THE BOUNDARY VALUE PROBLEM IN DISPLACEMENTS FOR CUSPED PRISMATIC SHELLS ON THE CASE OF THE FIRST APPROXIMATION OF I. VEKUA'S MODEL

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In the case of the first approximation of I. Vekua's version of the elastic shell theory the general system consists of six second order differential equations. This system is split into two autonomous systems. One of them has the form

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial}{\partial x}\left(h^3\frac{\partial v_1}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h^3\frac{\partial v_1}{\partial y}\right)+\lambda\frac{\partial}{\partial x}\left(h^3\frac{\partial v_2}{\partial y}\right)\\
&\qquad+\mu\frac{\partial}{\partial y}\left(h^3\frac{\partial v_2}{\partial x}\right)-\mu h\frac{\partial u_3}{\partial x}-3\mu h v_1+Y_1=0,\\
&\mu\frac{\partial}{\partial x}\left(h^3\frac{\partial v_2}{\partial x}\right)+(\lambda+2\mu)\frac{\partial}{\partial y}\left(h^3\frac{\partial v_2}{\partial y}\right)+\mu\frac{\partial}{\partial x}\left(h^3\frac{\partial v_1}{\partial y}\right)\\
&\qquad+\lambda\frac{\partial}{\partial y}\left(h^3\frac{\partial v_1}{\partial x}\right)-\mu h\frac{\partial u_3}{\partial y}-3\mu h v_2+Y_2=0,\\
&\mu\frac{\partial}{\partial x}\left(h\frac{\partial u_3}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_3}{\partial y}\right)+3\mu h\frac{\partial}{\partial x}(hv_1)+3\mu h\frac{\partial}{\partial y}(hv_2)\\
&\qquad+Y_3=0,
\end{aligned}\right\}\tag{1}$$

(The second one was investigated ealier and this article is a continuation of the paper [2]), where v_1, v_2 and u_3 are unknown functions and, namely, they are so called first (v_1, v_2) and zero (u_3) moments of the displacement vector; λ and $\mu > 0$ are Lame's constants; Y_1, Y_2 and Y_3 are the first (Y_1, Y_2) and zero (Y_3) Fourier-Legendre moments of the given volume force, $2h$ is the thickness of the shell. The system is considered on the so called middle surface ω of the shell, which is actually the orthogonal projection of the shell on the plane Oxy.

In this paper we study the case when the middle surface ω is a plane bounded domain with a smooth boundary $\partial\omega \in C^1$, where

$$\partial\omega = \Gamma_0 \cup \Gamma_1,$$

$$\Gamma_0 = \{(x,0):\ a \le x \le b,\ a,b \in R,\ a<b\},$$

$$\Gamma_1 = \{(x,y):\ (x,y)\in\partial\omega,\ y \ge 0\},$$

$$\Gamma_0 \cap \Gamma_1 = \{(a,0);\ (b,0)\}.$$

Let the thickness $2h$ be given by the function

$$h = h(y) = y^m,\ \ m>0.$$

Let $D(\omega)$ be a set of infinitely differentiable compactly supported functions on ω. We introduce the following bilinear form on $[D(\omega)]^3$:

$$(u,v)_{3,m} \equiv \int\limits_{\omega} y^m \left[\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right)\right] d\tau$$
$$+ \int\limits_{\omega} y^{\frac{m}{3}} \left[\left(\frac{\partial u_3}{\partial x} + 3u_1\right)\left(\frac{\partial v_3}{\partial x} + 3v_1\right) + \left(\frac{\partial u_3}{\partial y} + 3u_2\right)\left(\frac{\partial v_3}{\partial y} + 3v_2\right)\right] d\tau, \quad (2)$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in [D(\omega)]^3$.

