

SOME REMARKS ON THE NON-UNIQUENESS OF THE STATIONARY SOLUTIONS OF NAVIER-STOKES EQUATIONS

BY ELŻBIETA MOTYL

Abstract. We formulate some conditions when non-uniqueness of approximate solutions of the stationary Navier-Stokes equations occurs.

Introduction. In this paper we will be concerned with the stationary Navier-Stokes equations for incompressible liquid in a bounded domain in \mathbb{R}^n ($n \in \{2, 3\}$). Since a long time, it has been known that there exists a weak solution of this problem. For example, general existence results are proved in [8], Section 7 and in [11], Chapter II.1. As far as the problem of uniqueness or non-uniqueness is concerned, the situation is different. There are no general results. It is known that the solution is unique in a rather restrictive case when viscosity is large compared with external forces (see [8], Chapter I, Section 7, [11], Chapter II.1, [12], Part II, Section 9).

On the other hand, there are some specific results that assert non-uniqueness. For example, W. Velte proved a non-uniqueness result for the Taylor problem, i.e. the problem describing the flow of a viscous incompressible liquid in a domain of \mathbb{R}^3 bounded by two infinite cylinders with the same vertical axis (see [13], [14], [11]).

Let us also mention the Quette-Taylor experiment which indicates that if the Reynolds number of the flow increases, then the flow loses its stability and new steady states appear (see [12], Section 9).

There are some abstract results which assert that generically with respect to various parameters of the flow, the number of solutions of the Navier-Stokes problem is finite and odd (see [2], [3], [4], [7], [10]).

In the paper [5] some abstract criterion of non-uniqueness of the equation of the Navier-Stokes type is proved. This criterion is applied to the finite-dimensional Galerkin equations corresponding to the homogeneous boundary value problem for the stationary Navier-Stokes equations.

In this paper, we apply this abstract result concerning non-uniqueness to the Holly numerical method of solving the N-S problem. With references to the paper [7], we consider the approximate equations in a finite-dimensional subspace of the Sobolev space H_0^1 . From the results of this paper, it follows that generically with respect to viscosity and external forces the set of all stationary solutions of the N-S problem and the sets of solutions of appropriate approximate problems are finite and equinumerous. In this way the question of uniqueness or non-uniqueness has been reduced to the finite-dimensional space. Using the aforementioned criterion we give a sufficient condition for non-uniqueness.

1. General properties. Let us consider the stationary Navier-Stokes equations for incompressible fluid with homogeneous boundary conditions, i.e.

$$(1.1) \quad \partial^v v = \nu \Delta v + \tilde{f} - \nabla p,$$

$$(1.2) \quad \operatorname{div} v = 0,$$

$$(1.3) \quad v|_{\partial\Omega} = 0,$$

where Ω is a bounded domain (i.e. it is open and connected) in \mathbb{R}^n ($n = 2, 3$) with the Lipschitz boundary $\partial\Omega$. The number $\nu \in]0, \infty[$ (kinematic viscosity) and $\tilde{f} : \Omega \rightarrow \mathbb{R}^n$ (external forces) are given, while $v : \Omega \rightarrow \mathbb{R}^n$ (velocity) and $p : \Omega \rightarrow \mathbb{R}$ (pressure) are looked for. The symbol $\partial^u w$ stands for the vector field

$$(1.4) \quad \sum_{i=1}^n u_i \frac{\partial w}{\partial x_i} : \Omega \rightarrow \mathbb{R}^n$$

for any differentiable vector fields

$$u = (u_1, \dots, u_n) : \Omega \rightarrow \mathbb{R}^n, \quad w = (w_1, \dots, w_n) : \Omega \rightarrow \mathbb{R}^n$$

We consider the Sobolev space

$$H^1 := \{u \in L^2(\Omega, \mathbb{R}^n) : \text{there exist } \frac{\partial u}{\partial x_i} \text{ in the sense of Sobolev}$$

$$\text{and } \frac{\partial u}{\partial x_i} \in L^2(\Omega, \mathbb{R}^n), \text{ for each } 1 \leq i \leq n\}.$$

It is a Hilbert space with the scalar product

$$(u, w) \mapsto (u|w)_{L^2(\Omega, \mathbb{R}^n)} + ((u|w)),$$

where $((u|w)) := \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \middle| \frac{\partial w}{\partial x_i} \right)_{L^2(\Omega, \mathbb{R}^n)}$ is the Dirichlet scalar product of u and w . Let H_0^1 be the closure in H^1 of the subspace $\mathcal{D} := \mathcal{D}(\Omega, \mathbb{R}^n)$ of all \mathcal{C}^∞ mappings $\phi : \Omega \rightarrow \mathbb{R}^n$ with compact supports contained in Ω . From the Poincaré inequality, it follows that the bilinear form $((\cdot|\cdot))$ is a scalar product in H_0^1 inducing the topology inherited from H^1 .

