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ON THE LOCAL DEGREE OF PLANE ANALYTIC MAPPINGS

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Abstract. We give an axiomatic characterization of the local degree of real analytic mappings of $\mathbb{R}^2$.

1. Introduction. Let $(f_1, f_2) : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ be a real analytic mapping of a neighbourhood of zero in $\mathbb{R}^2$, which has an isolated zero i.e. such that $(f_1(x, y), f_2(x, y)) \neq (0, 0)$ for small $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. The local degree $\deg_0(f_1, f_2)$ is defined to be the topological degree of the mapping

$$S_\epsilon \ni (x, y) \to \frac{(f_1(x, y), f_2(x, y))}{\|(f_1(x, y), f_2(x, y))\|} \in S_1,$$

where S_ϵ is a circle centered at $(0, 0)$ with a small radius ϵ and S_1 is a unit circle. The notion of the local degree is a real counterpart of the multiplicity of complex holomorphic mappings, see **[3]**, Appendix B. It is well-known that the multiplicity can be characterized by axioms (see **[1]**). In **[2]** the authors have proved

Theorem. *Let us associate with each pair (f, g) of complex holomorphic functions with isolated zero a natural number $\mu(f, g)$ which satisfies the following conditions*

(i) $\mu(f, g) = \mu(g, f)$,
(ii) $\mu(fg, h) = \mu(f, h) + \mu(g, h)$,
(iii) $\mu(f + ag, g) = \mu(f, g)$, *for any holomorphic* a,
(iv) $\mu(X, Y) = 1$,

Then μ=*multiplicity.*

One can show that conditions *(iii)* and *(iv)* of the above theorem are valid for the local degree, i.e. $\deg_0(f + ag, g) = \deg_0(f, g)$, for any analytic a, and $\deg_0(X, Y) = 1$, but it is not difficult to see that the local degree does not satisfy properties *(i)* and *(ii)*. We will show that the local degree satisfies $\deg_0(f,g) = -\deg_0(g,f)$ and

$$\deg_0(fg,h) = \deg_0(f,gh) + \deg_0(g,fh).$$

We assume here that each of the pairs (f,g), (g,h), (f,h) has an isolated zero.

In the next section, an axiomatic characterization of the local degree will be given.

2. Main result. To present the main result we put

$$(f_1,f_2)\cdot(g_1,g_2) = (f_1g_1 - f_2g_2, f_1g_2 + f_2g_1).$$

Throughout the paper we also assume that each mapping has an isolated zero. The following theorem holds

THEOREM. *Let us associate with each pair* (f,g) *of plane analytic functions an integer* $\mathrm{I}_0(f,g)$ *satisfying the following conditions:*

(1) $\mathrm{I}_0((f_1,f_2)\cdot(g_1,g_2)) = \mathrm{I}_0(f_1,f_2) + \mathrm{I}_0(g_1,g_2)$,

(2) $\mathrm{I}_0(f+ag,g) = \mathrm{I}_0(f,g)$, *for any analytic* a,

(3) $\mathrm{I}_0(f, g\sum_{i=1}^k h_i^2) = \mathrm{I}_0(f,g)$, *for analytic* $h_i,\ i=1,\ldots,k$,

(4) $\mathrm{I}_0(X,Y) = 1$.

Then $\mathrm{I}_0 = \deg_0$.

We will prove the above theorem in the next section. Now let us note a very useful

COROLLARY. *If* I_0 *satisfies properties (1)–(3) of the theorem then*

(a) $\mathrm{I}_0(f,g) = -\mathrm{I}_0(g,f)$,

(b) $\mathrm{I}_0(f,-g) = -\mathrm{I}_0(f,g)$,

(c) $\mathrm{I}_0(f,gh) + \mathrm{I}_0(g,hf) + \mathrm{I}_0(h,fg) = 0$.

