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ON THE GENERALIZED RETRACT METHOD FOR DIFFERENTIAL INCLUSIONS WITH CONSTRAINTS

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Abstract. In the paper, we study the problem of existence of solutions to differential inclusions remaining in prescribed closed subsets of a Euclidean space. We find some new homological and homotopical sufficient conditions for existence of such trajectories. Strong deformations and multivalued admissible deformations are used as main tools.

Introduction. In the paper we study the problem of existence of trajectories of the first-order differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

remaining in a given closed set $K \subset \mathbb{R}^n$ for every $t \geq 0$. Such trajectory is said to be *viable* in K while the set $Viab_F(K)$ of all initial points of viable trajectories in K is called the *viability kernel*. So, our problem is equivalent to the question of non-emptiness of the viability kernel of a given set K.

The above notions have been studied by many authors in several contexts. We refer the reader to book [**3**] for rich (but not full) bibliography and examples of applications in such fields as optimal control, Hamilton–Jacobi equations, equilibria, etc. Note that viability corresponds to (semi-)invariance problems in dynamical systems and multivalued dynamical systems.

For differential equations having unique solutions, Ważewski ([**21**], Theorem 2) proved a powerful result which gives rather general conditions on behaviour of trajectories on ∂K implying the existence of a solution remaining

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forever in K. This famous result has been called the Ważewski retract method or the Ważewski topological principle.

Differential inclusions as well as differential equations without uniqueness bring us some difficulties. In particular, it may occur that there are some trajectories starting from a boundary point and leaving K immediately and some other which go inside, simultaneously. There are several papers dealing with the class of problems without the uniqueness of solutions (see e.g. [**14**], [**4**], [**5**]) but, as a necessary assumption, the authors have only considered situations where the sets of so-called "egress" and "strict egress" points are equal. The common point of these works lies in using the so-called multivalued retraction, which will be discussed in section 2. The difficulty is that, in a multivalued case, we usually meet two different exit sets:

$$K^-(F) := \{x_0 \in \partial K \mid \forall x \in S_F(x_0) : \ x \text{ leaves } K \text{ immediately}\},$$

$$K_e(F) := \{x_0 \in \partial K \mid \exists x \in S_F(x_0) : \ x \text{ leaves } K \text{ immediately}\},$$

where "immediately" means that for every $\varepsilon > 0$ there is $0 < t < \varepsilon$ such that $x(t) \notin K$. When there is no ambiguity, we shall write shortly K^- and K_e. Here $S_F(x_0)$ stands for the set of trajectories starting from x_0.

We have $K^- \subset K_e$ and K_e is usually essentially larger. The question arises: which of these exit sets is more important or more appropriate for our considerations? As we shall see, it depends on methods we would like to use. We shall describe it in the paper.

The second branch of the history of our problem is connected with the Conley index theory which has been developed since 70's in many directions (see [**17**] and references therein). Beside lots of results concerning continuous flows there have also been published some papers on the multivalued case (e.g. [**19, 15, 16**]).

The following result by Cardaliaguet (see [**6, 7**]) gives a connectedness type sufficient condition for existence of viable trajectories, which, while narrow from the topological point of view, concerns a large class of multivalued problems.

Proposition 0.1. *Let K be a closed convex subset of $\mathbb{R}^n$ or a closed connected C^1 n–manifold in $\mathbb{R}^n$ with a boundary ∂K and F be a compact convex valued upper semicontinuous map with at most a linear growth. If the set K^- is closed and not connected, then $Viab_F(K) \neq \emptyset$.*

The above result motivates us to try to weaken regularity assumptions on the set K and use more general homological or homotopical conditions instead of connectedness. This was the main aim of the paper [**11**]. Now, we also deal with this topic and develop some parts of [**11**].

Let us explain how the paper is organized.

After some background included in Section 1, we divide our considerations into two sections.

In Section 2 we present some new sufficient conditions for the existence of viable trajectories in terms of homology groups, admissible maps and multi-valued retractions and deformations. Theorem 2.1 is the main result of the section.

Formulations of results of Section 3 use the notion of a (single-valued) strong deformation but under stronger assumptions on the regularity of sets K and K^-. The main result of this section, Theorem 3.3, forms an essential progress in comparison with analogous results in [**11**]. We apply some useful selection and approximation lemmas, which are themselves interesting.

1. Preliminaries. In the paper we use the following notation: by $IntA$, $\overline{A}$ (or clA) and ∂A we denote the interior, closure and boundary of a subset A of a metric space X, an open ball centered at x_0 and with radius r is denoted by $B(x_0, r)$. The unit ball in a Euclidean space is denoted shortly by B_1. We use also notation $|\cdot|$ for the Euclidean norm. By $d_M(x)$ (or $dist(x, M)$) we denote the distance from a point x to a closed set M. The distance between two sets N, M will be always denoted by $dist(N, M) := \inf\{d_M(x) \mid x \in N\}$. All spaces are assumed to be metric.

The set-valued map $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is called a *Marchaud*[1] *map* if F is upper semicontinuous (in short: u.s.c.) with compact convex values and at most a linear growth (that is, there is a constant $c > 0$ such that $|F(x)| := \sup\{|y| \mid y \in F(x)\} \leq c(1 + |x|)$, for every x).