Finally, $\mathcal{V} := \mathcal{D} \cap \{\operatorname{div} = 0\}$ denotes the space of all divergence-free test vector fields on Ω , V – its closure in the Hilbert space $(H_0^1, ((\cdot|\cdot)))$ and V^\perp – its $((\cdot|\cdot))$ - orthogonal complement in H_0^1 .

J. Leray stated the following

DEFINITION 1.1. Suppose that $n \in \{2, 3, 4\}$, $f \in (H_0^1)'$. A vector field $v \in V$ is a (weak) solution of the problem (1.1)–(1.3) if

$$(1.5) \quad \int_{\Omega} (\partial^v v) \phi dm = -\nu((v|\phi)) + f(\phi).$$

for all $\phi \in V$.

From the Sobolev imbedding theorem, it follows that this definition is well-posed.

With the aid of the so called *acceleration functional*, identity (1.5) can be written as the operator equation in the space H_0^1 . In fact, we have the following

REMARK 1.2. (see Remark 2.8 in [7]). Suppose $\nu \in]0, \infty[$, $f \in (H_0^1)'$, $v \in H_0^1$. Then the following conditions are equivalent

- (i) $v \in V$ and $\int_{\Omega} (\partial^v v) \phi dm = -\nu((v|\phi)) + f(\phi)$ for any $\phi \in V$,
- (ii) $\nu v = P_V \mathcal{R}^{-1}(f - \mathcal{A}_{v,v})$.

The symbol \mathcal{R} stands for *the canonical Riesz isomorphism* in the space $(H_0^1, ((\cdot|\cdot)))$, i.e.

$$\mathcal{R} : H_0^1 \ni u \mapsto (H_0^1 \ni \phi \mapsto ((u|\phi)) \in \mathbb{R}) \in (H_0^1)'$$

The symbol $P_V : H_0^1 \rightarrow V$ stands for the $((\cdot|\cdot))$ - orthogonal projection onto V . Now, the acceleration functional is defined as

$$(1.6) \quad \mathcal{A}_{u,w} : H_0^1 \ni \phi \mapsto \int_{\Omega} (\partial^u w + \frac{\operatorname{div} u}{2} w) \phi dm \in \mathbb{R}$$

for any $u, w \in H_0^1$. We will use the following properties of this functional proved in [7] Lemma 2.7.

LEMMA 1.3. *The functional $\mathcal{A}_{u,w}$ is well-defined linear and continuous. Moreover,*

$$(1.7) \quad \mathcal{A}_{u,w}(\phi) = -\mathcal{A}_{u,\phi}(w) \text{ for any } \phi \in H_0^1;$$

in particular

$$(1.8) \quad \mathcal{A}_{u,w}(w) = 0.$$

By virtue of the Rellich-Kondrashev theorem, we infer that the acceleration functional has the following property

$$(1.9) \quad \begin{aligned} & \text{if } \overset{k}{u} \rightarrow u, \quad \overset{k}{w} \rightarrow w \text{ weakly in } H_0^1, \\ & \text{then } \mathcal{A}_{\overset{k}{u}, \overset{k}{w}} \rightarrow \mathcal{A}_{u,w} \text{ pointwise on } H_0^1 \text{ as } k \rightarrow \infty. \end{aligned}$$

(see (2.23) in [7]).

Let us recall the discretization of the problem (1.1)–(1.3) introduced by K. Holly (see [7], Section 2). The construction of the solution is based on the internal approximation $(H_N)_{N \in \mathbb{N}}$ of the space H_0^1 . The method is split into two steps. In the first step, the Navier-Stokes problem is approximated by the equation

$$((*)_s) \quad \nu u = P_V^s \mathcal{R}^{-1}(f_s - \mathcal{A}_{u, P_V^s u});$$

in the space H_0^1 , where $s \in \mathbb{N}$ is fixed. The second step involves the following discretization

$$((*)_{s,N}) \quad \nu w = P_V^{s,N} \mathcal{R}_N^{-1}(f_s - \mathcal{A}_{w, P_V^{s,N} w})$$

