UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIX
2001

ON THE IRREDUCIBILITY OF FIBRES OF COMPLEX POLYNOMIAL MAPPINGS

by Tadeusz Krasiński and Stanisław Spodzieja

Dedicated to Professor Tadeusz Winiarski
on the occasion of his 60th birthday

Abstract. We prove that for any open polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq 3$, there is a linear change of coordinates $\alpha : \mathbb{C}^m \to \mathbb{C}^m$ such that for each component f_i of $\alpha \circ f$, every fibre of f_i is irreducible. This is a generalization of the Kaliman result to the multidimensional case.

Introduction. Let X, Y be irreducible algebraic varieties over $\mathbb{C}$ in the sense of Weil-Serre (algebraic spaces in terms of [**5**], VII, §17) and $f : X \to Y$ be a regular dominating mapping. We call f *primitive* if the generic fibre of f is irreducible and we call f *totally primitive* if each fibre of f is irreducible. In the case that $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial, the primitivity of f is equivalent to the *indecomposability* or *non-compositness* of f considered by Schinzel [**7**] and Stein [**9**] (the equivalence was proved by Płoski in [**6**]).

Kaliman in [**4**] proved that in the two-dimensional jacobian conjecture for a polynomial mapping $(p, q) : \mathbb{C}^2 \to \mathbb{C}^2$, $Jac(p, q) \equiv 1$, one can assume that one component, say p, is totally primitive. He showed it by finding for (p, q) a polynomial automorphism $f = (f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ such that $p_1 := f_1(p, q)$ is totally primitive. In this paper we prove that in the multi-dimensional case the result is stronger. Namely, for any open polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq 3$, there is a linear change of coordinates $\alpha : \mathbb{C}^m \to \mathbb{C}^m$ such that each component of $\alpha \circ f$ is totally primitive (Theorem 3). The crucial role

---

2000 *Mathematics Subject Classification.* 14R15.

Key words and phrases. Polynomial mapping, primitive mapping, irreducible fibre.

This research was partially supported by KBN Grant No. 2 P03A 050 10 and KBN Grant No. 2 P03A 007 18.

in the proof plays a version of the Bertini Theorem (Proposition 3, cf. [**3**], Theorem 6.6). The last result follows from a criterion for the primitivity of the composition of a regular mapping with a projection (Theorem 2). Namely, we prove that for any regular dominating mapping $f : X \to \mathbb{C}^m$, and a projection $\pi : \mathbb{C}^m \to \mathbb{C}^{m-1}$ such that π is proper on the set of bad values of f, the composition $\pi \circ f$ is a primitive mapping.

In Section 1 we collect general facts on the primitivity of regular mappings (cf. [**8**], II.6, [**2**], 2.13,14, [**1**], III.4.3). Section 2 is devoted to the criterion for primitivity of the composition of a regular mapping with a projection. In the last Section 3 we prove the Kaliman Theorem in the multidimensional case.

1. Primitive mappings on algebraic varieties. Let $f : X \to Y$ be a regular dominating mapping between irreducible algebraic varieties X, Y over $\mathbb{C}$. The mapping f is called *primitive* if there exists a proper algebraic subset $\tilde{Y} \subset Y$ such that for each $\xi \in Y \setminus \tilde{Y}$, the fibre $f^{-1}(\xi)$ is an irreducible algebraic subvariety of X, and the mapping f is called *totally primitive* if each fibre $f^{-1}(\xi)$, $\xi \in Y$, is irreducible.

In the sequel we write "for the generic $\xi \in Y$" instead of "there exists a proper algebraic subvariety $\tilde{Y} \subset Y$ such that for each $\xi \in Y \setminus \tilde{Y}$".

From the definition of primitive mappings we immediately obtain

PROPOSITION 1. *Let X, Y be irreducible algebraic varieties. If $\tilde{X} \subset X$ is a proper algebraic subvariety of X, then a dominating mapping $f : X \to Y$ is primitive if and only if the mapping $f|_{X\setminus\tilde{X}} : X \setminus \tilde{X} \to Y$ is primitive.*

PROOF. Assume that f is primitive. Since $f|_{X\setminus\tilde{X}} : X \setminus \tilde{X} \to Y$ is dominating, then, for the generic $\xi \in Y$, $f^{-1}(\xi)$ is irreducible in X and $f^{-1}(\xi) \cap (X \setminus \tilde{X}) \neq \emptyset$. Then $f^{-1}(\xi) \cap (X \setminus \tilde{X})$ is irreducible in $X \setminus \tilde{X}$. This implies that $f|_{X\setminus\tilde{X}}$ is primitive.

