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THE INITIAL-BOUNDARY VALUE PROBLEM FOR SOME PSEUDOPARABOLIC SYSTEM IN UNBOUNDED DOMAIN

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Abstract. In the present paper the initial-boundary value problem for the system of pseudoparabolic equations in unbounded (in the respect to space variables) domain is considered under the assumption that the right-hand side and initial function of a given system have a rate of non-exponential growth. We prove that correctness classes for this problem do not depend on coefficients of the system.

Different problems for pseudoparabolic equations and their systems were investigated in [**1**]–[**8**]. For example, W. Rundell [**5**] showed that the uniqueness of the solution of the Cauchy problem for such equations hold only for functions with grow as $e^{\alpha|x|}$ and a constant α depends on the coefficients of the equation. In this case it is assumed that the right-hand side of the equation and initial functions have a rate of growth at most such as the specified exponent. Hilkevych [**6, 7**] obtained the same results.

Let Ω be an unbounded domain in $\mathbf{R}^n$ and Γ – its boundary; $Q_T = \Omega \times (0,T)$, $T < \infty$. Let there exists a sequence of bounded subdomains $\{\Omega^\tau\}$ of the domain Ω which has the following properties:

1) $\Omega = \bigcup\limits_{\tau\in\mathbf{N}} \Omega^\tau; \quad \tau \le \tau' \Rightarrow \Omega^\tau \subset \Omega^{\tau'}$;

2) $\partial\Omega^\tau = \Gamma_1^\tau \cup \Gamma_2^\tau$, where $\Gamma_1^\tau, \Gamma_2^\tau$ – are piece-wise smooth hypersurfaces; $\mathrm{mes}\{\Gamma_1^\tau \cap \Gamma_2^\tau\} = 0$, $\Gamma_1^\tau \ne \emptyset$, $\Gamma_1^\tau \cap \Gamma \ne \emptyset$, $\forall \tau \in \mathbf{N}$; $\quad \Gamma = \bigcup\limits_{\tau\in\mathbf{N}} \Gamma_1^\tau$.

Let us introduce functional spaces that we will use in the sequel. By $L^2_{\mathrm{loc}}(\Omega)$, we denote the space of functions belonging to $L^2(\Omega^\tau)$ for every $\tau \in \mathbf{N}$; $L^2_{\mathrm{loc}}(Q_T) := L^2\left((0,T); L^2_{\mathrm{loc}}(\Omega)\right)$.

Let $h(x)$, $d(x,t)$, $\psi(x)$ be positive functions, $h \in C(\Omega)$, $d \in C(Q_T)$, $\psi \in C^1(\Omega)$. By $V_\psi(Q_T)$ and $W_\psi(Q_T)$ we denote the closure of $C^\infty\left([0,T]; C_0^\infty(\Omega)\right)$

(the set of infinitely differentiable functions) in the norms

$$\|u\|_{V_\psi(Q_T)} = \left(\int\limits_{Q_T} \left[h(x)|u_t|^2 + d(x,t)|u|^2 + \sum_{i=1}^{n} \left(|u_{x_i}|^2 + |u_{x_it}|^2 \right) \right] \psi(x)dxdt \right)^{\frac{1}{2}}$$

and

$$\|u\|_{W_\psi(Q_T)} = \left(\int\limits_{Q_T} |u|^2 \psi(x)dxdt \right)^{\frac{1}{2}}$$

respectively and by $U_\psi(\Omega)$ – the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{U_\psi(\Omega)} = \left(\int\limits_{\Omega} \left[(h(x) + d(x,0))|u|^2 + \sum_{i=1}^{n} |u_{x_i}|^2 \right] \psi(x)dx \right)^{\frac{1}{2}}.$$

