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NEW REDUCTION IN THE JACOBIAN CONJECTURE

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Dedicated to Professor Tadeusz Winiarski
on the occasion of his 60th birthday

Abstract. It is sufficient to consider in the Jacobian Conjecture (for every $n > 1$) only polynomial mappings of cubic linear form $F(x) = x + (Ax)^{*3}$, i. e. $F(x) = (x_1 + (a_1^1 x_1 + ... + a_n^1 x_n)^3, ..., x_n + (a_1^n x_1 + ... + a_n^n x_n)^3)$ where the matrix $F'(x) - I = 3\,\Delta((Ax)^{*2})A$ is nilpotent for every $x = (x_1, ..., x_n)$. In the paper we give a new contributions to the Jacobian Conjecture, namely we show that it is sufficient in this problem to consider (for every $n > 1$) only cubic linear mappings $F(x) = x + (Ax)^{*3}$ such that $A^2 = 0$.

1. Introduction and notation. Let $\mathbb{K}$ denote either the field of complex numbers $\mathbb{K}$ or the field of reals $\mathbb{R}$. Basis in the domain and codomain vector spaces $\mathbb{K}^n$ are assumed to be fixed and identical, so a linear mapping A from $\mathbb{K}^n$ into $\mathbb{K}^n$ is identified with its matrix and denoted by the same letter A (I denotes the identity matrix). Let M_n denote the set of $n \times n$ square matrices with entries in $\mathbb{K}$. A vector $x \in \mathbb{K}^n$ is treated as one column matrix and x^T denotes its transpose, i. e. $x^T = (x_1, ..., x_n) \in \mathbb{K}^n$. Let $a_j, b_j, c_j : \mathbb{K}^n \to \mathbb{K}$ be linear forms and let the symbol $a_j x$ (resp. $b_j x, c_j x$) denote the value of the linear form a_j (resp. b_j, c_j) at a point $x \in \mathbb{K}^n$, i. e. $a_j x = a_j^1 x_1 + ... a_j^n x_n$, $j = 1, ..., n$. Denote for short the square matrix $A := [a_i^j : i, j = 1, ..., n]$ and the vector $(Ax)^T := (a_1 x, ..., a_n x)$, i.e. Ax is one column matrix. If $v = (v_1, ..., v_n)^T$ is a column vector, then we denote the k power of v by $v^{*k} := ((v_1)^k, ..., (v_n)^k)^T$ and by $\Delta(v^{*k})$ we denote the diagonal $n \times n$ matrix

$$\Delta(v^{*k}) := \begin{bmatrix} (v_1)^k & 0 & 0 & ... & 0 & 0 \\ 0 & (v_2)^k & 0 & ... & 0 & 0 \\ .... & & & & & \\ 0 & 0 & ... & 0 & (v_{n-1})^k & 0 \\ 0 & 0 & ... & 0 & 0 & (v_n)^k \end{bmatrix}.$$

If $F = (F_1, ..., F_n) : \mathbb{K}^n \to \mathbb{K}^n$ is a polynomial mapping, then we denote $\mathrm{Jac}\, F(x) := \det\, [\frac{\partial F_i}{\partial x_j}(x) : i, j = 1, ..., n]$. Let a polynomial mapping $F = (F_1, ..., F_n)$ have a cubic linear form $F(x) = x + (Ax)^{*3}$ that is $F_j(x) = x_j + (a_j x)^3$, $x = (x_1, ..., x_n) \in \mathbb{K}^n$, $j = 1, ..., n$.

We recall that the n-dimensional Jacobian Conjecture $(JC)_n$ $(n > 1)$ asserts

$(JC)_n$ If F is any polynomial mapping of $\mathbb{K}^n$ and $\mathrm{Jac}\, F(x) = \mathrm{const} \neq 0$, then F is injective.

By the Jacobian Conjecture (for short (JC)) we mean that $(JC)_n$ holds for each $n > 1$.

If F is injective polynomial transformation of $\mathbb{C}^n$, then F is a polynomial automorphism, cf. [**1, 8**]. Therefore the Jacobian Conjecture is sometimes formulated with the requirement that F has to be a polynomial automorphism. We have the following reduction theorem.

Theorem 1. [**2**] *In order to verify the Jacobian Conjecture (for every $n > 1$) it is sufficient to check the Jacobian Conjecture (for every $n > 1$) only for polynomial mappings $F = (F_1, ..., F_n)$ of a cubic linear form*

$$F(x) = x + (Ax)^{*3}, \quad \textit{i. e. } F_j(x) = x_j + (a_j x)^3, \quad j = 1, ..., n.$$

It is known ([**1, 2**]) that $\mathrm{Jac}\, F = 1$ if and only if the matrix $A_x := [(a_j x)^2 a^i_j : i, j = 1, ..., n] = \Delta((Ax)^{*2})\, A$ is nilpotent for every $x \in \mathbb{K}^n$. Some interesting applications of Th.1 to the Jacobian Conjecture can be found in [**4, 5, 7**]. Note that

$$F(x) = x + A_x(x) = x + \Delta((Ax)^{*2})\,(Ax)$$
$$F'(x) = I + 3A_x = I + 3\Delta((Ax)^{*2})A,$$

and call A the matrix of the cubic linear mapping F. Hence, for every $x \in \mathbb{K}^n$ there exists an index of nilpotency of the matrix A_x, i.e. a number $p(x) \in \mathbb{N}$ such that ${A_x}^{p(x)} = 0$ and ${A_x}^{p(x)-1} \neq 0$. We define the index of nilpotency of the mapping F to be the number $\mathrm{ind}\, F := \sup\, \{p(x) \in \mathbb{N} : x \in \mathbb{K}^n\}$. Obviously $\mathrm{ind}\, F \leq n$.