It is known ([**3**], Theorem 3.3.5) that for each $x_0 \in \mathbb{R}^n$ there is an absolutely continuous solution (which is called a *trajectory*) to the Cauchy problem

$$\dot{x}(t) \in F(x(t)) \qquad \text{for a.e. } t \geq 0, \;\; x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

Moreover, it satisfies the estimates:

$$|x(t)| \leq (|x_0| + 1)e^{ct} \text{ for all } t \geq 0$$

and

$$|\dot{x}(t)| \leq c(|x_0| + 1)e^{ct} \;\; \text{for a.e. } t \geq 0.$$

Take $b > c$. The set $S_F(x_0)$ of all absolutely continuous solutions to (2) is viewed as a subset of the Banach space

$$C := \{x \in C([0, \infty), \mathbb{R}^n) \mid \sup_{t \geq 0} |x(t)|e^{-bt} < \infty\}$$

[1]Let us note that in the same time (the 30's) that class of maps was independently introduced in the context of differential inclusions by Zaremba in his PhD thesis.

equipped with the norm

$$||x||_C := \sup_{t\geq 0} |x(t)|e^{-bt}.$$

Lemma 1.1. ([**3**], Theorem 2.4.4, Corollary 5.3.3, and [**1**]) *If F is a Marchaud map, then $S_F : \mathbb{R}^n \multimap C$ is u.s.c. with non-empty compact R_δ values*[2]*. Moreover, if F is Lipschitz, then S_F is also lower semicontinuous.*

We denote $S_F(K) := \bigcup_{x\in K} S_F(x)$.

Let $K \subset \mathbb{R}^n$ be a closed set.

Definition 1.2. We say that a trajectory $x(\cdot)$ for F starting from $x_0 \in K$ is *viable* in K if $x(t) \in K$ for every $t \geq 0$. A set K is said to be *viable*[3] *under* F, if, for each $x_0 \in K$, there is at least one trajectory $x \in S_F(x_0)$ which is viable in K. The largest closed subset of K viable under F (possibly empty, in general) is called a *viability kernel* of K, and is denoted by $Viab_F(K)$. It may be proved (see [**3**], Theorem 4.1.2) that $Viab_F(K)$ consists of all points $x_0 \in K$ such that there is a viable trajectory for F in K starting from x_0.

Using the notion of the viability kernel we can formulate the Viability Theorem as follows.

Proposition 1.3. ([**3**], Theorem 3.3.2) *One has $K = Viab_F(K)$ if and only if*

$$F(x) \cap T_K(x) \neq \emptyset \quad \textit{for every } x \in K.$$

Here $T_K(x)$ stands for the Bouligand contingent cone to K in a point $x \in K$ defined as:

$$T_K(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{h\to 0^+} \frac{dist(x+hv, K)}{h} = 0 \right\}.$$

When the set K is not viable, the viability kernel can be not only smaller than the whole set K, but even empty. In the paper we look for topological sufficient conditions for the non-emptiness of $Viab_F(K)$. We shall use exit sets defined in Introduction. It is important from the analytical point of view that it is possible (see [**8**], Lemma 5.2) to characterize the exit set in terms of tangent cones. In particular, one can check whether the exit set $K^-(F)$ is closed, which is a basic assumption in many results.

[2] A space X is a compact R_δ–set provided it is homeomorphic to an intersection of a decreasing sequence of compact contractible spaces. In particular, it is acyclic.

[3] In terms of multivalued dynamical systems, we can say that K is *weakly positively invariant*.

In the paper we use the so-called *exit function* $\tau_K : S_F(\mathbb{R}^n) \to [0,\infty]$ defined as follows:

$$\tau_K(x) := \begin{cases} \inf\{t > 0 \mid x(t) \notin K\}, & \text{for } x \text{ not viable,} \\ \infty, & \text{for } x \text{ viable.} \end{cases}$$

It is known (see [**3**], Lemma 4.2.2) that, for any closed set K, the map τ_K is upper semicontinuous (as a generalized real function).

Define also the function $\rho_K : S_F(K) \to [0,\infty]$,

$$\rho_K(x) := \inf\{t > 0 \mid x(t) \in K_e\}.$$

Lemma 1.4. ([**11**], Lemma 1.9) *Assume that $Z \subset K$ and no trajectory starting from Z remains in K. The function $\rho_K(\cdot)$ is lower semicontinuous (l.s.c.) on Z provided*

(3) *for each $x_0 \in \overline{K_e} \setminus K_e$ and $x \in S_F(x_0), x([0,\infty)) \cap K_e = \emptyset$.*

In particular, if the set K_e is closed, then $\rho_K(\cdot)$ is l.s.c.

Remark 1.5. There are important examples where (3) holds for non-closed K_e. For instance, in considerations in [**2**], $\overline{K_e} \setminus K_e$ is a singleton in which F is equal to zero. Second example can be found in [**18**], Section 5. Notice also that if $\overline{K_e} \setminus K_e \neq \emptyset$, then assumption (3) implies immediately that $Viab_F(K) \neq \emptyset$. Therefore (3) will be used only in results on localization of initial points of viable trajectories; in other results we will assume that the set K_e is closed.

Let us finally recall some important information on multivalued admissible maps which we use in the next section.

A map $p : X \to Y$ is said to be a *Vietoris map* provided p is onto, proper (i.e. $p^{-1}(A)$ is compact, for any compact subset A of Y), and the set $p^{-1}(y)$ is acyclic[4] for any $y \in Y$. A multivalued map $\varphi : X \multimap Y$ is called *admissible* (in the sense of Górniewicz [**12**]) if there exists a space Γ and two single-valued maps $p : \Gamma \to X$ and $q : \Gamma \to Y$ such that p is a Vietoris map, and $q(p^{-1}(x)) \subset \varphi(x)$, for every $x \in X$. We say that the pair (p,q) above is a *selected pair* of φ and denote it by $(p,q) \subset \varphi$. Of course, φ may have many selected pairs. From the Vietoris theorem (see [**13**], Theorem 8.9) it follows that a Vietoris map p induces an isomorphism $p_* : \check{H}(X) \to \check{H}(Y)$.