Let us assume that $f|_{X\setminus\tilde{X}}$ is primitive. Since the generic fibre of f has dimension $\dim X - \dim Y$ and the generic fibre of $f|_{\tilde{X}}$ has dimension at most $\dim X - \dim Y - 1$, then for the generic fibre of f, its all irreducible components intersect $X \setminus \tilde{X}$. In consequence, by the assumption, f is a primitive mapping. $\square$

PROPOSITION 2. *Let X, Y be irreducible algebraic varieties and $f : X \to Y$ be a dominating regular mapping. Then the mapping f is primitive if and only if there exist nonempty Zariski open subsets $X_0 \subset X$, $Y_0 \subset Y$, biregular to affine varieties (i.e. to irreducible algebraic subsets of some $\mathbb{C}^m$) such that $f(X_0) \subset Y_0$ and the mapping $f|_{X_0} : X_0 \to Y_0$ is primitive.*

PROOF. By Proposition 1, we may assume that X is an affine variety. Let $Y_0 \neq \emptyset$ be a nonempty Zariski open subset of Y and biregular to an affine variety. Let $X_1 = f^{-1}(Y_0)$. Then X_1 is a Zariski open subset of X. Hence it is an algebraic variety. So, there exists a nonempty Zariski open subset $X_0 \neq \emptyset$ of X_1, biregular to an affine variety. In consequence, $f(X_0) \subset Y_0$ and, by Proposition 1, f is primitive if and only if $f|_{X_0} : X_0 \to Y_0$ is primitive. This ends the proof. $\square$

From the proposition it follows that the investigation of the primitivity of regular mappings can be reduced to affine varieties. We shall use this proposition in the proof of general algebraic criterions for the primitivity. We start with definitions.

For an irreducible algebraic variety X, by $\mathbb{C}(X)$ we denote the field of rational functions on X.

Let $f : X \to Y$ be a dominating regular mapping between irreducible algebraic varieties X, Y. Let $f^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ be the homomorphism induced by f. Since f is dominating, then f^* is an embedding of $\mathbb{C}(Y)$ into $\mathbb{C}(X)$. Of course, the extension $\mathbb{C}(X)$ of $f^*(\mathbb{C}(Y))$ is finite generated, i.e. $\mathbb{C}(X) = f^*(\mathbb{C}(Y))(\varphi_1, ..., \varphi_m)$, for some $\varphi_1, ..., \varphi_m \in \mathbb{C}(X)$. Thus $\mathbb{C}(X)$ is isomorphic to the quotient field of the ring $f^*(\mathbb{C}(Y))[T_1, ..., T_m]/I_X$, where $T_1, ..., T_m$ are algebraically independent variables and I_X is the ideal of relations between $\varphi_1, ..., \varphi_m$, i.e. the kernel of the natural homomorphism

$$f^*(\mathbb{C}(Y))[T_1, ..., T_m] \to f^*(\mathbb{C}(Y))[\varphi_1, ..., \varphi_m].$$

The algebraic variety X is called *irreducible over* Y if $I_X^e \subset \overline{f^*(\mathbb{C}(Y))}[T_1, ..., T_m]$ is a prime ideal, where $\overline{f^*(\mathbb{C}(Y))}$ is an algebraic closure of $f^*(\mathbb{C}(Y))$ and I_X^e denotes the ideal in $\overline{f^*(\mathbb{C}(Y))}[T_1, ..., T_m]$ generated by I_X.

The following theorem gives the known equivalent conditions for the primitivity of f.

THEOREM 1. *Let $f : X \to Y$ be a regular dominating mapping between irreducible algebraic varieties X, Y. The following conditions are equivalent:*
(i) f is a primitive mapping,
(ii) $f^*(\mathbb{C}(Y))$ is algebraically closed in $\mathbb{C}(X)$,
(iii) X is irreducible over Y.

PROOF. (i)$\Rightarrow$(ii). By Proposition 2 one can assume that X and Y are affine varieties. Take any $\varphi \in \mathbb{C}(X)$ algebraic over $f^*(\mathbb{C}(Y))$. Since f is primitive, it is easy to show that φ is constant on the generic fibres of f. Hence, the minimal polynomial in $f^*(\mathbb{C}(Y))[T]$ for φ is of degree 1. Thus $\varphi \in f^*(\mathbb{C}(Y))$. Thus we have (ii).

The implication (ii)$\Rightarrow$(iii) follows from [**11**], Ch. VII, §11, Theorem 39.

One can find a proof of the implication (iii)$\Rightarrow$(i) in [**8**], Ch. 2, §6, Theorem 1. $\square$

From the above Theorem we immediately obtain

Corollary 1. *Let X, Y, Z be irreducible algebraic varieties. If $f : X \to Y$ and $g : Y \to Z$ are primitive mappings, then $g \circ f : X \to Z$ is primitive, too.* $\square$

2. Composition of a regular mappings with a projection. In this section we give a theorem on primitivity of the composition of a regular mapping with a projection (Theorem 2). Let us start with definitions.

Let X be an irreducible algebraic variety. By X^* we denote the set of singular points of X. Let $f : X \to \mathbb{C}^m$ be a regular dominating mapping. We say that $\xi \in \mathbb{C}^m$ is a *typical value* of f if there exists a neighbourhood $\Delta \subset \mathbb{C}^m$ of ξ such that $f|_{f^{-1}(\Delta)} : f^{-1}(\Delta) \to \Delta$ is a trivial topological bundle. We call the remaining points of $\mathbb{C}^m$ *bifurcation points* of f and denote by B_f the set of such points. Let $A_f = \{\xi \in \mathbb{C}^m :$ some irreducible component of $f^{-1}(\xi)$ is contained in $X^*\}$, C_f – the set of critical values of $f|_{X\setminus X^*} : X \setminus X^* \to \mathbb{C}^m$. We denote by E_f the Zariski closure of the set $A_f \cup B_f \cup C_f$ and call it *the set of bad values* of f.