According to the Korn' weighted inequality [see [2], p.35] it is easy to show that (2) is a scalar product on $[D(\omega)]^3$. If we complete $[D(\omega)]^3$ by the scalar product (2) we obtain the Hilbert space $\overset{0}{M}_m(\omega)$ with the norm

$$\|u\|^2_{3,m} = \int\limits_{\omega} y^m \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2\right] d\tau$$
$$+ \int\limits_{\omega} y^{\frac{m}{3}} \left[\left(\frac{\partial u_3}{\partial x} + 3u_1\right)^2 + \left(\frac{\partial u_3}{\partial y} + 3u_2\right)^2\right] d\tau, \quad (3)$$

where $u = (u_1, u_2, u_3) \in \overset{0}{M}_m(\omega)$.

Let

$$L_{2,(\sigma_m,\sigma_m,\sigma^1_m)}(\omega) := L_{2,\sigma_m}(\omega) \times L_{2,\sigma_m}(\omega) \times L_{2,\sigma^1_m}(\omega),$$

where $L_{2,\sigma_m}(\omega)$ is a Hilbert space of measurable functions ψ such, that the norm

$$\|\psi\|_{L_{2,\sigma_m}(\omega)} = \left(\int\limits_{\omega} \sigma_m(y)\psi^2(x,y)d\tau\right)^{\frac{1}{2}}$$

is finite,

$$\sigma_m(y) = \begin{cases} y^{2-m}, \ m \neq 1, \\ y^{1-\varepsilon}, \ \varepsilon > 0, \ m = 1, \end{cases}$$
$$\sigma^1_m = y^{4-m}.$$

We say that $v = (v_1, v_2, u_3) \in \overset{0}{M}_m(\omega)$ is a generalized solution of the system (1) if

$$B(v,w) := \int\limits_{\omega} \left\{ y^{3m} \left[(\lambda + 2\mu)\frac{\partial v_1}{\partial x}\frac{\partial w_1}{\partial x} + \mu\frac{\partial v_1}{\partial y}\frac{\partial w_1}{\partial y} + \lambda\frac{\partial v_2}{\partial y}\frac{\partial w_1}{\partial y}\right.\right.$$
$$\left. + \mu\frac{\partial v_2}{\partial x}\frac{\partial w_1}{\partial y}\right] + y^m \left[\mu\frac{\partial u_3}{\partial x}w_1 + 3\mu v_1 w_1\right]$$
$$+ y^{3m} \left[\mu\frac{\partial v_2}{\partial x}\frac{\partial w_2}{\partial x} + (\lambda + 2\mu)\frac{\partial v_2}{\partial y}\frac{\partial w_2}{\partial y} + \mu\frac{\partial v_1}{\partial y}\frac{\partial w_2}{\partial x} + \lambda\frac{\partial v_1}{\partial x}\frac{\partial w_2}{\partial y}\right]$$

$$+y^m\left[\mu\frac{\partial u_3}{\partial y}w_2+3\mu v_2w_2+\frac{1}{3}\mu\frac{\partial u_3}{\partial x}\frac{\partial w_3}{\partial x}+\frac{1}{3}\mu\frac{\partial u_3}{\partial y}\frac{\partial w_3}{\partial y}+\mu v_2\frac{\partial w_3}{\partial y}+\mu v_1\frac{\partial w_3}{\partial x}\right]\right\}d\tau$$

$$=\int\limits_{\omega}\left(Y_1w_1+Y_2w_2+\frac{1}{3}Y_3w_3\right)d\tau$$

for all $w=(w_1,w_2,w_3)\in[D(\omega)]^3$.

Theorem. *If $Y=(Y_1,Y_2,Y_3)\in L_{2,\left(\sigma_{3m},\sigma_{3m},\sigma^1_{3m}\right)}(\omega)$ then the system (1) has a unique generalized solution $v=(v_1,v_2,u_3)\in\overset{0}{M}{}_{3m}(\omega)$ and there holds the estimate*

$$\|v\|_{3,3m}\leq c\|Y\|_{L_{2,\left(\sigma_{3m},\sigma_{3m},\sigma^1_{3m}\right)}},$$

where c is a positive constant independent of Y and v.