It enables to consider for any selected pair (p,q) of φ a homomorphism

$$\check{H}(X) \xrightarrow{p_*^{-1}} \check{H}(\Gamma) \xrightarrow{q_*} \check{H}(Y).$$

We define

$$\varphi_* := \{q_* p_*^{-1} \mid (p,q) \subset \varphi\}.$$

[4] With respect to the Čech homology functor with compact carriers and coefficients in $\mathbb{Q}$. It means that $\check{H}_n(p^{-1}(y)) = 0$ for $n > 0$ and $\check{H}_0(p^{-1}(y)) = \mathbb{Q}$.

If $\varphi : X \multimap Y$ is acyclic, i.e. it has compact acyclic values, then for any two selected pairs $(p,q),(p',q')$ of φ, there is $q_*p_*^{-1} = q'_*(p')_*^{-1}$ ([**13**], Proposition 40.4). Hence, since every single-valued map is acyclic, we can obtain f_*, for $f : X \to Y$, considering both diagrams $X \xleftarrow{p_f} Gr(f) \xrightarrow{q_f} Y$ and $X \xleftarrow{id_X} X \xrightarrow{f} Y$, where $p_f(x,y) = x$ and $q_f(x,y) = y$.

Proposition 1.6. ([**13**], Theorem 40.5) *Let $\varphi : X \multimap X_1$ and $\psi : X_1 \multimap X_2$ be two admissible maps. Then the composition $\psi\circ\varphi : X \multimap X_2$ is an admissible map and, for each selected pairs $(p_1,q_1) \subset \varphi$ and $(p_2,q_2) \subset \psi$, there exists a selected pair $(p,q) \subset \psi\circ\varphi$ such that $(q_2)_*(p_2)_*^{-1}(q_1)_*(p_1)_*^{-1} = q_*p_*^{-1}$.*

We will need the following.

Proposition 1.7. ([**11**], Proposition 6.4) *Let $\varphi : X \multimap X_1$ and $\psi : X_1 \multimap X_2$ be two admissible maps. If $\varphi = i : X \hookrightarrow X_1$ (resp. $\psi = i : X_1 \hookrightarrow X_2$), then*

$$(\psi\circ i)_* = \{q_*p_*^{-1}i_* \mid (p,q) \subset \psi\}, \tag{4}$$

resp.

$$(i\circ\varphi)_* = \{i_*q_*p_*^{-1} \mid (p,q) \subset \varphi\}. \tag{5}$$

It is easy to see that, for any two admissible maps, if $\varphi \subset \psi$, then $\varphi_* \subset \psi_*$.

Two admissible maps $\varphi,\psi : X \multimap Y$ are *homotopic* (written $\varphi \sim \psi$) provided there exists an admissible map $\chi : X\times[0,1] \multimap Y$ such that $\chi(\cdot,0) \subset \varphi$ and $\chi(\cdot,1) \subset \psi$.

Proposition 1.8. ([**13**], Theorem 40.11, Corollary 40.12) *For any two admissible maps $\varphi,\psi : X \multimap Y$, if $\varphi \sim \psi$, then $\varphi_* \cap \psi_* \neq \emptyset$.*

In particular, if $\varphi : X \multimap X$ and $\varphi \sim id_X$, then $Id_{\check{H}(X)} = q_*p_*^{-1}$, for some selected pair (p,q) of φ.

2. First approach: homological one. The aim of this section is to prove the following result being a generalization of Theorem A in [**11**] (see also Theorem 12 in [**10**]).

Theorem 2.1. *Let K be a closed subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a Marchaud map. Assume that the set K_e is closed and*

$$\begin{aligned}&\text{there is a subset } A \subset K,\ K_e \subset A,\\ &\text{and there exists a retraction } r : A \to K_e\\ &\text{such that } x([0,\tau_K(x)]) \subset A\\ &\text{for every } x_0 \in K_e \text{ and every } x \in S_F(x_0).\end{aligned} \tag{6}$$

If the homomorphism $i_* : \check{H}(K_e) \to \check{H}(K)$ induced by the inclusion map is not an isomorphism, then $Viab_F(K) \neq \emptyset$.

Before the proof, let us give some comments.

Comment 1. Our assumption (6) excludes, roughly speaking, the situation where some trajectories starting from a component of K_e leave K through another one.

Comment 2. We may illustrate the importance of assumption (6) with the following example.

Let

$$K := ([-1,2] \times [-2,2]) \setminus \left\{(x,y) \in \mathbb{R}^2 \mid x > 0, -x^2 < y < x^2\right\}$$

and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x,y) := \begin{cases} (1, 2\sqrt{y}), & 0 \le y \le 1, \\ (1, -2\sqrt{-y}), & -1 \le y < 0, \\ (1, 2\sqrt{2-y}), & 1 < y \le 2, \\ (1, -2\sqrt{y-2}), & y > 2, \\ (1, -2\sqrt{y+2}), & -2 \le y < -1, \\ (1, 2\sqrt{-y-2}), & y < -2. \end{cases}$$

One can check that the set K_e consists of exactly three points, so $\check{H}(K_e) \ncong \check{H}(K)$ but assumption (6) is not satisfied (look at the origin which belongs to K_e). Obviously, $Viab_F(K) = \emptyset$.