Lemma 1. *If X is an irreducible algebraic variety and $f : X \to \mathbb{C}^m$ is a dominating regular mapping, then*

(a) E_f is a proper algebraic subset of $\mathbb{C}^m$,

(b) there exists a positive integer d_f such that for any $\xi \in \mathbb{C}^m \setminus E_f$, $f^{-1}(\xi)$ is the union of d_f irreducible components of dimension $\dim X - m$.

(c) for any $\xi \in \mathbb{C}^m \setminus E_f$ there exist a connected neighbourhood $\Delta \subset \mathbb{C}^m \setminus E_f$ of ξ and open connected sets $U_1, ..., U_{d_f} \subset X \setminus X^*$ such that for any j, $f|_{U_j} : U_j \to \Delta$ is a trivial holomorphic bundle and for any $\tilde{\xi} \in \Delta$, $f^{-1}(\tilde{\xi}) \cap U_j$, $j = 1, ..., d_f$, are subsets of different components of $f^{-1}(\tilde{\xi})$.

Proof. (a) From [**10**], Corollary 5.1 we conclude that B_f is contained in a proper algebraic subset of $\mathbb{C}^m$.

By the Sard Lemma, C_f is also contained in a proper algebraic subset of $\mathbb{C}^m$.

From the definition of B_f, for any $\xi \in \mathbb{C}^m \setminus B_f$, each irreducible component of $f^{-1}(\xi)$ has dimension $\dim X - m$. Let $Y \subset \mathbb{C}^m$ be the Zariski closure of $f(X^*)$. If $Y \neq \mathbb{C}^m$, then $A_f \subset Y$ is contained in a proper algebraic subset of $\mathbb{C}^m$. If $Y = \mathbb{C}^m$, then the mapping $\tilde{f} = f|_{X^*} : X^* \to \mathbb{C}^m$ is dominating. Then there exists a nonempty Zariski open subset $U \subset \mathbb{C}^m$ such that for all $\xi \in U$, each irreducible component of $\tilde{f}^{-1}(\xi)$ has dimension $\dim X - m - 1$, and so each irreducible component of $f^{-1}(\xi)$ intersects $X \setminus X^*$. Thus $A_f \subset B_f \cup (\mathbb{C}^m \setminus U)$ is contained in a proper algebraic subset of $\mathbb{C}^m$. Summing up we have (a).

(b) follows from the definition of A_f and B_f.

(c) follows from (b), the definition of E_f and the Implicit Function Theorem. $\square$

THEOREM 2. *Let X be an irreducible algebraic variety and $f : X \to \mathbb{C}^m$, $m \geq 2$, be a regular dominating mapping. If $\pi : \mathbb{C}^m \ni (\xi, \xi') \mapsto \xi \in \mathbb{C}^{m-1}$ is the canonical projection and $\pi|_{E_f} : E_f \to \mathbb{C}^{m-1}$ is proper, then $\tilde{f} = \pi \circ f : X \to \mathbb{C}^{m-1}$ is a primitive mapping.*

PROOF. Let $X_1 = X \setminus (f^{-1}(E_f) \cup X^*)$. From Lemma 1(a), $f^{-1}(E_f)$ is a proper algebraic subset of X, and so, by Proposition 1, it suffices to prove that $g = \tilde{f}|_{X_1} : X_1 \to \mathbb{C}^{m-1}$ is a primitive mapping. We shall use Theorem 1, implication (ii)$\Rightarrow$(i).

Let $\varphi \in \mathbb{C}(X_1)$ be algebraic over $g^*(\mathbb{C}(\lambda))$, $\lambda = (\lambda_1, ..., \lambda_{m-1})$. Then there exists an irreducible polynomial

$$p = u^s + a_1(\lambda)u^{s-1} + \ldots + a_s(\lambda) \in \mathbb{C}(\lambda)[u],$$

where $a_j \in \mathbb{C}(\lambda)$, such that

$$p(g, u) \in g^*(\mathbb{C}(\lambda))[u]$$

is the minimal polynomial of φ. Obviously

(1) $(g, \varphi)(X_1)$ is a dense constructible subset of

$$\Gamma = \{(\xi, t) \in \mathbb{C}^{m-1} \times \mathbb{C} : p(\xi, t) = 0\}.$$

To finish the proof we will show that $s = 1$. By the Monodromy Theorem it suffices to prove that for any $\xi \in \mathbb{C}^{m-1}$ there exist a neighbourhood $\Delta \subset \mathbb{C}^{m-1}$ of ξ and s different meromorphic functions $\psi_1, ..., \psi_s$ on Δ such that