Proof. Let us show that the form $B(v,w)$ is coercive on $\overset{0}{M}{}_{3m}(\omega)$. First we estimate $B(v,v)$ for $v=(v_1,v_2,u_3)\in\overset{0}{M}{}_{3m}(\omega)$.

$$B(v,v):=\int\limits_{\omega}y^m\left\{y^{2m}\left[(\lambda+2\mu)\left(\frac{\partial v_1}{\partial x}\right)^2+\mu\left(\frac{\partial v_1}{\partial y}\right)^2+\lambda\frac{\partial v_2}{\partial y}\frac{\partial v_1}{\partial x}+\mu\frac{\partial v_2}{\partial x}\frac{\partial v_1}{\partial y}\right]\right.$$

$$+\mu\frac{\partial u_3}{\partial x}v_1+3\mu v_1^2+y^{2m}\left[\mu\left(\frac{\partial v_2}{\partial x}\right)^2+(\lambda+2\mu)\left(\frac{\partial v_2}{\partial y}\right)^2+\mu\frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial x}+\lambda\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y}\right]$$

$$\left.+\mu\frac{\partial u_3}{\partial y}v_2+3\mu v_2^2+\frac{1}{3}\mu\left(\frac{\partial u_3}{\partial x}\right)^2+\frac{1}{3}\mu\left(\frac{\partial u_3}{\partial y}\right)^2+\mu v_1\frac{\partial u_3}{\partial x}+\mu v_2\frac{\partial u_3}{\partial y}\right\}d\tau$$

$$=\int\limits_{\omega}y^{3m}\left[\lambda\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}\right)^2+2\mu\left(\left(\frac{\partial v_1}{\partial x}\right)^2+\left(\frac{\partial v_2}{\partial y}\right)^2\right)+\mu\left(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}\right)^2\right]d\tau$$

$$+\frac{1}{3}\mu\int\limits_{\omega}y^m\left[\left(\frac{\partial u_3}{\partial x}+3v_1\right)^2+\left(\frac{\partial u_3}{\partial y}+3v_2\right)^2\right]d\tau$$

$$\geq\mu\int\limits_{\omega}y^{3m}\left[\left(\frac{\partial v_1}{\partial x}\right)^2+\left(\frac{\partial v_2}{\partial y}\right)^2+\left(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}\right)^2\right]d\tau$$

$$+\delta\int\limits_{\omega}y^m\left[\left(\frac{\partial u_3}{\partial x}+3v_1\right)^2+\left(\frac{\partial u_3}{\partial y}+3v_2\right)^2\right]d\tau,$$

where $0<\delta\leq\frac{1}{3}\mu$.

Hence,

$$B(v,\,v)\geq c_1\|v\|^2_{3,3m}.$$

Let us estimate $|B(v,w)|$, where $v=(v_1,v_2,u_3)$, $w=(w_1,w_2,w_3)\in \overset{0}{M}_{3m}(\omega)$.