Comment 3. One can easily find (even in the Lipschitz single-valued case) examples where assumption (6) holds, $\check{H}(K_e) \cong \check{H}(K)$ and $Viab_F(K) = \emptyset$.

Comment 4. Assumption (6) is weaker and more suitable than the following one considered in Theorem A in [**11**]:

(7) for every $x_0 \in K_e(F)$ and every $x \in S_F(x_0)$, $x([0, \tau_K(x)]) \subset K_e(F)$.

The importance of the replacing assumption (7) by (6) is visible in the following simple example.

Let $K := \overline{B_1} \subset \mathbb{R}^2$ and $F : \mathbb{R}^2 \multimap \mathbb{R}^2$,

$$F(x,y) := \{(x+u, -y+v) \in \mathbb{R}^2 \mid u \in \frac{1}{2}[-y,y], v \in \frac{1}{2}[-x,x]\}.$$

It is easy to check that

$$\begin{aligned} K^- &= cl\left\{(x,y) \in \partial B_1 \mid \inf_{u \in F(x,y)} \langle u, (x,y)\rangle > 0\right\} = \\ &= \left\{(x,y) \in \partial B_1 \mid x^2 \ge \frac{5+\sqrt{5}}{10}\right\} \end{aligned}$$

and

$$\begin{aligned} K_e &= cl\left\{(x,y)\in\partial B_1 \mid \sup_{u\in F(x,y)}\langle u,(x,y)\rangle > 0\right\} = \\ &= \left\{(x,y)\in\partial B_1 \mid x^2 \geq \frac{5-\sqrt{5}}{10}\right\}. \end{aligned}$$

To check (6) it is sufficient to notice that, for every $(x,y)\in\partial B_1$ with $x>\frac{1}{5}$ $(x<-\frac{1}{5})$, one has $\inf_{u\in F(x,y)}\langle u,(1,0)\rangle > 0$ (resp. $\inf_{u\in F(x,y)}\langle u,(-1,0)\rangle > 0$), and $K_e\subset A := \{(x,y)\in\overline{B_1} \mid |x|>\frac{1}{5}\}$. It is seen that K_e is a retract of A, and each trajectory x starting in K_e satisfies $x([0,\tau_K(x)])\subset A$. Note that there exist trajectories starting in K_e which do not satisfy condition (7).

Proof of Theorem 2.1. Assume, on the contrary, that there is no viable trajectory in K. Consider the multivalued homotopy $H : K\times[0,1]\multimap K$,

$$H(x_0,t) := \bigcup_{x\in S_F(x_0)} \begin{cases} x([t\rho_K(x), t\tau_K(x)]), & \text{if } t\tau_K(x)\leq\rho_K(x), \\ x([t\rho_K(x),\rho_K(x)])\cup r(x([\rho_K(x),t\tau_K(x)])), & \text{if } t\tau_K(x)>\rho_K(x). \end{cases}$$

The map H can be described as the following composition

$$K\times[0,1] \overset{S_F\times id}{\multimap} S_F(K)\times[0,1] \overset{J\times id}{\multimap} S_F(K)\times[0,\infty)\times[0,1] \xrightarrow{k} K,$$

where $(S_F\times id)(x_0,t) := S_F(x_0)\times\{t\}$, $(J\times id)(x,t) := \{x\}\times[\rho_K(x),\tau_K(x)]\times\{t\}$ and

$$k(x,s,t) := \begin{cases} x(st), & \text{if } st\notin[\rho_K(x),\tau_K(x)], \\ r(x(st)), & \text{if } st\in[\rho_K(x),\tau_K(x)]. \end{cases}$$

Since ρ_K is l.s.c. (see Lemma 1.4) and τ_K is u.s.c., one can see that the map J is a compact convex valued u.s.c. map. Properties of r and the solution map $S_F(\cdot)$ imply that H, as a composition of admissible maps, is admissible. It is seen that $H(x_0,t)\ni x_0$ for every $x_0\in K_e$. Moreover, H joins $H(\cdot,0)=id_K$ with $H(\cdot,1)=i\circ\Phi : K\multimap K$, where $\Phi : K\multimap K_e$,

$$\Phi(x_0) := \bigcup_{x\in S_F(x_0)} r(x([\rho_K(x),\tau_K(x)])). \tag{8}$$

Consider the diagram

$$\begin{tikzcd}
K_e \arrow[r, "i"] \arrow[d, multimap, "\Phi \circ i"] & K \arrow[dl, multimap, "\Phi"'] \arrow[d, multimap, "i \circ \Phi"] \\
K_e \arrow[r, "i"] & K
\end{tikzcd}$$

From Proposition 1.8 applied to H, $Id_{\check{H}(K)} \in (i \circ \Phi)_*$, which means that $Id_{\check{H}(K)} = i_* q_* p_*^{-1}$ for some selected pair (p, q) of Φ (comp. (5)) and hence, i_* is onto. On the other hand, since $id_{K_e} \subset \Phi \circ i$, from (4) one obtains $Id_{\check{H}(K_e)} = \bar{q}_* \bar{p}_*^{-1} i_*$ for some selected pair $(\bar{p}, \bar{q}) \subset \Phi$. This implies that i_* is injective and so, an isomorphism; a contradiction. $\square$

We can call the map H in the above proof a *strong admissible (multivalued) deformation* and $H(\cdot, 1)$ a *multivalued admissible retraction*. Hence, we can formulate the above statement in a slightly more general way, namely: the viability kernel is non-empty if there is no strong admissible deformation of K onto K_e. The notions given above lead us also to the following.