The problem

$$H(x)u_t - \sum_{i,j=1}^{n} (A_{ij}(x)u_{x_it})_{x_j} - \sum_{i,j=1}^{n} (B_{ij}(x,t)u_{x_i})_{x_j} -$$

$$- \sum_{i=1}^{n} C_i(x,t)u_{x_i} + D(x,t)u = F(x,t), \tag{1}$$

$$u|_{\Gamma\times[0,T]} = 0, \tag{2}$$

$$u|_{t=0} = u_0(x) \tag{3}$$

is considered in Q_T. Here A_{ij}, B_{ij}, C_i, D, H are square $m \times m$ matrices; $u = (u_1, ..., u_m)^t$, $F = (F_1, ..., F_m)^t$; $(\cdot,\cdot)$ is a scalar product in $\mathbf{R}^m$; $|\cdot|$ is a norm in $\mathbf{R}^m$.

Definition. By a solution of the problem (1) – (3) we mean such a function $u \in V_\psi(Q_T)$ which satisfies the integral equality

$$\int\limits_{Q_T} \left[(H(x)u_t, v) + \sum_{i,j=1}^{n} \left(A_{ij}(x)u_{x_it}, v_{x_j} \right) + \sum_{i,j=1}^{n} \left(B_{ij}(x,t)u_{x_i}, v_{x_j} \right) - \right.$$

$$\left. - \sum_{i=1}^{n} (C_i(x,t)u_{x_i}, v) + (D(x,t)u, v) \right] dxdt = \int\limits_{Q_T} (F(x,t), v)\, dxdt \tag{4}$$

for every function $v \in C_0^\infty(Q_T)$ and fulfil the condition (3) almost everywhere in Ω.

We say that the coefficients of the system (1) fulfil the conditions (A), (B), (D), (H) if:

$$
\begin{aligned}
(A): \quad & a\sum_{i=1}^{n}|\xi^i|^2 \le \sum_{i,j=1}^{n}\left(A_{ij}(x)\xi^i,\xi^j\right), \quad a>0, \quad \forall x\in\Omega;\\
& A_{ij}(x)=A_{ji}(x),\ A_{ij}(x)=A^t_{ij}(x),\ \forall x\in\Omega,\ \forall i,j\in\{1,...,n\};\\
& A_{ij}\in L^\infty(\Omega),\ \forall i,j\in\{1,...,n\};\\
(B): \quad & b\sum_{i=1}^{n}|\xi^i|^2 \le \sum_{i,j=1}^{n}\left(B_{ij}(x,t)\xi^i,\xi^j\right), \quad b>0, \quad \forall (x,t)\in Q_T;\\
& B_{ij}(x,t)=B_{ji}(x,t),\ B_{ij}(x,t)=B^t_{ij}(x,t),\ \forall (x,t)\in Q_T,\\
& \forall i,j\in\{1,...,n\};\\
& B_{ij}\in L^\infty(Q_T),\ B_{ijt}\in L^\infty(Q_T), \quad \forall i,j\in\{1,...,n\};\\
(D): \quad & d(x,t)|\xi|^2\le (D(x,t)\xi,\xi)\le \theta d(x,t)|\xi|^2, \quad \forall (x,t)\in Q_T;\\
& D\in L^\infty_{\mathrm{loc}}(Q_T),\ D_t\in L^\infty(Q_T);\\
(H): \quad & h(x)|\xi|^2\le (H(x)\xi,\xi)\le \theta h(x)|\xi|^2, \quad \forall x\in\Omega; \quad H\in L^\infty_{\mathrm{loc}}(\Omega);
\end{aligned}
$$

for all vectors ξ, ξ^i, ξ^j in $\mathbf{R}^m$, $1\le i,j\le n$; $\theta>1$.

