PLURIREGULARITY IN POLYNOMIALLY BOUNDED
O–MINIMAL STRUCTURES

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Abstract. Given a polynomially bounded o–minimal structure $\mathfrak{S}$ and a set $A \subset \mathbb{R}^n$ belonging to $\mathfrak{S}$, we show that A (considered as a subset of $\mathbb{C}^n$) is pluriregular at every point $a \in \overline{\operatorname{int} A}$ that can be attained by a C^∞ arc $\gamma : [0, \epsilon] \to \mathbb{R}^n$ belonging to $\mathfrak{S}$, such that $\gamma(0) = a$ and $\gamma((0, \epsilon]) \subset \operatorname{int} A$. In particular, if $\mathfrak{S}$ is a recently found in [**22**] polynomially bounded o–minimal structure of quasianalytic functions in the sense of Denjoy–Carleman, then any set $A \subset \mathbb{R}^n$ that belongs to $\mathfrak{S}$ is pluriregular at every point $a \in \overline{\operatorname{int} A}$.

1. Introduction. Let E be a subset of the space $\mathbb{C}^n$. We set

$$V_E(z) = \sup\{u(z) : \ u \in \mathcal{L}(\mathbb{C}^n), \ u \leq 0 \text{ on } E\},$$

where $\mathcal{L}(\mathbb{C}^n) = \{u \in PSH(\mathbb{C}^n) : \ \sup_{z \in \mathbb{C}^n}[u(z) - \log(1 + |z|)] < \infty\}$ is the *Lelong class* of plurisubharmonic functions with *minimal growth*. The function V_E is called the (*plurisubharmonic*) *extremal function* associated with E (see [**27**]). By the pluripotential theory due to E. Bedford and B.A. Taylor (see [**9**]), if E is *nonpluripolar* in $\mathbb{C}^n$ (*i.e.* there is no plurisubharmonic function u on $\mathbb{C}^n$, $u(z) \not\equiv -\infty$ such that $E \subset \{u(z) = -\infty\}$) then the upper semicontinuous regularization V_E^* of V_E belongs to $\mathcal{L}(\mathbb{C}^n)$ and is a solution (in $\mathbb{C}^n \setminus \hat{E}$, where $\hat{E}$ denotes the polynomial hull of E) of the homogeneous complex *Monge–Ampère equation* $(dd^c V_E^*)^n = 0$, which reduces in the one-dimensional case to the *Laplace equation*. Therefore V_E^* is a multidimensional counterpart of the

classical *Green function* for $\mathbb{C} \setminus \hat{E}$. It is a result of Siciak [27] that if $E \subset \mathbb{C}^n$ is compact then

(1.1)
$$V_E(z) = \sup\{\frac{1}{\deg p} \log |p(z)| : \ p \text{ is a polynomial of } \deg p \geq 1$$
$$\text{and } \|p\|_E \leq 1\} = \log \Phi_E(z),$$

where Φ_E is the (*polynomial*) *extremal function* of E introduced by Siciak in [26].

Suppose now that E is a nonempty bounded open subset of $\mathbb{R}^n$, where (in the whole paper) $\mathbb{R}^n$ is treated as a subset of $\mathbb{C}^n$ such that $\mathbb{R}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \ \Im z_j = 0, \ j = 0, \ldots, n\}$. It has been proved in [20] that if $a \in \bar{E}$ can be attained by a semianalytic arc $h : \ [0,1] \to \bar{E}$ such that $h(0) = a$ and $h((0,1]) \subset E$ then the function V_E is continuous at a. From the above result it follows that if E is a subanalytic set in $\mathbb{R}^n$ then E is *pluriregular* at every point a of $\overline{\text{int}\, E}$ (*see* [20]). This means, by definition, that the extremal function V_E is continuous at a, which is equivalent to saying that $V_E^*(a) := \limsup_{z \to a} V_E(z) = 0$.

