UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXXIX
2001

ON THE DARBOUX EQUATION

by Zbigniew Błocki

Abstract. Given a Riemannian surface M with a metric tensor g, we compute the Gauss curvature of a metric $g - du \otimes du$, where u is a smooth function on M.

Introduction. In his celebrated monograph [**2**], Darboux, among other things, considered the problem of embedding abstract Riemannian surfaces in $\mathbb{R}^3$. If g is a metric tensor on M then one looks for three functions u, v, w on M, such that

$$g = du \otimes du + dv \otimes dv + dw \otimes dw.$$

Locally, two of them, say v and w, must satisfy

$$\widetilde{g} := dv \otimes dv + dw \otimes dw > 0.$$

For the Gauss curvature $\widetilde{K}$ of the new metric $\widetilde{g}$ we thus have

$$\widetilde{K} = 0, \tag{0.1}$$

which is in fact an equation just for the first component u, known as the Darboux equation. Tedious calculations (see e.g. [**3**]) show that this equation, is, in modern terms, equivalent to

$$M(u) = K(1 - |\nabla u|^2), \tag{0.2}$$

where M is the Monge-Ampère operator and K the Gauss curvature (with respect to the original metric g).

The aim of this note is to give the precise formula for $\widetilde{K}$, which will in particular show that (0.1) and (0.2) are equivalent. Namely we shall prove the following result.

2000 *Mathematics Subject Classification.* 53C42.

Supported by KBN Grant 7 T07A 003 16.

THEOREM. *Suppose that M is a Riemannian manifold with metric tensor g and $\dim M = 2$. For $u \in C^\infty(M)$ let*

$$\widetilde{g} = g - du \otimes du.$$

Then $\widetilde{g} > 0$ if and only if $|\nabla u| < 1$. In such a case the Gauss curvature with respect to $\widetilde{g}$ is given by

$$\widetilde{K} = \frac{K(1-|\nabla u|^2) - M(u)}{(1-|\nabla u|^2)^2}. \tag{0.3}$$

1. Preliminaries. Here we collect the basic definitions and some formulas, which we will use in the proof of the theorem. For details we refer for example to [**1**]. Let M be a Riemannian manifold with the metric tensor $g = \langle \cdot, \cdot \rangle$. The tensor g induces the unique symmetric, metric connection ∇ on M. This means that ∇ satisfies

$$\nabla_X Y = \nabla_Y X + [X,Y], \quad X, Y \in \mathcal{X}(M) \tag{1.1}$$

and $\nabla g = 0$, that is

$$X\langle Y, Z\rangle - \langle \nabla_X Y, Z\rangle - \langle Y, \nabla_X Z\rangle = 0, \quad X, Y, Z \in \mathcal{X}(M). \tag{1.2}$$

(1.1) and (1.2) are equivalent to

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ &+ \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle, \quad X, Y, Z \in \mathcal{X}(M). \end{aligned} \tag{1.3}$$

In particular, we may compute

$$\langle \nabla_X X, Y\rangle = \tfrac{1}{2} Y|X|^2 = \langle \nabla_Y X, X\rangle, \quad X, Y \in \mathcal{X}(M). \tag{1.4}$$

If $u \in C^\infty(M)$ then $\nabla u \in \mathcal{X}(M)$ is uniquely defined by

$$\langle \nabla u, X\rangle = Xu, \quad X \in \mathcal{X}(M).$$

From (1.4) it follows that

$$\langle \nabla_{\nabla u} \nabla u, X\rangle = \tfrac{1}{2} X|\nabla u|^2 = \tfrac{1}{2} X \nabla u\, u = \tfrac{1}{2}\langle X \nabla u, \nabla u\rangle, \quad X \in \mathcal{X}(M). \tag{1.5}$$

Set

$$\nabla^2 u : \mathcal{X}(M) \ni X \longmapsto \nabla_X \nabla u \in \mathcal{X}(M).$$

Then $\nabla^2 u$ is a $C^\infty(M)$-linear endomorphism of the $C^\infty(M)$-module $\mathcal{X}(M)$. The Monge-Ampere operator is defined by

$$M(u) = \det \nabla^2 u.$$

On M, we have the Riemannian curvature tensor

$$R(X,Y;W,Z) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W\rangle, \quad X, Y, W, Z \in \mathcal{X}(M).$$

If $\dim M = 2$, the Gauss curvature of M is defined by

$$K = \frac{R(X,Y;X,Y)}{|X|^2|Y|^2 - \langle X,Y\rangle^2},$$

provided that, at a given point, $X, Y \in \mathcal{X}(M)$ span the tangent space.