For the sake of simplicity, let us set

$$
\begin{aligned}
\hat{A} &= \sup_{\Omega}\sum_{i,j=1}^{n}\|A_{ij}(x)\|^2; \quad \hat{B}=\sup_{Q_T}\sum_{i,j=1}^{n}\|B_{ij}(x,t)\|^2;\\
\hat{C} &= \sup_{Q_T}\sum_{i=1}^{n}\|C_i(x,t)\|^2; \qquad \omega_c=\hat{C}\sup_{Q_T}\left(\frac{1}{h(x)}+\frac{1}{d(x,t)}\right).
\end{aligned}
$$

THEOREM 1. *Let the coefficients of the system* (1) *satisfy conditions* (A), (B), (D), (H), $C_i\in L^\infty(Q_T)$, $i=1,\dots,n$ *and let there exists a positive function* $\psi\in C^1(\overline{\Omega})$ *such that for every* $x\in\Omega$

$$
\sum_{i=1}^{n}\frac{\psi^2_{x_i}(x)}{\psi^2(x)}\le \gamma\min\left\{h(x),\inf_{[0,T]}d(x,t)\right\}, \quad \gamma>0. \tag{5}
$$

Let also $\hat{A}<\frac{a}{4n^2\gamma}$, $\omega_c<\infty$. *Then the problem* (1) – (3) *has no more than one solution.*

PROOF. Let u_1, u_2 be solutions of the problem (1) – (3). For each of them we write the integral equality (4), deduct these equalities and put $u=u_1-u_2$,

$v = (u_t + u)\psi(x)e^{-\mu t}$, $\mu > 0$. Using the assumptions, we estimate

$$
\begin{aligned}
I_1 &= \int\limits_{Q_T} \left(H(x)u_t + D(x,t)u, u_t + u\right)\psi(x)e^{-\mu t}dxdt \geq \\
&\geq \int\limits_{Q_T} \left[h(x)|u_t|^2 + \left(d(x,t) - \frac{d^1(x,t)}{2} + \frac{\mu}{2}(h(x) + d(x,t))\right)|u|^2\right] \\
&\quad \psi(x)e^{-\mu t}dxdt; \\
I_2 &= \int\limits_{Q_T} \sum_{i,j=1}^{n} \left(A_{ij}(x)u_{x_i t} + B_{ij}(x,t)u_{x_i}, ((u_t + u)\psi(x))_{x_j}\right) e^{-\mu t}dxdt \geq \\
&\geq \int\limits_{Q_T} \left[\left(a - \frac{n\hat{A}}{\delta_1}\right)\sum_{i=1}^{n}|u_{x_i t}|^2 + \left(b - \frac{b^1}{2} + \frac{\mu}{2}(a+b) - \frac{n\hat{B}}{\delta_1}\right) \times \right. \\
&\times \left. \sum_{i=1}^{n}|u_{x_i}|^2 - n\delta_1 \sum_{i=1}^{n}\frac{\psi_{x_i}^2(x)}{\psi^2(x)}\left(|u_t|^2 + |u|^2\right)\right]\psi(x)e^{-\mu t}dxdt; \\
I_3 &= \int\limits_{Q_T} \sum_{i=1}^{n}\left(C_i(x,t)u_{x_i}, u_t + u\right)\psi(x)e^{-\mu t}dxdt \leq \\
&\leq \int\limits_{Q_T} \left[\frac{\hat{C}}{2\delta_2}\left(\frac{1}{h(x)} + \frac{1}{d(x,t)}\right)\sum_{i=1}^{n}|u_{x_i}|^2 + \frac{n\delta_2}{2}\left(h(x)|u_t|^2 + d(x,t)|u|^2\right)\right] \\
&\quad \psi(x)e^{-\mu t}dxdt
\end{aligned}
$$