Corollary 2.2. *Let K be a closed subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a Marchaud map. Assume that K_e is closed and (6) is satisfied. Then, if there is no multivalued admissible retraction of K onto K_e, $Viab_F(K) \neq \emptyset$. In particular, if K is connected and K_e is disconnected, then $Viab_F(K) \neq \emptyset$.*

Note that the notions of admissible deformation and admissible retraction are more appropriate for invariance problems than the notion of a multivalued retraction[5] considered e.g. in [**4, 5, 14**]. Indeed, one can easily find a multivalued retraction of a finite dimensional ball onto its boundary (!). Admissibility of a map is just a suitable property, which is useful in topological fixed point theory and some related topics (see [**13**] and references therein).

Using similar arguments as in the proof of Theorem 2.1 we can also prove the result on localization of initial states of viable trajectories in K, generalizing analogous results by Ważewski and others (see [**4, 5, 14**]).

[5] We say that a multivalued map $\Phi : X \multimap A$, $A \subset X$ is a *multivalued retraction* provided Φ is a compact connected valued u.s.c. map with $x \in \Phi(x)$ for every $x \in A$.

Corollary 2.3. *Let K be a closed subset of $\mathbb{R}^n$ and $Z \subset K$ be an arbitrary subset. Assume that $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is a Marchaud map satisfying (3) and*

$$\begin{array}{l} \textit{there is a subset } A \subset K,\ K_e \subset A, \\ \textit{and there exists a retraction } r : A \to K_e \\ \textit{such that } x([\rho_K(x), \tau_K(x)]) \subset A \\ \textit{for every } x_0 \in Z \textit{ and every } x \in S_F(x_0). \end{array} \tag{9}$$

If there is an admissible multivalued retraction of K_e onto $Z \cap K_e$ and there is no admissible multivalued retraction of Z onto $Z \cap K_e$, then there is a trajectory $x(\cdot)$ starting from $Z \setminus K_e$ and viable in K.

In the proof, we construct an admissible map $\Phi : Z \multimap K_e$ given by (8) and compose it with the admissible retraction given in assumptions. We obtain an admissible retraction of Z onto $K_e \cap Z$ and finish the proof. Note that in proving the upper semicontinuity of the map Φ, we have to use Lemma 1.4, since the set K_e does not have to be closed.

3. Second approach: through deformation retracts. It is seen that every strong deformation (single-valued) is a strong admissible deformation. Therefore, it would be better to formulate sufficient conditions for non-emptiness of the viability kernel in terms of strong deformations. The aim of this section is to study when it is possible in a multivalued case.

We start with a preliminary result which is rather obvious because of well-known selection theorems.

Proposition 3.1. ([**11**], Proposition 3.1) *Let K be a closed subset of $\mathbb{R}^n$ and F be a locally Lipschitz Marchaud map such that $K_e = K^-$ is closed.*

If K^- is not a strong deformation retract of K, then $Viab_F(K) \neq \emptyset$.

To prove the proposition, it is sufficient to take any locally Lipschitz selection f of F, which surely has the same exit set as F. Assuming $Viab_F(K) = \emptyset$, we can perform a standard construction of a strong deformation of K onto K^- using regularity of f. Some immediate consequences of Proposition 3.1 are given in [**11**].

Unfortunately, the situation (essentially multivalued) where $K_e = K^-$ is rare. Usually $K^- \subset K_e$ and $K^- \neq K_e$. The important question arises: is it possible to find a locally Lipschitz selection or arbitrarily near approximation f of F such that $K^-(f) = K^-(F)$?

Looking for a suitable selection we would like to find the one with $f(x) \in F(x) \cap T_K(x)$ on $K_e \setminus K^-$. Let us note the main difficulty in what we need. Even for a very regular set K (when $T_K(\cdot)$ is locally Lipschitz) and for a locally Lipschitz map F, the map $F(\cdot) \cap T_K(\cdot)$ may be not lower semicontinuous (l.s.c.). So, there are no appropriate general selection theorems. Nevertheless, in [**11**], the authors proved some interesting results under suitable regularity assumptions on K and the map F. Let us give a slight restriction of one of the results in [**11**].

THEOREM 3.2. (comp. [**11**], Theorem 3.16) *Let K be a $C^{1,1}$ n–manifold in $\mathbb{R}^n$ with a boundary ∂K and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a Marchaud locally Lipschitz map such that K^- is closed, $M = \overline{K_e \setminus K^-}$ is a compact Lipschitz neighborhood retract (the retraction is Lipschitz) and the following conditions are satisfied:*

(i) For each $x \in M$ and $y \in \partial F(x)$, the cone $T_{F(x)}(y)$ is a half-space;

(ii) For the Hamiltonian

$$\mathcal{H}(x,p) := \min\{\langle v,p\rangle \mid v \in F(x)\},$$

the derivative $\frac{\partial \mathcal{H}}{\partial p}(x,p) = ArgMin_{v\in F(x)}\langle v,p\rangle$ exists and is locally Lipschitz on $M \times (\mathbb{R}^n \setminus \{0\})$.

If K^- is not a strong deformation retract of K, then $Viab_F(K) \neq \emptyset$.

We refer to [**11**] for the proof and other related results. Let us only give two comments.

Comment 1. Values of F are very regular (a bit more than strictly convex). As an example of such situation we can consider the control problem

$$\left\{ \begin{array}{l} \dot{x} = f(x) + A(x)u \\ u \in U = \overline{B_1}, \end{array} \right. \tag{10}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $A : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$ are locally Lipschitz. Then the map $F(x) := f(x) + A(x)U$ satisfies (i)–(ii).