PROOF. From property *(1)* we get that $\mathrm{I}_0(1,0) + \mathrm{I}_0(1,0) = \mathrm{I}_0(1,0)$, so $\mathrm{I}_0(1,0) = 0$. Similarly we check that $\mathrm{I}_0(-1,0) = 0$. Thus we have

$$\mathrm{I}_0(0,1) + \mathrm{I}_0(0,1) = \mathrm{I}_0((0,1)\cdot(0,1)) = \mathrm{I}_0(-1,0) = 0,$$

hence $\mathrm{I}_0(0,1) = 0$. Using above observations and properties *(1)* and *(3)* of the theorem we get

$$\mathrm{I}_0(f,g) + \mathrm{I}_0(g,f) = \mathrm{I}_0(0, f^2+g^2) = \mathrm{I}_0(0,1) = 0,$$

so *(a)* follows.

The proof of *(b)* is similar. Using *(1)*, *(3)* and *(a)* we have

$$\mathrm{I}_0(f,-g) + \mathrm{I}_0(f,g) = \mathrm{I}_0(f^2+g^2,0) = -\mathrm{I}_0(0,f^2+g^2) = \mathrm{I}_0(0,1) = 0.$$

To prove *(c)* let us notice, that

$$\begin{aligned}&\mathrm{I}_0(f,gh)+\mathrm{I}_0(g,hf)+\mathrm{I}_0(h,fg)=\\&\quad=\mathrm{I}_0(fg-fgh^2,h(f^2+g^2))+\mathrm{I}_0(h,fg)=\\&\quad=\mathrm{I}_0(fg-fgh^2,h)+\mathrm{I}_0(h,fg)=\mathrm{I}_0(fg,h)+\mathrm{I}_0(h,fg),\end{aligned}$$

and using *(a)* we are done. □

We end this section with a simple application of the main theorem and corollary. We will compute the local degree of the mapping

$$(X,Y)\to(X^2-Y,XY-Y^3)$$

$$\begin{aligned}&\deg_0(X^2-Y,XY-Y^3)=\deg_0(X^2-Y,XY-Y^3-Y^2(X^2-Y))=\\&\quad=\deg_0(X^2-Y,XY-X^2Y^2)=\deg_0(X^2-Y,XY(1-XY))=\\&\quad=\deg_0(X^2-Y,XY)=\\&\quad=-\deg_0(X,Y(X^2-Y))-\deg_0(Y,(X^2-Y)X)=\\&\quad=-\deg_0(X,-Y^2)-\deg_0(Y,X^3)=\deg_0(X,Y)=1.\end{aligned}$$

3. Proof of the main theorem. In the first step we prove that the local degree satisfies conditions *(1)–(4)*, where I_0 is replaced by $\deg_0$. For that purpose we will without proof recall the argument principle. If F is an analytic mapping of the neighbourhood of zero with isolated zero in $\mathbb{R}^2$, then for sufficiently small $\epsilon>0$ there exists a differentiable function (the argument function) $\phi_F:[0,2\pi]\to\mathbb{R}$ such that

$$\frac{F(\epsilon\cos t,\epsilon\sin t)}{\|F(\epsilon\cos t,\epsilon\sin t\|}=(\cos\phi_F(t),\sin\phi_F(t)).$$

With the above symbols the following formula holds

Theorem. **(Argument principle)**

$$\deg_0 F=\frac{1}{2\pi}(\phi_F(2\pi)-\phi_F(0)).$$

Let us denote $F=(f_1,f_2)$ and $G=(g_1,g_2)$. Using the argument principle we easily check that

$$\deg_0(F\cdot G)=\deg_0 F+\deg_0 G.$$

In fact, if ϕ_F and ϕ_G are the argument functions of the mappings F and G, respectively, then $\phi_F + \phi_G$ is the argument function of $F \cdot G$. Hence

$$\begin{aligned}\deg_0(F \cdot G) &= \frac{1}{2\pi}\left[(\phi_F + \phi_G)(2\pi) - (\phi_F + \phi_G)(0)\right] = \\ &= \frac{1}{2\pi}\left[\phi_F(2\pi) - \phi_F(0)\right] + \frac{1}{2\pi}\left[\phi_G(2\pi) - \phi_G(0)\right] = \\ &= \deg_0 F + \deg_0 G.\end{aligned}$$