2. We will prove the following.

Theorem 2. (new reduction theorem) *In order to verify the Jacobian Conjecture (for every $n > 1$) it is sufficient to check the Jacobian Conjecture (for every $n > 1$) only for polynomial mappings $F = (F_1, ..., F_n)$ of the cubic linear form*

$$F_j(x) = x_j + (a_j x)^3, \qquad j = 1, ..., n,$$

having an additional nilpotent property of the matrix $A := [a^j_i : i, j = 1, ..., n]$, namely $A^2 = 0$.

PROOF. Due to Th.1 we can take $F : \mathbb{K}^n \to \mathbb{K}^n$ of the form $F(x) = x + (Ax)^{*3}$, $x \in \mathbb{K}^n$. Evidently F is a polynomial automorphism if and only if $x + \delta(Ax)^{*3}$ is a polynomial automorphism for every (some) $\delta \in \mathbb{K} \setminus \{0\}$. Put $\widehat{F}(x,y) := (x + \delta(Ax)^{*3}, y)$, $\delta \neq 0$, $(x,y) \in \mathbb{K}^n \times \mathbb{K}^n$. Obviously F is a polynomial automorphism of $\mathbb{K}^n$ if and only if $\widehat{F} : \mathbb{K}^{2n} \to \mathbb{K}^{2n}$ is an automorphism of $\mathbb{K}^{2n}$. We define polynomial automorphisms of $\mathbb{K}^{2n}$ by the formulas:

$$Q(x,y) := \big(\alpha x - \beta y, y + (\alpha Ax - \beta Ay)^{*3}\big) \quad \text{where } \alpha\beta \neq 0,$$

and

$$P(x,y) := \left(\tfrac{1}{\alpha}x + \tfrac{\beta}{\alpha}y, y\right) \quad \text{where } \alpha\beta \neq 0.$$

Put $G := P \circ \widehat{F} \circ Q : \mathbb{K}^{2n} \to \mathbb{K}^{2n}$. It not difficult to verify that

$$G(x,y) = \left(x + \tfrac{(\delta+\beta)\alpha^2}{\beta^3}(\beta Ax - \tfrac{\beta^2}{\alpha}y)^{*3}, y + (\alpha Ax - \beta y)^{*3}\right).$$

The mapping F is a polynomial automorphism if and only if G is a polynomial automorphism. Now we choose $\alpha \neq 0$, $\beta \neq 0$ such that $\frac{(\delta+\beta)\alpha^2}{\beta^3} = 1$ (it is always possible if $\frac{\alpha^2}{\beta^2} \neq 1$). Hence we get

$$G(x,y) = \left(x + (\beta Ax - \tfrac{\beta^2}{\alpha}y)^{*3}, y + (\alpha Ax - \beta y)^{*3}\right).$$

Denote by N a block matrix (with entries in M_n) of the form

$$N := \begin{pmatrix} \beta A & -\frac{\beta^2}{\alpha}A \\ \alpha A & -\beta A \end{pmatrix}.$$

Observe that we can write $G(w) = w + (Nw)^{*3}$, $w \in \mathbb{K}^{2n}$. It is easy to check that $N^2 = 0$. Therefore the theorem is proved. $\square$

REMARK 1. Since $A^2 = 0$, rank $A \leq \frac{n}{2}$.
In the example given in [**3**, Ex. 7.8], and also investigated in [**6**, Ex. 6.1], the matrix A of an automorphism $F(x) = x + (Ax)^{*3} : \mathbb{K}^{15} \to \mathbb{K}^{15}$ has the form

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&-1&1&1&1&0&0&-1&0&0&-1&0\\
0&0&-1&0&-1&0&\frac{1}{2}&0&0&\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{2}&0&0\\
0&0&1&-2&0&0&0&1&-1&-1&-1&0&0&0&1\\
1&0&1&-2&0&0&0&1&-1&-1&-1&0&0&0&1\\
0&1&1&-2&0&0&0&1&-1&-1&-1&0&0&0&1\\
1&0&-1&0&-1&0&\frac{1}{2}&0&0&\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{2}&0&0\\
1&0&0&-2&-1&1&1&1&0&0&-1&0&0&-1&0\\
0&1&0&-2&-1&1&1&1&0&0&-1&0&0&-1&0\\
1&0&1&0&1&0&-\frac{1}{2}&0&0&-\frac{1}{2}&0&\frac{1}{2}&\frac{1}{2}&0&0\\
0&1&-1&2&0&0&0&-1&1&1&1&0&0&0&-1\\
0&1&0&2&1&-1&-1&-1&0&0&1&0&0&1&0\\
1&1&1&-2&0&0&0&1&-1&-1&-1&0&0&0&1\\
1&1&0&-2&-1&1&1&1&0&0&-1&0&0&-1&0
\end{pmatrix}$$

It is easy to check that ind $A = 2$, rank $A = 5$ and ind $F = 5$.

REMARK 2. It was proved earlier ([**2**]) that in Th.1 we can additionally assume that $(*)$ the matrix $A = A_c$ for some point $c \in \mathbb{K}^n$ and ind $A =$ ind F. If we investigated the Jacobian Conjecture for cubic linear assuming ind $A = 2$, then the property $(*)$ usually does not hold (cf. the mentioned above example where ind $A = 2 < 5 =$ ind F).

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