of the equation $(*)_s$ in the space H_N , where $s, N \in \mathbb{N}$

Let us explain the symbols which appear above. K. Holly introduced the following operators

$$\begin{aligned} P_{V^\perp}^s &:= \operatorname{div}^* \circ \left(\sum_{j=0}^s (\operatorname{id} - \operatorname{div} \operatorname{div}^*)^j \right) \circ \operatorname{div} \\ P_{V^\perp}^{s,N} &:= \operatorname{div}_N^* \circ \left(\sum_{j=0}^s (\operatorname{id} - \operatorname{div} \operatorname{div}_N^*)^j \right) \circ \operatorname{div} \circ P_N, \\ P_V^s &:= \operatorname{id} - P_{V^\perp}^s, \quad P_V^{s,N} := \operatorname{id} - P_{V^\perp}^{s,N}. \end{aligned}$$

Here

$$\operatorname{div}^* : \{f = 0\} \rightarrow H_0^1$$

is the adjoint operator to the divergence operator $\operatorname{div} : (H_0^1, ((\cdot|\cdot))) \rightarrow (\{f = 0\}, (\cdot|\cdot)_{L^2(\Omega)})$, where $\{f = 0\} := \{q \in L^2(\Omega) : \int_\Omega q \, dm = 0\}$. Next, $\operatorname{div}_N^* q$ is defined as the $((\cdot|\cdot))$ – Riesz representation of the functional

$$H_N \ni \phi \mapsto \int_\Omega q \operatorname{div} \phi \, dm \in \mathbb{R}$$

for a given $q \in \{f = 0\}$. Similarly, for the functional $l \in (H_0^1)'$, the symbol $\mathcal{R}_N^{-1}(l)$ denotes the $((\cdot|\cdot))$ – Riesz representation of the restriction $l|_{H_N}$ of l to H_N . Finally, $f_s \in (H_0^1)'$ stands for the approximate external forces.

We collect key properties of the operators P_V^s and $P_V^{s,N}$ in the following

REMARK 1.4. (see Remark 2.6 in [7])

- (a) The operators $P_V^s, P_V^{s,N}$ are selfadjoint endomorphisms of the space H_0^1 for any $s, N \in \mathbb{N}$,
- (b) $|P_V^s - P_V|_{\text{End}H_0^1} \leq \frac{1}{\theta}(1 - \theta)^{s+1}$ and $P_V \leq P_V^s \leq \text{id}_{H_0^1}$ for any $s \in \mathbb{N}$,
- (c) for fixed $s \in \mathbb{N} : P_V^{s,N} \rightarrow P_V^s$ pointwise as $N \rightarrow \infty$,
- (d) $\|\text{div } P_V^s u\|_{L^2(\Omega)} \leq \frac{1}{\theta}(1 - \theta)^{s+1} \|\text{div } u\|_{L^2(\Omega)} \leq \frac{1}{\theta}(1 - \theta)^{s+1} \|u\|_{H_0^1}$ for any $u \in H_0^1$, where $\theta \in]0, 1[$ is a constant depending only on Ω .

From Theorem 2.9 in [7], it follows that there exist solutions of the problems $(*)_s, (*)_s, N$. Moreover, the sequence $(\overset{s,N}{v})_{N \in \mathbb{N}}$ of the solutions of the equations $(*)_{s,N}$ contains a subsequence convergent in H_0^1 to a solution of the equation $(*)_s$. Similarly, from the sequence $(\overset{s}{v})_{s \in \mathbb{N}}$ of the solutions of the equations $(*)_s$, we can select a subsequence convergent to a solution of the Navier-Stokes problem. In general, we do not know whether these solutions are unique, thus we consider the sets of solutions. To be more specific

- $\mathfrak{R}(\nu, f)$ – the set of all solutions of $(*)$,
- $\mathfrak{R}^s(\nu, f_s)$ – the set of all solutions of $(*)_s$ for fixed $s \in \mathbb{N}$,
- $\mathfrak{R}^{s,N}(\nu, f_s)$ – the set of all solutions of $(*)_{s,N}$ for fixed $(s, N) \in \mathbb{N}^2$.

In Section 4 of the paper [7], some properties of the above sets are investigated, connected with the problem of stability of the considered method. Let us recall two main results which spurred further investigations concerning the problem of non-uniqueness.