$$p(\xi, \psi_j(\xi)) = 0 \qquad \text{on } \Delta,\ j = 1, ..., s.$$

Take any $\xi_0 \in \mathbb{C}^{m-1}$. Since $\pi|_{E_f}$ is proper, then there exists $\xi_0' \in \mathbb{C}$ such that $(\xi_0, \xi_0') \in \mathbb{C}^m \setminus E_f$. Let d_f be the number defined in Lemma 1(b) for the mapping f. Since φ is constant on each irreducible component of the generic fibre $g^{-1}(\xi)$, then φ is constant on each irreducible component of the generic fibre $f^{-1}(\xi, \xi') \cap X_1$, and so, by (1), $s \leq d_f$. By Lemma 1(c), there exist a neighbourhood $\Delta_0 = \Delta \times D \subset \mathbb{C}^m \setminus E_f$ of (ξ_0, ξ_0'), where $\Delta \subset \mathbb{C}^{m-1}$, $D \subset \mathbb{C}$, and open sets $U_j \subset X_1$, $j = 1, ..., d_f$ (in the natural topology) such that for any j, $f|_{U_j} : U_j \to \Delta_0$ is a trivial holomorphic bundle, and for any $(\xi, \xi') \in \Delta_0$, $f^{-1}(\xi, \xi') \cap U_j$ are subsets of different components of $f^{-1}(\xi, \xi')$. In consequence there exist meromorphic functions ψ_1,...,ψ_{d_f} on Δ_0 such that

$$\varphi|_{U_j} = \psi_j \circ f|_{U_j}, \qquad j = 1, ..., d_f$$

and so,

$$\psi_j^s(\xi, \xi') + a_1(\xi)\psi_j^{s-1}(\xi, \xi') + \ldots + a_s(\xi) = 0 \quad \text{for } (\xi, \xi') \in \Delta_0,\ j = 1, ..., d_f.$$

Hence we see that ψ_j does not depend on ξ', and so,

$$\psi_j^s(\xi) + a_1(\xi)\psi_j^{s-1}(\xi) + \ldots + a_s(\xi) = 0 \qquad \text{on } \Delta,\ j = 1, ..., d_f. \tag{2}$$

Moreover

$$\varphi|_{U_j} = \psi_j \circ g|_{U_j}, \qquad j = 1, ..., d_f. \tag{3}$$

Note that for any $\xi \in \Delta$, each irreducible component of $g^{-1}(\xi)$ intersects $U = U_1 \cup \ldots \cup U_{d_f}$. Indeed, for fixed $\xi \in \Delta$, the set $g^{-1}(\xi)$ is non-empty. Let V_0 be an irreducible component of $g^{-1}(\xi)$. Obviously $\dim V_0 \geq 1$. Since $g^{-1}(\xi) = f^{-1}(\{\xi\} \times \mathbb{C}) \cap X_1$ then from the definition of X_1 and the Remmert Open Mapping Theorem there follows that $f|_{V_0} : V_0 \to \{\xi\} \times \mathbb{C}$ is open and thence dominating mapping. In consequence, there exists $\xi' \in \mathbb{C}$ such that $(\xi, \xi') \in f(V_0) \cap \Delta_0$. From the choice of U_j there follows that $f^{-1}(\xi, \xi') \cap U_j$, $j = 1, ..., d_f$ are subsets of different components of $f^{-1}(\xi, \xi')$. But at least one of them is contained in V_0. This gives the announced observation. From (3) and the above observation we have

$$\bigcup_{j=1}^{d_f} \{(\xi, \psi_j(\xi)) : \xi \in \Delta\} = \bigcup_{j=1}^{d_f} (g, \varphi)(U_j) = (g, \varphi)(g^{-1}(\Delta)).$$

Since, by (1), $(g, \varphi)(g^{-1}(\Delta))$ is a dense subset of $\{(\xi, t) \in \Delta \times \mathbb{C};\ p(\xi, t) = 0\}$, then we see that there exist s different functions $\psi_{j_1}, ..., \psi_{j_s} \in \{\psi_1, ..., \psi_{d_f}\}$ satisfying (2).

This ends the proof. □

3. Irreducibility of components of polynomial mappings. In this section we prove a theorem on irreducibility of components of polynomial mappings (Theorem 3). It is a generalization of the Kaliman result [**4**] from the two-dimensional case of locally diffeomorphic mappings to the multidimensional case of open mappings.

Let us start with a proposition and two lemmas. This proposition is a version of the Bertini Theorem (cf. [**3**], Theorem 6.6).

For positive integers k, m we shall denote by $\mathbb{M}^{(k,m)}$ the set of all complex matrices $\alpha = [\alpha_{i,j}]_{\substack{i=1,\ldots,k \\ j=1,\ldots,m}}$ with k rows and m columns. Since $\mathbb{M}^{(k,m)}$ can be identified with $\mathbb{C}^{km}$, then $\mathbb{M}^{(k,m)}$ is an algebraic variety. For any $\alpha \in \mathbb{M}^{(k,m)}$, we shall also denote by α the linear mapping of $\mathbb{C}^m$ into $\mathbb{C}^k$ defined by α. If $k = m$ and α is invertible, then α is called a *linear change of coordinates in* $\mathbb{C}^k$.

PROPOSITION 3. *Let X be an irreducible algebraic variety over $\mathbb{C}$ and $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$, $m \geq 2$, be a regular mapping. Let $0 < k < \dim f(X)$.*

Then the mapping $\tau : X \times \mathbb{M}^{(k,m)} \to \mathbb{C}^k \times \mathbb{M}^{(k,m)}$ *of the form*

$$\tau(z,\alpha) = (\alpha(f(z)), \alpha)$$

is primitive.