On the other hand, by an example due to Sadullaev [24] (see also [3]), there exist bounded domains E in $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary except for a single point $a \in \partial E$ such that there exists a $\mathcal{C}^\infty$ curve $h : \ [0,1] \ni t \to h(t) \in \mathbb{R}^2$ with $h([0,1)) \subset E$ and $h(1) = a$ for which $V_E^*(a) > 0$. It follows that E is not pluriregular at a being pluriregular at any other point of $\bar{E}$. Actually, Sadullaev [24] proved the following

LEMMA 1.1. *Let* $f(z) = \sum\limits_{j=0}^{\infty} a_{k_j} z^{k_j}$ *be a gap series with* $k_j/k_{j+1} \to 0$ *as* $j \to \infty$ *and with the radius of convergence* $R = 1$. *Then the graph* $A = \{w - f(z) = 0\} \subset \mathbb{C}^2$ *of the function* f *is plurithin at every boundary point* $(z_0, w_0) \in \bar{A}$, *where* $|z_0| = 1$. *This means, by definition, that there exists a neighbourhood* U *of* (z_0, w_0) *and a plurisubharmonic function* u *in* U *such that*

$$\limsup_{\substack{(z,w) \to (z_0,w_0) \\ (z,w) \in A}} u(z,w) < u(z_0, w_0).$$

Moreover, the function u *can be chosen to be in the Lelong class in* $\mathbb{C}^2$.

This lemma permits one to show that in the above mentioned semianalyticity accessibility criterion of pluriregularity, the (semi)analytic arc cannot be in general replaced not only by a C^∞ arc but even by a quasianalytic one. We have

Example 1.2. Let
$$f(x) = \sum_{j=0}^{\infty} 2^{-k_j^2} x^{k_{j+1}},$$
where $k_0 = 2$ and $k_{j+1} = k_j^3$ for $j = 0, 1, \ldots$. Then the function f is $\mathcal{C}^{\infty}$ on $[0, 1]$ and
$$M_{k_j}(f) := \max_{x \in [0,1]} |f^{(k_j)}(x)| = O(k_j!).$$
Hence f is quasianalytic on $[0, 1]$ in the sense of Denjoy–Carleman (*see* [**7**], [**18**]). By Lemma 1.1 the set $\Gamma = \{(x, y) \in \mathbb{R}^2 : \ y = f(x), \ x \in [0, 1)\}$ is plurithin at the point $(1, f(1))$. Hence, by a known procedure (*see* [**9**, Proposition 4.8.2]), one can choose a function $u \in \mathcal{L}(\mathbb{C}^2)$ and $\delta \in (0, 1)$ such that $u(x, f(x)) \leq -1$ if $x \in [\delta, 1)$ and $u(1, f(1)) > 0$. Then $u(x, y) < 0$ in an open neighbourhood G of $\Gamma \cap \{\delta \leq x < 1\}$ in $\mathbb{R}^2$. Set $D = \{(x, y) : \ |y - f(x)| < \frac{1}{2} \inf\{\text{dist}(f(t), \mathbb{R}^2 \setminus G) : \ t \in [\delta, x], \ x \in (\delta, 1)\}$. Then $\bar{D} \setminus \{(1, f(1))\} \subset G$. Since every point b of $\bar{D} \setminus \{(1, f(1))\}$ can be attained by an interval I_b such that $I_b \setminus \{b\} \subset D$, and since $V_{\bar{D}}^* = V_{\bar{D}} = 0$ in D, we have $V_{\bar{D}}^*(b) = 0$ in such a point b. On the other hand, by [**27**, Prop. 3.11], $V_{\bar{D}}^* = V_{\bar{D} \setminus \{(1, f(1))\}}^*$, whence $V_{\bar{D}}^*(1, f(1)) \geq u(1, f(1)) > 0$. This means that the set $\bar{D}$ is pluriregular at every point $b \in \bar{D}$ except for $(1, f(1))$.

Remark 1.3. It has been proved in [**19**] that the pluriregularity of compact subsets of $\mathbb{C}^n$ is invariant under nondegenerate analytic mappings from $\mathbb{C}^n$ to $\mathbb{C}^k$ (with $1 \leq k \leq n$). By Example 1.2 it is clear that this is not the case for quasianalytic diffeomorphisms. For, let $F(x, y) = (x, y - f(x))$, where f is the function of Example 1.2. Then $F(D)$ is pluriregular at $(1, 0)$ while $F^{-1}(F(D)) = D$ is not pluriregular at $(1, f(1))$.