$$
\begin{aligned}
| B(v,w) | := \Bigg| \int_\omega y^{3m} \Bigg[(\lambda+2\mu)\frac{\partial v_1}{\partial x}\frac{\partial w_1}{\partial x} + \mu \left(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}\right)\left(\frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x}\right) \\
+\lambda \frac{\partial v_2}{\partial y}\frac{\partial w_1}{\partial x} + (\lambda+2\mu)\frac{\partial v_2}{\partial y}\frac{\partial w_2}{\partial y} + \lambda\frac{\partial v_1}{\partial x}\frac{\partial w_2}{\partial y}\Bigg] d\tau \\
+\frac{\mu}{3}\int_\omega y^m \Bigg[\frac{\partial u_3}{\partial x}\frac{\partial w_3}{\partial x} + \frac{\partial u_3}{\partial y}\frac{\partial w_3}{\partial y} + 3\frac{\partial u_3}{\partial x}w_1 + 3\frac{\partial u_3}{\partial y}w_2 + 3v_1\frac{\partial w_3}{\partial x} \\
+3v_2\frac{\partial w_3}{\partial y} + 9v_1w_1 + 9v_2w_2\Bigg] d\tau\Bigg| = \Bigg|\int_\omega y^{3m}\Bigg[(\lambda+2\mu)\frac{\partial v_1}{\partial x}\frac{\partial w_1}{\partial x} \\
+\mu\left(\frac{\partial v_1}{\partial y}+\frac{\partial v_2}{\partial x}\right)\left(\frac{\partial w_1}{\partial y}+\frac{\partial w_2}{\partial x}\right) + \lambda\frac{\partial v_2}{\partial y}\frac{\partial w_1}{\partial x} + (\lambda+2\mu)\frac{\partial v_2}{\partial y}\frac{\partial w_2}{\partial y} \\
+\lambda\frac{\partial v_1}{\partial x}\frac{\partial v_2}{\partial y}\Bigg] d\tau + \frac{\mu}{3}\int_\omega y^m\Bigg[\left(\frac{\partial u_3}{\partial x}+3v_1\right)\left(\frac{\partial w_3}{\partial x}+3w_1\right) \\
+\left(\frac{\partial u_3}{\partial y}+3v_2\right)\left(\frac{\partial w_3}{\partial y}+3w_2\right)\Bigg] d\tau\Bigg|.
\end{aligned}
$$

Hence, if we use the Hölder inequality, we obtain

$$
|B(v,w)| \le c_2 \|v\|_{3,3m}\|w\|_{3,3m}.
$$

It is evident that (see [2], Lemma1, p.28)

$$
\left|\int_\omega Y\cdot w d\tau\right| \le c_3\|w\|_{3,3m}\|Y\|_{L_{2,\left(\sigma_{3m},\sigma_{3m},\sigma^1_{3m}\right)}}.
$$

Thus, we have shown, that $B(v,w)$ is coercive and bounded, and the functional $\int_\omega Y\cdot w d\tau$ is bounded on $\overset{0}{M}_{3m}(\omega)$. Now, the Lax-Milgram theorem (see, e.g. [3]) completes the proof.□

In what follows we make some ramarks concerning the theorem.

At first we remark, that $\forall u\in \overset{0}{M}_{3m}(\omega)$

$$
u|_{\Gamma_1} = 0,
$$

and

$$
\overset{0}{M}_{3m}(\omega) \supset \overset{0}{H}_{2,3m}(\omega)\times \overset{0}{H}_{2,3m}(\omega)\times \overset{0}{H}_{1,3}(\omega)
$$

(Definition and properties of space $\overset{0}{H}_{2,m}(\omega)$ one can see in the paper [2]).

Let $v = (v_1, v_2, u_3) \in \overset{0}{M}{}_{3m}(\omega)$ be a generalized solution of system (1).

If $0 < m < \frac{1}{3}$, then

$$v_1|_{\partial\omega} = v_2|_{\partial\omega} = u_3|_{\partial\omega} = 0$$

in the sense of the trace.

Consequently in this case v must be given on the whole boundary if we consider boundary conditions in displacements.

If $\frac{1}{3} < m < 1$, then

$$v_1|_{\Gamma_1} = v_2|_{\Gamma_1} = 0, \; u_3|_{\partial\omega} = 0.$$

v_1 and v_2 do not have traces on Γ_0, in general. In this case v_1 and v_2 must be given only on Γ_1, and u_3 must be given on the whole boundary.

If $m \geq 1$, then

$$v_1|_{\Gamma_1} = v_2|_{\Gamma_1} = u_3|_{\Gamma_1} = 0$$

and v_1, v_2 and u_3 do not have traces on Γ_0, in general. Therefore, we must give v_1, v_2 and u_3 only on Γ_0, and have them free from boundary conditions on Γ_1.

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Received September 3, 2002; revised November 15, 2002; accepted December 3, 2002.