Proof. By Corollary 1, it suffices to prove this proposition in the case $k = \dim f(X) - 1$. Let $Y \subset \mathbb{C}^m$ be the Zariski closure of $f(X)$. It is known that there exists a dense subset $W \subset \mathbb{M}^{(k+1,m)}$ such that for any $\beta \in W$, $\beta|_Y : Y \to \mathbb{C}^{k+1}$ is proper. Moreover, for any $\beta \in W$ there exists a dense subset $U_\beta \subset \mathbb{M}^{(k,k+1)}$ such that for any $\eta \in U_\beta$, $\eta : \mathbb{C}^{k+1} \to \mathbb{C}^k$ restricted to $E_{\beta \circ f}$ is proper. Hence

$$U = \{\alpha \in \mathbb{M}^{(k,m)} : \exists \beta \in W, \exists \eta \in U_\beta,\ \alpha = \eta \circ \beta\}$$

is dense in $\mathbb{M}^{(k,m)}$. By Theorem 2, for any $\alpha \in U$ the mapping $\alpha \circ f$ is primitive. Thus

$$\{(\xi,\alpha) \in \mathbb{C}^k \times \mathbb{M}^{(k,m)} : \tau^{-1}(\xi,\alpha) \text{ is irreducible}\}$$

is dense in $\mathbb{C}^k \times \mathbb{M}^{(k,m)}$. Hence, by Lemma 1(b), for the generic $(\xi,\alpha) \in \mathbb{C}^k \times \mathbb{M}^{(k,m)}$, the fibre $\tau^{-1}(\xi,\alpha)$ is irreducible. Thus the mapping τ is primitive. $\square$

In the proof of Theorem 3 we will need a lemma on a family of algebraic sets.

We say that an algebraic set $V \subset \mathbb{C}^k$ is *in general position* if for any $s \in \{1,...,k\}$ the projection

$$V \ni (\xi_1,...,\xi_k) \mapsto (\xi_1,...,\xi_{s-1},\xi_{s+1},...,\xi_k) \in \mathbb{C}^{k-1}$$

is proper.

Take any invertible $\eta \in \mathbb{M}^{(k,k)}$. Put $L_\eta : \mathbb{C}^k \times \mathbb{M}^{(k,m)} \to \mathbb{C}^k \times \mathbb{M}^{(k,m)}$,

$$L_\eta(\xi,\alpha) = (\eta(\xi), \eta\alpha),$$

where $\eta\alpha$ denotes the multiplication of matrices. Obviously L_η is a linear automorphism of $\mathbb{C}^k \times \mathbb{M}^{(k,m)}$.

Lemma 2. *Let* $V \subsetneq \mathbb{C}^k \times \mathbb{M}^{(k,m)}$ *be an algebraic set. If for any invertible* $\eta \in \mathbb{M}^{(k,k)}$ *there is*

$$L_\eta(V) = V,$$

then for the generic $\alpha \in \mathbb{M}^{(k,m)}$,

$$V_\alpha = \{\xi \in \mathbb{C}^k : (\xi,\alpha) \in V\}$$

is in general position.

Proof. Let $\mathbb{P}^k$ be the k-dimensional projective space over $\mathbb{C}$, and $H_\infty = \mathbb{P}^k \setminus \mathbb{C}^k$ – the hyperplane at infinity. Denote by $P_s \in H_\infty$, $s = 1, ..., k$ the points at infinity of the coordinate axes $X_s = \{(z_1, ..., z_k) \in \mathbb{C}^k : z_1 = \ldots = z_{s-1} = z_{s+1} = \ldots = z_k = 0\}$. It is easy to see that, if V_a and X_s have no common points at infinity for $s = 1, ..., k$, then V_α is in general position. Thus, it suffices to prove that for the generic $\alpha \in \mathbb{M}^{(k,m)}$, $P_s \notin \overline{(V_\alpha)}$ for $s = 1, ..., k$, where $\overline{(V_\alpha)} \subset \mathbb{P}^k$ denotes the closure of V_α.

Let $\overline{V}$ be the closure of V in $\mathbb{P}^k(\mathbb{C}) \times \mathbb{M}^{(k,m)}$. Take an invertible $\eta \in \mathbb{M}^{(k,k)}$. Let $\tilde{\eta} : \mathbb{P}^k \to \mathbb{P}^k$ be the canonical extension of η, i.e.

$$\tilde{\eta}(z_0 : ... : z_k) = (z_0 : \eta(z_1, ..., z_k)),$$

and $\tilde{L}_\eta : \mathbb{P}^k \times \mathbb{M}^{(k,m)} \to \mathbb{P}^k \times \mathbb{M}^{(k,m)}$ be the automorphism of $\mathbb{P}^k \times \mathbb{M}^{(k,m)}$ generated by L_η, i.e.

$$\tilde{L}_\eta(P, \alpha) = (\tilde{\eta}(P), \eta\alpha).$$

By the assumption, $\tilde{L}_\eta(\overline{V}) = \overline{V}$. Since $V \neq \mathbb{C}^k \times \mathbb{M}^{(k,m)}$, then

$$V_\infty = \overline{V} \cap (H_\infty \times \mathbb{M}^{(k,m)}) \subsetneq (H_\infty \times \mathbb{M}^{(k,m)}). \tag{4}$$