THEOREM 1.5. (see Theorem 4.5 in [7]) If $(\nu, f) \in \mathcal{G}$, then

- (i) $\lim_{s \rightarrow \infty} \mathfrak{R}^s(\nu, f_s) = \mathfrak{R}(\nu, f)$ in the Hausdorff metric;
- (ii) for almost all $s \in \mathbb{N} : \#\mathfrak{R}^s(\nu, f_s) = \#\mathfrak{R}(\nu, f) < \infty$.

Here, \mathcal{G} is some subset of the data: (viscosity, external forces). In fact,

$$\mathcal{G} := \{(\nu, f) \in]0, \infty[\times (H_0^1)' : P_V \mathcal{R}^{-1} f \text{ is a regular value of the mapping } V \ni \phi \mapsto \nu \phi + P_V \mathcal{R}^{-1} \mathcal{A}_{\phi, \phi} \in V\}.$$

In Section 3 of the paper [7], there is proved that the set \mathcal{G} is open and dense in $]0, \infty[\times (H_0^1)'$. A similar result concerning the sequence $(\mathfrak{R}^{s,N}(\nu, f_s))_{N \in \mathbb{N}}$ is expressed in the following theorem

THEOREM 1.6. (see Theorem 4.7 in [7]) If $(\nu, f) \in \mathcal{G}$, then for almost all $s \in \mathbb{N}$:

- (i) $\lim_{N \rightarrow \infty} \mathfrak{R}^{s,N}(\nu, f_s) = \mathfrak{R}^s(\nu, f_s)$ in the Hausdorff metric;
- (ii) for almost all $N \in \mathbb{N} : \#\mathfrak{R}^{s,N}(\nu, f_s) = \#\mathfrak{R}^s(\nu, f_s) < \infty$.

The properties of the sets of solutions stated in (ii) of Theorems 1.5, 1.6 are of fundamental importance for further considerations. They guarantee that for the data $(\nu, f) \in \mathcal{G}$, the sets $\mathfrak{R}(\nu, f)$, $\mathfrak{R}^s(\nu, f_s)$ and $\mathfrak{R}^{s,N}(\nu, f_s)$ are finite and equinumerous. In this way the problem of uniqueness or non-uniqueness of the N-S equations has been reduced to the corresponding problem in the finite-dimensional space H_N for sufficiently large $s, N \in \mathbb{N}$.

In the sequel, we shall concentrate on the equations $(*)_{s,N}$. To be more specific, we shall prove that the criterion for non-uniqueness mentioned in Introduction can be applied to the problems $(*)_{s,N}$.

2. The problem of non-uniqueness. For the reader's convenience we recall the main result of the paper [5].

Let us consider a real separable Hilbert space $(H, (\cdot|\cdot))$ and a homogeneous polynomial $Q : H \rightarrow H$. Assume that Q is *vortex*, i.e.

$$(2.1) \quad (Q(x)|x) = 0, \quad x \in H.$$

Moreover, we assume that

$$(2.2) \quad \begin{aligned} & \text{if } x_k \rightarrow x \text{ weakly in } H, \\ & \text{then } Q(x_k) \rightarrow Q(x) \text{ weakly in } H. \end{aligned}$$

A polynomial Q satisfying the above conditions is called the *nonlinearity of the Navier-Stokes type*.

For a given $c \in H \setminus \{0\}$ we consider the following *equation of the Navier-Stokes type*

$$(2.3) \quad Q(x) + x = c.$$

We say that a half-space $\{x \in H : (x|c) \geq 0\}$ is *attractive* in the field Q if

$$(Q(h)|c) > 0$$

for any $h \in (\mathbb{R} \cdot c)^\perp \setminus \{0\}$ and it is *uniformly attractive* in the field Q if

$$\inf\{(Q(h)|c), \quad h \in (\mathbb{R} \cdot c)^\perp, |h| = 1\} > 0.$$

It is easy to see that if $\dim H < \infty$, then the attractive half-space is uniformly attractive. The main result of the above cited paper is the following

THEOREM 2.1. (see Theorem 1.35 in [5]) *Suppose that $\dim H = 2k$ and the half-space $\{(\cdot|e) \geq 0\}$ is attractive in the field Q ; $k \in \mathbb{N}$, $e \in \{x \in H : |x| = 1\}$. If a number $R \in \mathbb{R}$ is such as*

$$R > \frac{1}{\delta}, \quad |Q(e)| < 2\sqrt{R} \left(\frac{\delta - R^{-1}}{3} \right)^{\frac{3}{2}},$$

where $\delta := \inf\{(Q(\zeta)|e), \quad \zeta \in (\mathbb{R} \cdot e)^\perp, |\zeta| = 1\}$, then the equation (2.3) with the right-hand side $c = Re$ has at least two different solutions.