and obtain

$$
\begin{aligned}
&\int\limits_{Q_T} \left[\left(a - \frac{n\hat{A}}{\delta_1}\right)\sum_{i=1}^{n}|u_{x_i t}|^2 + \right. \\
&+ \left(b - \frac{b^1}{2} + \frac{\mu}{2}(a+b) - \frac{n\hat{B}}{\delta_1} - \frac{\hat{C}}{2\delta_2}\left(\frac{1}{h(x)} + \frac{1}{d(x,t)}\right)\right)\sum_{i=1}^{n}|u_{x_i}|^2 + \\
&+ \left(d(x,t) - \frac{d^1(x,t)}{2} + \frac{\mu}{2}(h(x) + d(x,t)) - n\delta_1\sum_{i=1}^{n}\frac{\psi_{x_i}^2(x)}{\psi^2(x)} - \right. \\
&- \left.\left.\frac{n\delta_2}{2}d(x,t)\right)|u|^2 + \left(h(x) - n\delta_1\sum_{i=1}^{n}\frac{\psi_{x_i}^2(x)}{\psi^2(x)} - \frac{n\delta_2}{2}h(x)\right)|u_t|^2\right] \\
&\psi(x)e^{-\mu t}dxdt \leq 0, \qquad (6)
\end{aligned}
$$

where coefficients b^1 and $d^1(x,t)$ depend on $\|B_{ijt}(x,t)\|$ and $\|D_t(x,t)\|$, respectively. Number δ_1 is chosen so that

$$2n\frac{\hat{A}}{a} < \delta_1 < \frac{1}{2n\gamma}.$$

Let μ and δ_2 be such that

$$\begin{aligned} 1 - 2n\delta_1\gamma - n\delta_2 &\geq 0, \\ d(x,t) - d^1(x,t) + \mu\,(h(x) + d(x,t)) - d(x,t)\,(2n\delta_1\gamma + n\delta_2) &\geq 0, \\ b - b^1 + \mu(a+b) - \frac{2n\hat{B}}{\delta_1} - \frac{\omega_c}{2\delta_2} &\geq 0. \end{aligned}$$

Then (6) implies $\|u\|_{V_\psi(Q_T)} \leq 0$, i.e. $u = 0$ almost everywhere. The theorem is proved. □

THEOREM 2. *Let the coefficients of the system* (1) *fulfil all the assumptions of Theorem 1,* $\left(\frac{1}{h} + \frac{1}{d}\right) F \in W_\psi(Q_T)$, $u_0 \in U_\psi(\Omega)$. *Then the problem* (1) – (3) *has at least one solution.*

PROOF. Let us consider the problem

$$\begin{aligned} H(x)u_t \quad &- \sum_{i,j=1}^{n} (A_{ij}(x)u_{x_i t})_{x_j} - \sum_{i,j=1}^{n} (B_{ij}(x,t)u_{x_i})_{x_j} - \\ &- \sum_{i=1}^{n} C_i(x,t)u_{x_i} + D(x,t)u = F^*(x,t), \end{aligned} \tag{7}$$

$$u|_{\Gamma^* \times [0,T]} = 0, \tag{8}$$

$$u|_{t=0} = u_0^*(x), \tag{9}$$

in the domain $Q_T^* = \Omega^* \times (0,T)$ where $\Omega^* \in \{\Omega^\tau\}$, Γ^* is a boundary of Ω^*. Here

$$F^*(x,t) = \begin{cases} F(x,t), & (x,t) \in Q_T^*, \\ 0, & (x,t) \in Q_T \setminus Q_T^*; \end{cases} \qquad u_0^*(x) = \begin{cases} u_0(x), & x \in \Omega^*, \\ 0, & x \in \Omega \setminus \Omega^*. \end{cases}$$