Moreover,

$$\tilde{L}_\eta(V_\infty) = V_\infty. \tag{5}$$

Let $V_{\infty,\alpha} = V_\infty \cap (H_\infty \times \{\alpha\})$. Observe that

$$W_s = \{\alpha \in \mathbb{M}^{(k,m)} : \ (P_s, \alpha) \in V_{\infty,\alpha}\}, \qquad s = 1, ..., k$$

are proper algebraic subsets of $\mathbb{M}^{(k,m)}$. Indeed, take $s \in \{1, ..., k\}$. Obviously W_s is an algebraic set. By (4), there exists $\alpha^0 \in \mathbb{M}^{(k,m)}$ such that $V_{\infty,\alpha^0} \neq H_\infty \times \{\alpha^0\}$. Thus there exists a point $Q \in H_\infty$ such that $(Q, \alpha^0) \notin V_{\infty,\alpha^0}$. Moreover, there exists an invertible $\eta \in \mathbb{M}^{(k,k)}$ such that $\tilde{\eta}(Q) = P_s$. Hence and from (5),

$$(P_s, \eta\alpha^0) = \tilde{L}_\eta(Q, \alpha^0) \notin \tilde{L}_\eta(V_{\infty,\alpha^0}) = V_{\infty,\eta\alpha^0}.$$

Thus $\eta\alpha^0 \notin W_s$, i.e. W_s is a proper algebraic subset of $\mathbb{M}^{(k,m)}$.

Since, for any $\alpha \in \mathbb{M}^{(k,m)}$, $(\overline{(V_\alpha)} \cap H_\infty) \times \{\alpha\} \subset V_{\infty,\alpha}$, then from the above there follows that for $\alpha \in \mathbb{M}^{(k,m)} \setminus (W_1 \cup ... \cup W_s)$, $P_s \notin \overline{(V_\alpha)}$ for $s = 1, ..., k$. This ends the proof. $\square$

Lemma 3. *Let $f : \mathbb{C}^n \to \mathbb{C}^m$ be an open polynomial mapping. If f is totally primitive, then the all components of f are totally primitive, too.*

Proof. Let $f = (f_1, ..., f_m) : \mathbb{C}^n \to \mathbb{C}^m$. For the simplicity of notations we prove that f_1 is totally primitive. Fix $t \in \mathbb{C}$. Since $\Gamma_t = f_1^{-1}(t) = f^{-1}(\{t\} \times \mathbb{C}^{m-1})$, then, by the Remmert Open Mapping Theorem, $f|_{\Gamma_t} : \Gamma_t \to \{t\} \times \mathbb{C}^{m-1}$ is an open mapping. So, it is dominating on each irreducible component of Γ_t.

By the assumptions, for each $\xi \in \mathbb{C}^{m-1}$, $f^{-1}(t,\xi)$ is irreducible. Hence, for the generic $\xi \in \mathbb{C}^{m-1}$, $f^{-1}(t,\xi)$ is contained in each irreducible component of Γ_t. This implies that the intersection of the components of Γ_t has dimension $\dim \Gamma_t$. Thus Γ_t is irreducible. □

THEOREM 3. *Let $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq 3$, be an open polynomial mapping. For the generic linear change of coordinates $\alpha : \mathbb{C}^m \to \mathbb{C}^m$, all components of $\alpha \circ f$ are totally primitive.*

PROOF. Define mappings $\kappa : \mathbb{C}^n \times \mathbb{M}^{(m,m)} \to \mathbb{C}^m \times \mathbb{M}^{(m,m)}$, $\tau : \mathbb{C}^n \times \mathbb{M}^{(m-1,m)} \to \mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}$, by

$$\kappa(z,\alpha) = (\alpha(f(z)),\alpha), \qquad \tau(z,\beta) = (\beta(f(z)),\beta),$$

and projections $\pi_s : \mathbb{C}^m \to \mathbb{C}^{m-1}$, $\Pi_s : \mathbb{M}^{(m,m)} \to \mathbb{M}^{(m-1,m)}$, $s = 1,...,m$,

$$\pi_s(\xi_1,...,\xi_m,) = (\xi_1,...,\xi_{s-1},\xi_{s+1},...,\xi_m,),$$
$$\Pi_s([\alpha_{i,j}]_{i,j=1,...,m}) = [\alpha_{i,j}]_{\substack{i=1,...,s-1,s+1,...,m\\ j=1,...,m}}.$$

Denote by id the identity mapping on $\mathbb{C}^n$. Then the diagram

$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{M}^{(m,m)} & \xrightarrow{\kappa} & \mathbb{C}^m \times \mathbb{M}^{(m,m)} \\ {\scriptstyle (\mathrm{id},\Pi_s)}\big\downarrow & & {\scriptstyle (\pi_s,\Pi_s)}\big\downarrow \\ \mathbb{C}^n \times \mathbb{M}^{(m-1,m)} & \xrightarrow{\tau} & \mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)} \end{array} \tag{6}$$

is commutative for $s = 1,...,m$. By the assumption, $\dim f(\mathbb{C}^n) = m$. Thus, by Proposition 3, τ is a primitive mapping. Let $V \subset \mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}$ be the minimal algebraic set, such that for any $(\xi,\beta) \in (\mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}) \setminus V$, the fibre $\tau^{-1}(\xi,\beta)$ is irreducible. Observe that V satisfies the assumptions of Lemma 2. Indeed, by primitivity of τ, V is a proper algebraic subset of $\mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}$. Take any invertible $\eta \in \mathbb{M}^{(m-1,m-1)}$. Let us observe that

