^{\kappa_2 - 1} \tag{12}$$

in ω_δ *and* $a_2(x_1, x_2) = O(x_1^{\kappa_1})$, $x_1 \to 0_+$ (0 *is the Landau symbol), the Keldysh problem is well-posed.*

Equation (10) is a particular case of equation (11), therefore condition (12) has the form

$$\kappa \geq x_1 \ \ in \ \ \omega_\delta,$$

which is fulfilled for sufficiently small δ.

Since in case of Problem 2 conditions of Theorem 3.1 are satisfied, it follows that the Keldysh problem is well-posed.

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