By the solution of the problem (7) – (9) we mean a function $u^* \in H^1\left((0,T); H_0^1(\Omega^*)\right)$ that satisfies the integral equality

$$\int\limits_{Q_T^*} \left[(H(x)u_t, v) + \sum_{i,j=1}^{n} \left(A_{ij}(x)u_{x_i t}, v_{x_j}\right) + \sum_{i,j=1}^{n} \left(B_{ij}(x,t)u_{x_i}, v_{x_j}\right) - \right.$$
$$\left. - \sum_{i=1}^{n} (C_i(x,t)u_{x_i}, v) + (D(x,t)u, v) \right] dxdt = \int\limits_{Q_T^*} (F^*(x,t), v)\, dxdt,$$

for every function $v \in C^\infty([0,T]; C_0^\infty(\Omega^*))$ and fulfils the condition (9) almost everywhere in Ω^*. We shall approximate a solution of (7) – (9) using the Galerkin method. Let $\{\varphi^{*,k}(x)\}$ be a basis of $H_0^1(\Omega^*)$. We orthogonalize this system with respect to the scalar product

$$(u,v) = \int\limits_{\Omega^*} \left[(H(x)u, v) + \sum_{i,j=1}^{n} \left(A_{ij}(x) u_{x_i}, v_{x_j} \right) \right] dx$$

and put $u^{*,N} = \sum\limits_{k=1}^{N} c_k^N(t) \varphi^{*,k}(x)$ where $c_k^N(t)$, $k = 1, ..., N$ may be found from the system of equations

$$\begin{aligned} &\int\limits_{\Omega^*} \Bigg[\Big(H(x) u_t^{*,N}, \varphi^{*,k}(x) \Big) + \sum_{i,j=1}^{n} \Big(A_{ij}(x) u_{x_i t}^{*,N}, \varphi_{x_j}^{*,k}(x) \Big) + \\ &+ \sum_{i,j=1}^{n} \Big(B_{ij}(x,t) u_{x_i}^{*,N}, \varphi_{x_j}^{*,k}(x) \Big) - \sum_{i=1}^{n} \Big(C_i(x,t) u_{x_i}^{*,N}, \varphi^{*,k}(x) \Big) + \\ &+ \Big(D(x,t) u^{*,N}, \varphi^{*,k}(x) \Big) \Bigg] dx = \int\limits_{\Omega^*} \Big[\Big(F^*(x,t), \varphi^{*,k}(x) \Big) \Big] dx, \qquad k = \overline{1,N}, \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\sum_{s=1}^{N} \left(c_s^N(t) \right)' \Bigg\{ \int\limits_{\Omega^*} \Big[\Big(H(x) \varphi^{*,s}(x), \varphi^{*,k}(x) \Big) + \\ &+ \sum_{i,j=1}^{n} \Big(A_{ij}(x) \varphi_{x_i}^{*,s}(x), \varphi_{x_j}^{*,k}(x) \Big) \Big] dx \Bigg\} = \Phi\left(c_1^N(t), ... c_N^N(t) \right), \qquad k = \overline{1,N}, \end{aligned}$$

and conditions

$$c_k^N(0) = \int\limits_{\Omega^*} \left[\Big(H(x) u_0^*, \varphi^{*,k}(x) \Big) + \sum_{i,j=1}^{n} \Big(A_{ij}(x) u_{0x_i}^*, \varphi_{x_j}^{*,k}(x) \Big) \right] dx.$$

After multiplying each equation of the system (10) by $\left(c_k^N(t) + \left(c_k^N\right)'(t)\right) e^{-\mu t}$, $\mu > 0$, summing over k and integrating over the interval $[0, T]$, we obtain

$$\begin{aligned}
\int\limits_{Q_T^*} \Big[\Big(H(x)u_t^{*,N}, u_t^{*,N} + u^{*,N}\Big) &+ \sum_{i,j=1}^{n} \Big(A_{ij}(x)u_{x_it}^{*,N}, u_{x_jt}^{*,N} + u_{x_j}^{*,N}\Big) + \\
&+ \sum_{i,j=1}^{n} \Big(B_{ij}(x,t)u_{x_i}^{*,N}, u_{x_jt}^{*,N} + u_{x_j}^{*,N}\Big) - \\
&- \sum_{i=1}^{n} \Big(C_i(x,t)u_{x_i}^{*,N}, u_t^{*,N} + u^{*,N}\Big) + \\
&+ \Big(D(x,t)u^{*,N}, u_t^{*,N} + u^{*,N}\Big)\Big] e^{-\mu t} dxdt = \\
&= \int\limits_{Q_T^*} \Big(F^*(x,t), u_t^{*,N} + u^{*,N}\Big) e^{-\mu t} dxdt. \qquad (11)
\end{aligned}$$