Equalities *(2)* and *(3)* for the local degree can be easily obtained from the following

Lemma. *Let U be a sufficiently small neighbourhood of zero. If* $\operatorname{sgn} f(x,y) = \operatorname{sgn} \tilde{f}(x,y)$ *on the set* $\{(x,y) \in U : g(x,y) = 0\}$*, then*

$$\deg_0(f,g) = \deg_0(\tilde{f},g).$$

Proof of the lemma. The mapping

$$S_\epsilon \times [0,1] \ni (x,y,t) \to \frac{(tf(x,y) + (1-t)\tilde{f}(x,y), g(x,y))}{\|(tf(x,y) + (1-t)\tilde{f}(x,y), g(x,y))\|} \in S_1$$

is a smooth homotopy between the mappings $\frac{(f,g)}{\|(f,g)\|}\big|_{S_\epsilon}$ and $\frac{(\tilde{f},g)}{\|(\tilde{f},g)\|}\big|_{S_\epsilon}$. In fact, it suffices to show that

$$(tf(x,y) + (1-t)\tilde{f}(x,y), g(x,y)) \neq (0,0)$$

for $(x,y) \in S_\epsilon$ and $t \in [0,1]$. Let us assume that there exist $(x_0,y_0) \in S_\epsilon$ and $t_0 \in [0,1]$ such that $g(x_0,y_0) = 0$. Since $f(x_0,y_0) \neq 0$ and $\tilde{f}(x_0,y_0) \neq 0$ have the same sign and $t_0 \geqslant 0$, we have

$$\operatorname{sgn}(t_0 f(x_0,y_0) + (1-t_0)\tilde{f}(x_0,y_0)) = \operatorname{sgn} f(x_0,y_0) \neq 0.$$

The lemma follows because homotopic mappings have the same topological degree.

Property *(2)* for the local degree follows by taking $\tilde{f} = f + ag$ in the above lemma.

To prove *(3)* we use the fact that

$$\deg_0(f, g\sum_{i=1}^{k} h_i^2) = -\deg_0(g\sum_{i=1}^{k} h_i^2, f)$$

and *(3)* follows by applying the lemma again.

Property *(4)* can be checked by using the argument principle. The argument function of the identity mapping is $\phi(t) = t$, thus $\deg_0(X,Y) = 1$.

Now we are ready to show that $\deg_0(f,g) = \mathrm{I}_0(f,g)$ for any pair of analytic functions with isolated zero. There exist natural numbers k, l such that

$f = X^k\tilde{f}$ and $g = X^l\tilde{g}$ where $\tilde{f}, \tilde{g}$ are analytic functions for which $\tilde{f}(0, Y) \not\equiv 0$ and $\tilde{g}(0, Y) \not\equiv 0$. We apply induction with respect to

$$\min\{\operatorname{ord} \tilde{f}(0, Y), \tilde{g}(0, Y)\}.$$

In the first step let us see that if $\min\{\operatorname{ord} \tilde{f}(0, Y), \tilde{g}(0, Y)\} = 0$, then either $\tilde{f}(0, 0) \neq 0$ or $\tilde{g}(0, 0) \neq 0$. Let us assume the second inequality. By *(b)* and *(3)* we have

$$\mathrm{I}_0(f, X^l\tilde{g}) = \operatorname{sgn} \tilde{g}(0, 0)\mathrm{I}_0\left(f, X^l \sqrt{\operatorname{sgn} \tilde{g}(0, 0)\tilde{g}}^2\right) = \operatorname{sgn} \tilde{g}(0, 0)\mathrm{I}_0(f, X^l),$$