$$L_\eta(V) = V.$$

Indeed, take any $(\xi,\beta) \in (\mathbb{C}^{m-1} \times \mathbb{M}^{(m-1,m)}) \setminus L_\eta(V)$. Then $(\xi,\beta) = L_\eta(\xi^1,\beta^1)$ and $(\xi^1,\beta^1) \notin V$. Therefore,

$$\begin{aligned}\tau^{-1}(\xi,\beta) =& \{z \in \mathbb{C}^n : \beta(f(z)) = \xi\} \times \{\beta\} \\ =& \{z \in \mathbb{C}^n : \eta \circ \beta^1(f(z)) = \eta(\xi^1)\} \times \{\beta\} \\ =& \{z \in \mathbb{C}^n : \beta^1(f(z)) = \xi^1\} \times \{\beta\}\end{aligned}$$

is irreducible, because

$$\tau^{-1}(\xi^1,\beta^1) = \{z \in \mathbb{C}^n : \beta^1(f(z)) = \xi^1\} \times \{\beta^1\}$$

is irreducible. Hence, by definition of V, $V \subset L_\eta(V)$. Since L_η is an automorphism, then $L_\eta(V) = V$. So, by Lemma 2, there exists a Zariski open subset

$U \subset \mathbb{M}^{(m-1,m)}$ such that for each $\beta \in U$, the set $V_\beta = \{\xi \in \mathbb{C}^{m-1} : (\xi, \beta) \in V\}$ is in general position in $\mathbb{C}^{m-1}$.

From the above, the set $W = \bigcap_{s=1}^{m} \Pi_s^{-1}(U)$ is a nonempty Zariski open subset of $\mathbb{M}^{(m,m)}$. Thus the set D of all invertible $\alpha \in W$ is also a nonempty Zariski open subset of $\mathbb{M}^{(m,m)}$. Fix any $\alpha \in D$ and let

$$\alpha \circ f = (\tilde{f}_1, ..., \tilde{f}_m).$$

Take any $i \in \{1, ..., m\}$. Since $m \geq 3$, then there exist $j_1, j_2 \in \{1, ..., m\}$, $j_1 \neq j_2$ such that $i \in \{1, ..., m\} \setminus \{j_1, j_2\}$. Without loss of generality we may assume that $i \in \{1, ..., m-2\}$. To prove that $\tilde{f}_i$ is totally primitive, by Lemma 3, it suffices to show that the mapping

$$(\tilde{f}_1, ..., \tilde{f}_{m-2}) : \mathbb{C}^n \to \mathbb{C}^{m-2}$$

is totally primitive. Take $\xi \in \mathbb{C}^{m-2}$. Then

$$\Gamma_\xi = (\tilde{f}_1, ..., \tilde{f}_{m-2})^{-1}(\xi) = (\tilde{f}_1, ..., \tilde{f}_{m-1})^{-1}(\{\xi\} \times \mathbb{C}).$$

By the Remmert Open Mapping Theorem,

$$(\tilde{f}_1, ..., \tilde{f}_{m-1})|_{\Gamma_\xi} : \Gamma_\xi \to \{\xi\} \times \mathbb{C}$$

is an open mapping. Thus, it is dominating on each irreducible component of Γ_ξ. By the choice of α, $\beta = \Pi_m(\alpha) \in U$. By the definition of U, V_β is in general position, thus, for the generic $t \in \mathbb{C}$, $(\xi, t, \beta) \notin V$. Moreover, by (6), for any $z \in \mathbb{C}^n$,

$$((\tilde{f}_1, ..., \tilde{f}_{m-1})(z), \beta) = (\pi_m, \Pi_m) \circ \kappa(z, \alpha) = \tau(z, \beta).$$

Thus, for the generic $t \in \mathbb{C}$,

$$(\tilde{f}_1, ..., \tilde{f}_{m-1})^{-1}(\xi, t) \times \{\beta\} = \tau^{-1}(\xi, t, \beta)$$

is irreducible. Hence, for the generic $t \in \mathbb{C}$, the fibre $(\tilde{f}_1, ..., \tilde{f}_{m-1})^{-1}(\xi, t)$ is irreducible, and, consequently, is contained in each irreducible component of Γ_ξ. This implies that the intersection of the components of Γ_ξ has dimension $\dim \Gamma_\xi$. Thus Γ_ξ is irreducible. This proves Theorem 3. $\square$

REMARKS. 1. The above theorem does not hold for $m = 2$. A simple example is the mapping: $f : \mathbb{C}^2 \to \mathbb{C}^2$, $f(x_1, x_2) = (x_1^2, x_2^2)$.

2. The assumption of openness of f in the theorem is essential and cannot be replaced by the weaker one that f is dominating. An example: $f : \mathbb{C}^n \to \mathbb{C}^n$, $f(x_1, ..., x_n) = (x_1 \ldots x_n,\, x_2 \ldots x_n, \ldots, x_{n-1}x_n,\, x_n)$.

3. From the proof it follows that for any $1 \leq k \leq m-2$ and $\xi \in \mathbb{C}^k$ the fibre $(\tilde{f}_{i_1}, \ldots, \tilde{f}_{i_k})^{-1}(\xi)$, $1 \leq i_1 < \ldots < i_k \leq m$, is irreducible. It is not true for $k = m-1$ as shown by the example: $f : \mathbb{C}^m \to \mathbb{C}^m$, $f(x_1, ..., x_m) = (x_1^2, ..., x_m^2)$.