Comment 2. The class of neighborhood locally Lipschitz retracts is quite large. It contains, e.g., all proximate retracts, that is, sets M with a neighborhood U of M such that the projection

$$\pi_M(x) := \{y \in M \mid |y - x| = \inf_{u\in M} |u - x|\}$$

is single-valued. In particular, it contains all $C^{1,1}$ manifolds.

Now, we prove the main result of this section and compare it with Theorem 3.2.

THEOREM 3.3. *Let $K = \overline{IntK}$ be a closed sleek[6] subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ be a Marchaud map such that K^- is a closed strong deformation retract of its certain open neighborhood V in K. Assume that $Int\, T_K(x) \neq \emptyset$ for every $x \in K \setminus K^-$.*

If K^- is not a strong deformation retract of K, then $Viab_F(K) \neq \emptyset$.

[6]We say that a set K is *sleek*, if the cone map $T_K(\cdot)$ is lower semicontinuous.

The comparison we mentioned above will be given again in the form of some comments.

Comment 1. Recall that sleekness means that $T_K(\cdot)$ is l.s.c. which is essentially less than being Lipschitzean.

Comment 2. F may be not Lipschitzean (only u.s.c.).

Comment 3. Assumption $Int\, T_K(x) \neq \emptyset$ eliminates "too sharp corners" of the set K.

Comment 4. Retractness assumption says that K^- is an NDR (neighborhood deformation retract) in K (see [**20**]). This situation very often appears if K^- is a neighborhood retract of K.

To prove Theorem 3.3, we need some lemmas.

Lemma 3.4. ([**9**], Lemma 3.2) *Let $K \subset \mathbb{R}^n$ be a compact set and F be such that K_s is closed. Then, for any open neighborhood V_0 of K_s in $\mathbb{R}^n$, there exist an open neighborhood V_F of K_s in $\mathbb{R}^n$ and $\varepsilon_0 > 0$ such that, for every $p \in V_F \cap K$, $0 < \varepsilon \leq \varepsilon_0$ and every locally Lipschitz ε–approximation[7] f of F, there is $p \notin Viab_f(K)$ and $S_f(p)([0, \tau_K(p)]) \subset V_0 \cap K$, where τ_K is the exit function for f.*

Lemma 3.5. *Let A be a closed subset of $\mathbb{R}^n$. Assume that $F : \mathbb{R}^n \multimap \mathbb{R}^n$, $\Psi : A \multimap \mathbb{R}^n$ are convex valued, F is u.s.c., and Ψ satisfies the following condition:*

$$\text{(11)} \qquad \begin{array}{l} \textit{For every } x \in A \textit{ there exist } y_x \in F(x) \cap Int\, \Psi(x) \\ \textit{and an open neighborhood } V(x) \textit{ of } x \textit{ in } X \\ \textit{such that } y_x \in Int\, \Psi(z) \textit{ for each } z \in V(x) \cap A. \end{array}$$

Then, for every $\varepsilon > 0$, there exist an open neighborhood Ω_ε of A in $\mathbb{R}^n$ and a locally Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) f is an ε–approximation of F,
(ii) f is a selection of $Int\, \psi(\cdot)$ on A.

Let us note that assumption (11) is satisfied if, e.g., Ψ is l.s.c. and

$$F(x) \cap Int\, \Psi(x) \neq \emptyset \text{ for every } x \in A.$$

Proof of Lemma 3.5. Denote $A_m := A \cap \overline{B(0,m)}$, $m \geq 1$. For a given $\varepsilon > 0$, consider the open covering of A_1 in X,

$$U(x) := B(x, \varepsilon/2) \cap \{x' \in \mathbb{R}^n \mid F(x') \subset F(x) + \varepsilon/2B_1\},\ x \in A_1.$$

[7] By an ε–approximation of F we mean a single-valued map f such that $d_{\text{conv } F(B(x,\varepsilon))} f(x) < \varepsilon$ for every x. This condition is slightly weaker than the usual one considered in approximation techniques ("conv" is added, comp. [**13**]).

Since A_1 is compact, we find a finite open star-refinement $\mathcal{V}_1=\{V_1,\dots,V_{k_1}\}$ of $\{U(x)\}_{x\in A_1}$ i.e., for every $i\in\{1,\dots,k_1\}$, there is $\bar{x}\in A_1$ such that

$$st(V_i,\mathcal{V}_1):=\bigcup\{V_j\in\mathcal{V}_1|\ V_j\cap V_i\neq\emptyset\}\subset U(\bar{x}).$$

For $m\geq 2$ and $x\in A_m\setminus A_{m-1}$ we consider also

$$U(x)\subset B(x,\varepsilon/2)\cap\{x'\in\mathbb{R}^n\mid F(x')\subset F(x)+\varepsilon/2B_1\},\ U(x)\cap A_{m-1}=\emptyset,$$

and find, analogously, a finite open star-refinement $\mathcal{V}_m$ of $\{U(x)\}_{x\in A_m}$ such that

$$\begin{aligned}\mathcal{V}_m=&\{V_1,\dots,V_{k_{m-1}},V_{k_{m-1}+1},\dots,V_{k_m}\},\\ V_{k_{m-1}+i}\cap A_{m-1}=&\emptyset, i=1,\dots,k_m-k_{m-1}.\end{aligned}$$

By assumption (11), we find, for every $x\in A$ ($x\in A_m\setminus A_{m-1}$), a point $y_x\in F(x)\cap Int\,\Psi(x)$ and open neighborhoods $V(x)\subset U(x)$, $V_i\in\mathcal{V}_m$ of x in X such that $V(x)\subset V_i$ and $y_x\in Int\,\Psi(z)$ for each $z\in V(x)\cap A$.