2. Proof of the theorem.

We have

$$\widetilde{g}(X,Y) = \langle X,Y\widetilde{\rangle} = \langle X,Y\rangle - Xu\,Yu, \quad X,Y \in \mathcal{X}(M).$$

Thus

$$\widetilde{g}(\nabla u, \nabla u) = (1 - |\nabla u|^2)|\nabla u|^2. \tag{2.1}$$

If $\widetilde{g} > 0$ (i.e. $\widetilde{g}(X,X) > 0$ for $X \neq 0$), then it follows that $|\nabla u| < 1$. On the other hand, if $|\nabla u| < 1$ then for $X \neq 0$ we get

$$\widetilde{g}(X,X) = |X|^2 - \langle \nabla u, X\rangle^2 > 0$$

by the Schwartz inequality and (2.1). This proves the first statement.

Now assume that $|\nabla u| < 1$. In the interior of the set $\{\nabla u = 0\}$, (0.3) is clear. Since the result is purely local, and because both sides of (0.3) belong to $C^\infty(M)$, we may assume that $\nabla u \neq 0$ everywhere on M. We may also assume that there is $W \in \mathcal{X}(M)$ with $W \neq 0$ and $Wu = 0$ (in local coordinates we may choose $W = (\partial u/\partial x_2)\partial_1 - (\partial u/\partial x_1)\partial_2$). This means that

$$\langle \nabla u, W\rangle = \langle \nabla u, W\widetilde{\rangle} = 0. \tag{2.2}$$

Since $\nabla u/|\nabla u|, W/|W|$ form an orthonormal basis of the tangent space, we have

$$\begin{aligned} M(u) &= \frac{\langle \nabla_{\nabla u}\nabla u, \nabla u\rangle\langle \nabla_W \nabla u, W\rangle - \langle \nabla_{\nabla u}\nabla u, W\rangle\langle \nabla_W \nabla u, \nabla u\rangle}{|\nabla u|^2|W|^2} \\ &= \frac{\langle \nabla_{\nabla u}\nabla u, \nabla u\rangle\langle \nabla_W \nabla u, W\rangle - \langle \nabla_{\nabla u}\nabla u, W\rangle^2}{|\nabla u|^2|W|^2}, \end{aligned} \tag{2.3}$$

where the last equality follows from (1.4).

From (1.3) we obtain

$$\langle \widetilde{\nabla}_X Y, Z\widetilde{\rangle} = \langle \nabla_X Y, Z\rangle - XYu\,Zu, \quad X,Y,Z \in \mathcal{X}(M).$$

This, (2.1) and (2.2) give

$$\widetilde{\nabla}_X Y - \nabla_X Y = \left(\frac{\langle \widetilde{\nabla}_X Y, \nabla u \widetilde{\rangle}}{(1-|\nabla u|^2)|\nabla u|^2} - \frac{\langle \nabla_X Y, \nabla u\rangle}{|\nabla u|^2} \right) \nabla u$$
$$(2.4) \qquad + \left(\frac{\langle \widetilde{\nabla}_X Y, W \widetilde{\rangle}}{|W|^2} - \frac{\langle \nabla_X Y, W\rangle}{|W|^2} \right) W$$
$$= \frac{\langle \nabla_X Y - XY, \nabla u\rangle}{1-|\nabla u|^2} \nabla u, \quad X, Y \in \mathcal{X}(M).$$

In particular, by (1.5),

$$\widetilde{\nabla}_X \nabla u - \nabla_X \nabla u = -\frac{\langle \nabla_{\nabla u} \nabla u, X\rangle}{1-|\nabla u|^2} \nabla u, \quad X \in \mathcal{X}(M).$$

Hence, by (2.4), (2.2) and (2.3),

$$\begin{aligned} &\widetilde{R}(W, \nabla u; W, \nabla u) - R(W, \nabla u; W, \nabla u) \\ &\quad = \langle \nabla_W (\widetilde{\nabla}_{\nabla u} \nabla u - \nabla_{\nabla u} \nabla u) - \nabla_{\nabla u} (\widetilde{\nabla}_W \nabla u - \nabla_W \nabla u), W\rangle \\ &\quad\quad - \frac{\langle \nabla_{\nabla u} \nabla u, \nabla u\rangle}{1-|\nabla u|^2} \langle \nabla_W \nabla u, W\rangle + \frac{\langle \nabla_{\nabla u} \nabla u, W\rangle}{1-|\nabla u|^2} \langle \nabla_{\nabla u} \nabla u, W\rangle \\ &\quad = -\frac{|\nabla u|^2 |W|^2 M(u)}{1-|\nabla u|^2}. \end{aligned}$$

This, together with (2.1), easily gives (0.3). □

References

1. do Carmo M., *Riemannian Geometry,* Birkkäuser, 1992.
2. Darboux G., *Leçons sur la theorie générale des surfaces, vol. III,* Gauthier-Villars, 1894.
3. Spivak M., *A Comprehensive Introduction to Differential Geometry, vol. V,* Publish or Perish, 1979.

Received September 5, 2000

Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków
Poland
e-mail: blocki@im.uj.edu.pl