4. Theorem 3 really is a generalization of the Kaliman Theorem, because every polynomial mapping with non-zero constant jacobian is open.

Corollary 2. *Let* $f : \mathbb{C}^n \to \mathbb{C}^m$, $m \geq 3$, *be an open polynomial mapping. For the generic linear change of coordinates* $\alpha : \mathbb{C}^m \to \mathbb{C}^m$, *where* $\alpha \in \mathbb{M}^{(m,m)}$, *and for any component* $\tilde{f}_j$ *of the mapping* $\alpha \circ f$,

$$\tilde{f}_j - t$$

is an irreducible polynomial in $\mathbb{C}[x_1, ..., x_n]$ *for any* $t \in \mathbb{C}$.

Proof. By Theorem 3, for the generic $\alpha \in \mathbb{M}^{(m,m)}$, any component of $\alpha \circ f$ is totally primitive. Fix such α. Suppose to the contrary that there exist a component $\tilde{f}_j$ of $\alpha \circ f$ and $t_0 \in \mathbb{C}$ such that the polynomial $\tilde{f}_j - t_0$ is reducible. Since $\tilde{f}^{-1}(t_0)$ is an irreducible algebraic set and $\tilde{f} - t_0$ is a reducible polynomial, then there exist a polynomial $g \in \mathbb{C}[x_1, ..., x_n]$ and $k > 1$ such that $f_j - t_0 = g^k$. Thus, for any $t \neq 0$, $f_j - t_0 - t = (g - \varepsilon_1) \dots (g - \varepsilon_k)$, where ε_i, $i = 1, ..., k$ are all k-th roots of t. Thus $f_j^{-1}(t_0 + t)$ is a reducible algebraic set for $t \neq 0$. This is impossible. $\square$

Corollary 3. *Let* $f \in \mathbb{C}[x_1, ..., x_n]$, $n \geq 3$. *If* f *is monic with respect to* x_1, *then for the generic* $(\alpha, \beta) \in \mathbb{C}^2$,

$$f + \alpha x_2 + \beta x_3 - t$$

is an irreducible polynomial for any $t \in \mathbb{C}$.

Proof. If $\deg f = 0$, then the assertion is obvious. Let $\deg f > 0$. Since f is monic with respect to x_1, then for any $(t_1, t_2, t_3) \in \mathbb{C}^3$, the set

$$\{z = (z_1, ..., z_n) \in \mathbb{C}^n : f(z) = t_1,\ z_2 = t_2,\ z_3 = t_3\}$$

has dimension $n - 3$. Thus, by the Remmert Open Mapping Theorem, $(f, x_2, x_3)\colon \mathbb{C}^n \to \mathbb{C}^3$ is an open mapping. Hence, by Theorem 3, the assertion follows. $\square$

Analogously to the above we obtain

Corollary 4. *Let* $f \in \mathbb{C}[x_1, ..., x_n]$, $n \geq 3$ *be a nonconstant polynomial. If* f *is monic with respect to* x_1, *and* $g \in \mathbb{C}[x_2]$, $h \in \mathbb{C}[x_3]$ *are nonconstant polynomials, then for the generic* $(\alpha, \beta) \in \mathbb{C}^2$,

$$f + \alpha g + \beta h - t$$

is an irreducible polynomial for any $t \in \mathbb{C}$. $\square$

References

1. Grothendieck A., *Éléments de Géométrie Algébrique III. Étude Cohomologique des Faisceaux Cohérents (Première partie),* Publ. Math. IHES **11**, 1961.
2. Iitaka S., *Algebraic Geometry,* Graduate Text in Mathematics **76**, Springer–Verlag, New York–Heidelberg–Berlin, 1982.

3. Jouanolou J.P., *Théorèmes de Bertini et Applications,* Progress in Mathematics **42**, Birkhäuser, Boston–Basel–Stuttgart, 1983.
4. Kaliman S., *On the jacobian conjecture,* Proc. Amer. Math. Soc. **117** (1993), 45–51.
5. Łojasiewicz S., *Introduction to Complex Analytic Geometry,* Birkhäuser, Basel–Boston–Berlin, 1991.
6. Płoski A., *On the irreducibility of polynomials in several complex variables,* Bull. Polish Acad. Sci. Math. **39** (1991), 241–247.
7. Schinzel A., *Selected topics on polynomials,* The University of Michigan Press, Ann Arbor, 1982.
8. Shafarevich I.R., *Basic Algebraic Geometry,* Nauka, Moskwa, 1988 (in Russian).
9. Stein Y., *The total reducibility order of a polynomial in two variables,* Israel J. Math. **68** (1989), 109–122.
10. Verdier J-L., *Stratifications de Whitney et théoréme de Bertini–Sard,* Invent. Math. **36** (1976), 295–312.
11. Zariski O., Samuel P., *Commutative Algebra,* Princeton, New Jersey, 1958.

Received March 26, 2001

University of Łódź
Faculty of Mathematics
Banacha 22
90-238 Łódź
Poland
e-mail: krasinsk@krysia.uni.lodz.pl
e-mail: spodziej@imul.uni.lodz.pl