Let $\mathcal{V}:=\{V(x_1),\dots,V(x_l),\dots\}$ be a locally finite covering of A chosen so that

$$\{V(x_1),\dots,V(x_{l_1})\}\text{ is a covering of }A_1,$$

$$\{V(x_1),\dots,V(x_{l_2})\},l_2\geq l_1,\text{ is a covering of }A_2\text{ and }A_1\cap\bigcup_{i=l_1+1}^{l_2}V(x_i)=\emptyset,$$

and, for any other m,

$$\begin{aligned}\{V(x_1),\dots,V(x_{l_m})\},l_m\geq l_{m-1}\text{ is a covering of }A_m&\\ \text{and }A_{m-1}\cap\bigcup_{i=l_{m-1}+1}^{l_m}V(x_i)=\emptyset.&\end{aligned}$$

Take a locally Lipschitz partition of unity $\{\lambda_i\}_{i=1}^{\infty}$ subordinated to $\mathcal{V}$. Denote $\Omega_\varepsilon:=\bigcup_{i=1}^{l}V(x_i)$ and define $f:\Omega_\varepsilon\to\mathbb{R}^n$,

$$f_0(x):=\sum_{i=1}^{\infty}\lambda_i(x)y_i,$$

where $y_i:=y_{x_i}$. Of course, f_0 is locally Lipschitzean. Moreover, by convexity of values of Ψ, f_0 is a selection of $Int\,\Psi(\cdot)$ on A.

Let $x\in\bigcup_{i=1}^{\infty}V(x_i)$. Then, in fact, there is $m\geq 1$ such that $x\in\bigcup_{i=1}^{l_m}V(x_i)$. Since $\mathcal{V}$ is a star-refinement of $\{U(x)\}_{x\in A}$, there is $\bar{x}\in A$ (in fact, $\bar{x}\in A_m$) such that $x,x_i\in U(\bar{x})$ for each $i\in\{1,\dots,l_m\}$ with $x\in V(x_i)$. Therefore, $y_i\in F(x_i)\subset F(\bar{x})+\varepsilon/2B_1$ and, since $F(\bar{x})$ is convex, $f_0(x)\in F(\bar{x})+\varepsilon/2B_1$. Hence, $f_0(x)\in F(B(x,\varepsilon))+\varepsilon B_1$ which means that f_0 is an ε–approximation of F.

Let $f_1 : \mathbb{R}^n \to \mathbb{R}^n$ be any locally Lipschitz ε–approximation of F. Using a locally Lipschitz function $u : \mathbb{R}^n \to [0,1]$ such that $u \equiv 1$ on A and $u \equiv 0$ on $\mathbb{R}^n \setminus \Omega_\varepsilon$, we join f_0 and f_1 obtaining $f : \mathbb{R}^n \to \mathbb{R}^n$,

$$f(x) := u(x)f_0(x) + (1 - u(x))f_1(x),$$

which is still an ε–approximation of F and a selection of $Int\,\Psi(\cdot)$ on A. The proof is complete. $\square$

Lemma 3.6. *Let $X \subset \mathbb{R}^n$ and $A \subset X$ be a closed subset. Assume that $\Psi : X \multimap \mathbb{R}^n$ is convex valued, and satisfies the following condition:*

(12) *For every $x \in X$ there exist $y_x \in \Psi(x)$ and an open neighborhood $V(x)$ of x in X such that $y_x \in \Psi(z)$ for each $z \in V(x)$ with $y_x = 0$ for every $x \in A$.*

Then there exists a locally Lipschitz selection $f : \Omega_\varepsilon \to \mathbb{R}^n$ of Ψ such that $f(x) = 0$ for every $x \in A$.

Proof. Without loss of generality we may assume that the covering $\{V(x)\}_{x\in X}$ is countable and locally finite, and $V(x) \cap A = \emptyset$ for every $x \notin A$. Let $\{\lambda_i\}_{i=1}^{\infty}$ be a locally Lipschitz partition of unity subordinated to this covering.

Define $f : X \to \mathbb{R}^n$,

$$f(x) := \sum_{i\in I(x)} \lambda_i(x) y_{x_i},$$

where $I(x) := \{i \in \mathbb{N} \mid x \in V(x_i)\}$. Obviously, f is locally Lipschitzean. Moreover, by convexity of values of Ψ, f is its selection. Since, for every $x \in A$, $\{y_{x_i} \mid i \in I(x)\} = \{0\}$, we obtain that $f(A) = \{0\}$. $\square$

Proof of Theorem 3.3. Assume that $Viab_F(K) = \emptyset$. Let $V_F \subset V$ be as in Lemma 3.4, chosen for V. Take an open neighborhood Ω_0 of K^- in K such that $\overline{\Omega_0} \subset V_F$.

For an arbitrary small $\varepsilon > 0$ define the following auxiliary map $F_\varepsilon : \mathbb{R}^n \multimap \mathbb{R}^n$,

$$F_\varepsilon(x) := F(x) + \delta_\varepsilon(x)\overline{B_1},$$

where $\delta_\varepsilon(x) := \min\{d_{\overline{\Omega_0}}(x), \varepsilon\}$. Then $K^-(F_\varepsilon) = K^-(F)$ and

$$F_\varepsilon(x) \cap Int\,T_K(x) \neq \emptyset \quad \text{for every } x \in K \setminus \overline{\Omega_0}.$$

Let $\Omega \supset \overline{\Omega_0}$ be an open subset in K such that $\overline{\Omega} \subset V_F$. From Lemma 3.5 it follows that there exists a locally Lipschitz ε–approximation f of F_ε such that $f(x) \in Int\,T_K(x)$ for every $x \in K \setminus \Omega$. Therefore, $K^-(f) \subset \Omega$.