The estimates

$$\begin{aligned}
I_4 &= -\int\limits_{Q_T^*} \sum_{i=1}^{n} \Big(C_i(x,t)u_{x_i}^{*,N}, u_t^{*,N} + u^{*,N}\Big) e^{-\mu t} dxdt \geq \\
&\geq -\int\limits_{Q_T^*} \left[\delta_2 \sum_{i=1}^{n} \left|u_{x_i}^{*,N}\right|^2 + \frac{n\hat{C}}{2\delta_2}\left(\left|u_t^{*,N}\right|^2 + \left|u^{*,N}\right|^2\right)\right] e^{-\mu t} dxdt, \\
I_5 &= \int\limits_{Q_T^*} \Big(F^*(x,t), u_t^{*,N} + u^{*,N}\Big) e^{-\mu t} dxdt \leq \\
&\leq \int\limits_{Q_T^*} \left[\frac{|F^*(x,t)|^2}{\varepsilon} + \frac{\varepsilon}{2}\left(\left|u_t^{*,N}\right|^2 + \left|u^{*,N}\right|^2\right)\right] e^{-\mu t} dxdt
\end{aligned}$$

and equality (11) imply

$$\int\limits_{\Omega^*_T} \left[(a+b) \sum_{i=1}^{n} |u^{*,N}_{x_i}|^2 + (h(x) + d(x,T)) \, |u^{*,N}|^2 \right] \frac{e^{-\mu T}}{2} dx -$$
$$-\frac{1}{2} \int\limits_{\Omega^*_0} \left[(a^0 + b^0) \sum_{i=1}^{n} |u^{*,N}_0{}_{x_i}| \theta \left(h(x) + d(x,0) \right) |u^{*,N}_0| \right] dx +$$
$$+ \int\limits_{Q^*_T} \left[\left(h(x) - \frac{n\hat{C}}{2\delta_2} - \frac{\varepsilon}{2} \right) |u^{*,N}_t|^2 + a \sum_{i=1}^{n} |u^{*,N}_{x_i t}|^2 + \right.$$
$$+ \left(d(x,t) - \frac{d^1(x,t)}{2} + \frac{\mu}{2}(h(x) + d(x,t)) - \frac{n\hat{C}}{2\delta_2} - \frac{\varepsilon}{2} \right) |u^{*,N}|^2 +$$
$$\left. + \left(b - \frac{b^1}{2} + \frac{\mu}{2}(a+b) - \delta_2 \right) \sum_{i=1}^{n} |u^{*,N}_{x_i}|^2 \right] e^{-\mu t} dx dt \le$$
$$\le \frac{1}{\varepsilon} \int\limits_{Q^*_T} |F^*(x,t)|^2 \, e^{-\mu t} dx dt,$$

(coefficients a^0 and b^0 are finite and depend on $\|A_{ij}(x)\|$ and $\|B_{ij}(x,t)\|$, respectively). Last inequality may be read as

$$\left\| u^{*,N} \right\|_{H^1\left((0,T); H^1_0(\Omega^*) \right)} \le M.$$

It means that there are exist a subsequence $\{u^{*,N_k}\}$ and function u^* such that $u^{*,N_k} \to u^*$ weakly in $H^1\left((0,T); H^1(\Omega^*)\right)$. It is easy to see that this function u^* is a solution of the problem (7) – (9).