so if l is even then according to *(3)* and *(2)* we get

$$\mathrm{I}_0(f, X^l) = \mathrm{I}_0(f, 1) = \mathrm{I}_0(0, 1) = 0.$$

If l is odd, then

$$\mathrm{I}_0(f, X^l) = \mathrm{I}_0(f, X) = \mathrm{I}_0(f(0, Y), X).$$

Now let us write $f(0, Y) = \sum_{i=p}^{\infty} f_i Y^i$, $f_p \neq 0$. Then

$$\mathrm{I}_0(f(0, Y), X) = -\mathrm{I}_0(X, f_p Y^p \sum_{i=p}^{\infty} \frac{f_i}{f_p} Y^{i-p}) = -\mathrm{I}_0(X, f_p Y^p) =$$

$$= \begin{cases} -\mathrm{I}_0(X, f_p Y) & \text{if } p \text{ is odd,} \\ -\mathrm{I}_0(X, f_p) & \text{otherwise} \end{cases} = \begin{cases} -\operatorname{sgn} f_p & \text{if } p \text{ is odd,} \\ 0 & \text{otherwise} \end{cases}.$$

The same equality holds for the local degree $\deg_0$. Hence we have proved that $\deg_0(X^k\tilde{f}, X^l\tilde{g}) = \mathrm{I}_0(X^k\tilde{f}, X^l\tilde{g})$ provided $\tilde{g}(0, 0) \neq 0$. Because of the antysymmetry property, $\mathrm{I}_0(f, g) = -\mathrm{I}_0(g, f)$, the same is valid in the case $\tilde{f}(0, 0) \neq 0$.

In the next induction step let us denote $m = \operatorname{ord} \tilde{g}(0, Y)$ and let $0 < m \leqslant \operatorname{ord} \tilde{f}(0, Y)$. By induction we assume that for all mappings (f', g'), such that $\min\{\operatorname{ord} \tilde{f}'(0, Y), \operatorname{ord} \tilde{g}'(0, Y)\} < m$ the equality $\mathrm{I}_0(f', g') = \deg_0(f', g')$ holds. We will show that $\mathrm{I}_0(f, g) = \deg_0(f, g)$.

From property *(c)* of the corollary we have

$$\mathrm{I}_0(f, X^l\tilde{g}) + \mathrm{I}_0(X^l, \tilde{g}f) + \mathrm{I}_0(\tilde{g}, fX^l) = 0$$

hence

$$\mathrm{I}_0(f, g) = \mathrm{I}_0(\tilde{g}f, X^l) + \mathrm{I}_0(fX^l, \tilde{g}). \qquad (*)$$

From the previously proved part we have that $\mathrm{I}_0(\tilde{g}f, X^l) = \deg_0(\tilde{g}f, X^l)$, hence it suffices to show that $\mathrm{I}_0(fX^l, \tilde{g}) = \deg_0(fX^l, \tilde{g})$. By virtue of the Weierstrass preparation theorem there exist analytic functions a, r such that $fX^l = a\tilde{g} + r$. Moreover, r is a polynomial in variable Y of degree less than m. So by applying property *(2)* we get

$$\mathrm{I}_0(fX^l, \tilde{g}) = \mathrm{I}_0(a\tilde{g} + r, \tilde{g}) = \mathrm{I}_0(r, \tilde{g}).$$

It is easy to see, that if $r = X^s\tilde{r}$ with $\tilde{r}(0,Y) \not\equiv 0$ then

$$\min\{\operatorname{ord}\tilde{r}(0,Y), \operatorname{ord}\tilde{g}(0,Y)\} \leqslant \operatorname{ord}\tilde{r}(0,Y) \leqslant \deg_Y r \leqslant m-1.$$

From our inductive assumption we have the equality $\mathrm{I}_0(r,\tilde{g}) = \deg_0(r,\tilde{g})$ and from property *(2)* we get $\mathrm{I}_0(fX^l,\tilde{g}) = \deg_0(fX^l,\tilde{g})$. Now from equality (*) we have

$$\mathrm{I}_0(f,g) = \deg_0(\tilde{g}f, X^l) + \deg_0(fX^l,\tilde{g})$$

and applying *(c)* with I_0 replaced by the local degree $\deg_0$ we get the equality $\mathrm{I}_0(f,g) = \deg_0(f,g)$. □

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