Take an open set U in K such that $\overline{\Omega} \subset U \subset \overline{U} \subset V_F$. Consider the map $\Gamma : K \multimap [0, \infty)$,

$$\Gamma(x) := [\tau_{K \setminus U}(x), \tau_K(x)].$$

This map does not have to be l.s.c. Nevertheless, it satisfies the following condition:

For every $x \in K$, there exist $\gamma_x \in \Gamma(x)$ and an open neighborhood $V(x)$ of x in K such that $\gamma_x \in \Gamma(z)$ for any $z \in V(x)$.

Indeed, it is sufficient to take $\gamma_x \in \Gamma(x)$ such that $S_f(x)(\gamma_x) \in U \setminus \overline{K^-(f)}$ if $x \notin \overline{K^-(f)} \cup K^-$, and $\gamma_x = 0$ if $x \in \overline{K^-(f)} \cup K^-$, and use regularity of f. From Lemma 3.6 it follows that there exists a continuous selection $\gamma : K \to [0, \infty)$ of Γ with $\gamma(x) = 0$ for every $x \in \overline{K^-(f)} \cup K^-$. Notice that $S_f(x)(\gamma(x)) \in V$ and $\gamma(x) \leq \tau_K(x)$ for every $x \in K$.

Define the homotopy $h : K \times [0, 1] \to K$,

$$h(x,t) := \begin{cases} S_f(x)(2t\gamma(x)), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ k\left(S_f(x)(\gamma(x)), 2t - 1\right), & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

One can see that h is continuous, $h(\cdot, 0) = id_K$ and $h(x, 1) \in K^-$ for every $x \in K$. Moreover, for every $x \in K^-$, there is $\gamma(x) = 0$ and hence $h(x,t) = k(x,t) = x$ for any $t \in [0, 1]$. We conclude that K^- is a strong deformation retract of K; a contradiction. □

References

1. Andres J., Gabor G., Górniewicz L., *Boundary value problems on infinite intervals*, Trans. Amer. Math. Soc., **351** (1999), 4861–4903.
2. Arronson G., *On two theorems by Hartman and Wintner. An Application of the Ważewski retract method*, J. Math. Anal. Appl., **66** (1978), 232–236.
3. Aubin J.-P., *Viability Theory*, Birkhäuser, Boston, 1991.
4. Bebernes J.W., Schuur J.D., *Investigations in the topological method of Ważewski*, Atti Accad. Naz. Lincei, Rend., Cl. Sci. Fis. Mat., VIII Ser., **49** (1970), 39–42.
5. ______, *The Ważewski topological methods for contingent equations*, Ann. Mat. Pura Appl., Ser. IV, **87** (1970), 271–279.
6. Cardaliaguet P., *Sufficient conditions of nonemptiness of the viability kernel*, PhD Thesis, Chapter 8, Université Paris IX, Dauphine, 1992.
7. ______, *Conditions suffisantes de non-vacuité du noyau de viabilité*, C. R. Acad. Sci., Paris, Ser. I, **314**, 11 (1992), 797–800.
8. Cardaliaguet P., Gabor G., Quincampoix M., *Equilibria and strict equilibria of multivalued maps on noninvariant sets*, Ann. Polon. Math., **82** (2003), 19–37.
9. Gabor G., *Homotopy index for multivalued maps on sleek sets*, submitted.
10. Gabor G., Quincampoix M., *De l'existence de solutions d'équations différentielles assujetties à rester dans un ensemble fermé*, C. R. Acad. Sci. Paris, Série I, **333** (2001), 913–918.

11. Gabor G., Quincampoix M., *On existence of solutions to differential inclusions remaining in a prescribed closed subset of a finite-dimensional space*, J. Differential Equations, **185**, No. **2** (2002), 483–512.
12. Górniewicz L., *Homological methods in fixed point theory of multivalued mappings*, Dissertationes Math., **129** (1976), 1–71.
13. ______, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer Academic Publishers, Dordrecht, 1999.
14. Jackson L.K., Klaasen G., *A variation of the topological method of Ważewski*, SIAM J. Appl. Math., **20** (1971), 124–130.
15. Kaczyński T., Mrozek M., *Conley index for discrete multi-valued dynamical systems*, Topology Appl., **65** (1995), 83–96.
16. Kunze M., *Non-Smooth Dynamical Systems*, Lect. Notes Math., **1744**, Springer, Berlin, 2000.
17. Mischaikow K., Mrozek M., Zgliczyński P. (editors), *Conley index theory*, Banach Center Publ., **47**, PWN, Warszawa, 1999.
18. Mrozek M., *Periodic and stationary trajectories of flows and ordinary differential equations*, Univ. Iagel. Acta Math., **27** (1988), 29–37.
19. ______, *A cohomological index of Conley type for multi-valued admissible flows*, J. Differential Equations, **84** (1990), 15–51.
20. Salamon D., *Connected simple systems and the Conley index of isolated invariant sets*, Trans. Amer. Math. Soc., **291** (1985), 1–41.
21. Ważewski T., *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math., **20** (1947), 279–313.

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