Let us consider the sequence $Q^\tau_T = \Omega^\tau \times (0,T)$, $\tau \in \mathbf{N}$. In each of this domains, there are exists a solution u^τ, which we extend as zero on Q_T. Then for every function $v \in C^\infty_0(Q_T)$ and for choosing function $\psi(x)$ the following equality holds:

$$\int\limits_{Q_T}\Bigg[(H(x)u_t^\tau, v\psi) + \sum_{i,j=1}^{n}\Big(A_{ij}(x)u_{x_it}^\tau, (v\psi)_{x_j}\Big) +$$

$$+\sum_{i,j=1}^{n}\Big(B_{ij}(x,t)u_{x_i}^\tau, (v\psi)_{x_j}\Big) - \sum_{i=1}^{n}\big(C_i(x,t)u_{x_i}^\tau, v\psi\big) +$$

$$+ (D(x,t)u^\tau, v\psi)\Big] e^{-\mu t}dxdt = \int\limits_{Q_T}(F^\tau(x,t), v\psi)\, e^{-\mu t}dxdt.$$

We put $v = u_t^\tau + u^\tau$ and estimating as above we get

$$\frac{1}{2}\int\limits_{\Omega_T}\left[(a+b)\sum_{i,j=1}^{n}\left|u_{x_i}^\tau\right|^2 + (h(x)+d(x,T))\,|u^\tau|^2\right]\psi(x)e^{-\mu T}dx -$$

$$-\frac{1}{2}\int\limits_{\Omega_0}\left[(a^0+b^0))\sum_{i=1}^{n}|u_{0x_i}^\tau|^2 + \theta(h(x)+d(x,0))|u_0^\tau|^2\right]\psi(x)dx +$$

$$+\int\limits_{Q_T}\Bigg[\left(a - \frac{n\hat{A}}{\delta_3}\right)\sum_{i=1}^{n}|u_{x_it}^\tau|^2 +$$

$$+\left(b - \frac{b^1}{2} + \frac{\mu}{2}(a+b) - \frac{n\hat{B}}{\delta_3} - \frac{\hat{C}}{2\delta_4}\left(\frac{1}{h(x)} + \frac{1}{d(x,t)}\right)\right)\sum_{i=1}^{n}|u_{x_i}^\tau|^2 +$$

$$+\Bigg(d(x,t) - \frac{d^1(x,t)}{2} + \frac{\mu}{2}(h(x)+d(x,t)) - n\delta_3\sum_{i=1}^{n}\frac{\psi_{x_i}^2(x)}{\psi^2(x)} - \frac{n\delta_4 d(x,t)}{2} -$$

$$-\frac{\varepsilon d(x,t)}{2}\Bigg)\times|u^\tau|^2 + \left(h(x) - n\delta_3\sum_{i=1}^{n}\frac{\psi_{x_i}^2(x)}{\psi^2(x)} - \frac{n\delta_4}{2}h(x) - \frac{\varepsilon}{2}h(x)\right)|u_t^\tau|^2\Bigg]$$

$$\psi(x)e^{-\mu t}dxdt \le \frac{1}{2\varepsilon}\int\limits_{Q_T}|F^\tau(x,t)|^2\left(\frac{1}{h(x)} + \frac{1}{d(x,t)}\right)\psi(x)e^{-\mu t}dxdt.$$

Given the assumptions of the theorem for F and u_0 we find that sequence $\{u^\tau\}$ is bounded in the norm of the space $V_\psi(Q_T)$. From this sequence we select a subsequence $\{u^{\tau_k}\}$ that converges weakly to some function u in the space $V_\psi(Q_T)$. The limit function u is a solution of the original problem. The theorem is proved. □

Notation. If the function ψ satisfies the condition

$$\lim_{|x|\to\infty}\sum_{i=1}^{n}\frac{\psi_{x_i}^2(x)}{\psi^2(x)}=\nu=\text{const}$$

then the solution will increase not faster than $e^{\nu|x|}$ when $x\to\infty$. However, this fact is well known (see, e.g. [**5**]).

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