2.1. *Application to the Navier-Stokes problem.* Let us fix $s, N \in \mathbb{N}$. Let H_N be a finite-dimensional subspace of H_0^1 and $P_H : H \rightarrow H_N$ be the $(\cdot|\cdot)$ -orthogonal projection. Putting $H := H_N$,

$$Q := Q^{s,N} := \frac{1}{\nu} P_V^{s,N} \mathcal{R}_N^{-1} \mathcal{A}^{P_V^{s,N}} : H_N \rightarrow H_N$$

and

$$c_*^{s,N} = \frac{1}{\nu} P_V^{s,N} \mathcal{R}_N^{-1} f_s$$

we can write the equation $(*)_{s,N}$ as

$$(2.4) \quad Q^{s,N}(w) + w = c_*^{s,N}.$$

Here, $\mathcal{A}^{P_V^{s,N}}(w) := \mathcal{A}_{w, P_V^{s,N} w}$ for any $w \in H_N$. Let us remark that directly from the definition of the operator $P_V^{s,N}$, it follows that $Q^{s,N}(H_N) \subset H_N$.

Applying Theorem 2.1, we can formulate the following sufficient condition for non-uniqueness of the solutions of the equation $(*)_{s,N}$.

THEOREM 2.2. *Assume that $\dim H_N = 2k$, ($k \in \mathbb{N}$). Let $e \in H_N$ be such a vector that $\|P_V^{s,N} e\|_{H_0^1} = 1$ and the half-space $\{((\cdot|P_V^{s,N} e)) \geq 0\}$ is attractive in the field $Q^{s,N}$. If a number $R \in \mathbb{R}$ is such as*

$$R > \frac{1}{\delta}, \quad \|Q^{s,N}(P_V^{s,N} e)\|_{H_0^1} < 2\sqrt{R} \left(\frac{\delta - R^{-1}}{3} \right)^{\frac{3}{2}},$$

where $\delta := \inf\{((Q^{s,N}(\zeta)|P_V^{s,N} e)), \zeta \in (\mathbb{R} \cdot P_V^{s,N} e)^\perp, \|\zeta\|_{H_0^1} = 1\}$, then the equation (2.4) with the right-hand side $c_*^{s,N} := R \cdot P_V^{s,N} e$ has at least two different solutions.

PROOF. It is sufficient to check that $Q^{s,N}$ satisfies conditions (2.1), (2.2). Let $w \in H_N$. According to (a) of Remark 1.4, $P_V^{s,N}$ is selfadjoint. By virtue of (1.8), we obtain

$$\begin{aligned} ((Q^{s,N}(w)|w)) &= \frac{1}{\nu} ((P_V^{s,N} \mathcal{R}_N^{-1} \mathcal{A}^{P_V^{s,N}}(w)|w)) \\ &= \frac{1}{\nu} ((\mathcal{R}^{-1} \mathcal{A}_{w, P_V^{s,N} w} | P_N(P_V^{s,N} w))) = \frac{1}{\nu} \mathcal{A}_{w, P_V^{s,N} w}(P_V^{s,N} w) = 0, \end{aligned}$$

Thus, $Q^{s,N}$ is vortex.

Now, let $\overset{k}{w} \rightarrow w$ weakly in H_N . Let $\phi \in H_N$. Then

$$\begin{aligned} ((Q^{s,N}(\overset{k}{w})|\phi)) &= \frac{1}{\nu} \mathcal{A}_{\overset{k}{w}, P_V^{s,N} \overset{k}{w}}(P_V^{s,N} \phi) \rightarrow \frac{1}{\nu} \mathcal{A}_{w, P_V^{s,N} w}(P_V^{s,N} \phi) \\ &= ((Q^{s,N}(w)|\phi)) \text{ as } k \rightarrow \infty, \end{aligned}$$

since by (1.9) and (c) of Remark 1.4, the sequence of operators $(\mathcal{A}_{w, P_V^s, N_w^k})_k$ is pointwise convergent to $\mathcal{A}_{w, P_V^s, N_w}$. Thus, condition (2.2) is also fulfilled, which completes the proof. \square

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Cracow University of Technology
 Institute of Mathematics
 Poland