Remark 1.4. In connection with Lemma 1.1. Sadullaev [**25**] has posed the question as to whether the arcs
$$E_1 = \{(x, y) \in \mathbb{R}^2 : \ y = x^{\alpha}, \ x \in (0, 1)\}$$
with α irrational, and
$$E_2 = \{(x, y) \in \mathbb{R}^2; \ y = e^{-1/x}, \ x \in (0, 1)\}$$
are plurithin at the origin (as subsets of $\mathbb{C}^2$). The question has appeared difficult and it took nearly 20 years to answering it (in the affirmative) by Levenberg and Poletsky [**11**] (case of E_1) and Wiegerinck [**30**] (case of E_2).

In spite of discouraged Example 1.2 and Remark 1.3, we are going to show that under certain conditions quasianalytic mappings do yield new examples of pluriregular sets. This will be closely related with the recently briskly progressing theory of o–minimal structures.

2. O–minimal structures. Let $\mathfrak{S} = \bigcup_{n \in \mathbb{N}} \mathfrak{S}_n$, where each $\mathfrak{S}_n$ is a family of subsets of $\mathbb{R}^n$. Following [**5**], we shall say that the collection $\mathfrak{S}$ is a structure on the field $(\mathbb{R}, +, \cdot)$ if:

(S1) Each $\mathfrak{S}_n$ is a boolean algebra with $\mathbb{R}^n \in \mathfrak{S}_n$;

(S2) $\mathfrak{S}_n$ contains the diagonal $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i = x_j \text{ for } 1 \le i < j \le n\}$;

(S3) If $A \in \mathfrak{S}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathfrak{S}_{n+1}$;

(S4) If $A \in \mathfrak{S}_{n+1}$, then $\pi(A) \in \mathfrak{S}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates;

(S5) $\mathfrak{S}_3$ contains the graphs of addition and multiplication.

If in addition the structure $\mathfrak{S}$ satisfies

(S6) $\mathfrak{S}_1$ consists exactly of the finite union of intervals of all kinds (including singletons),

then it is said to be *o–minimal* (short for "order–mimimal").

For a fixed structure $\mathfrak{S}$ on $(\mathbb{R}, +, \cdot)$, we say that a set $A \subset \mathbb{R}^n$ *belongs* to $\mathfrak{S}$ (or that A is *definable* in $\mathfrak{S}$) if $A \in \mathfrak{S}_n$. A function $f : A \to \mathbb{R}^m$ with $A \subset \mathbb{R}^n$ *belongs* to $\mathfrak{S}$ (or is *definable* in $\mathfrak{S}$) if its graph $\Gamma(f) \subset \mathbb{R}^{n+m}$ belongs to $\mathfrak{S}$. For other set–theoretical and topological properties of structures we refer the reader to [**5**].

Given structures $\mathfrak{S} = (\mathfrak{S}_n)$ and $\mathfrak{S}' = (\mathfrak{S}'_n)$ on $(\mathbb{R}, +, \cdot)$ we put $\mathfrak{S} \subset \mathfrak{S}'$ if $\mathfrak{S}_n \subset \mathfrak{S}'_n$ for all $n \in \mathbb{N}$. Given functions $f_j : \mathbb{R}^{n(j)} \to \mathbb{R}$ with j in some index set J, we let $\mathfrak{S}(\mathbb{R}, +, \cdot, (f_j)_{j \in J})$ denote the smallest structure on $(\mathbb{R}, +, \cdot)$ containing the graphs of all functions f_j. In the sequel, we shall be interested in o–minimal structures that are *polynomially bounded*. This means that for every function $f : \mathbb{R} \to \mathbb{R}$ belonging to the structure, there exists some $N \in \mathbb{N}$ (depending on f) such that $f(t) = O(t^N)$ as $t \to +\infty$. If $\mathfrak{S} = (\mathfrak{S}_n)$ is a polynomially bounded o–minimal structure and if $U \in \mathfrak{S}_n$ is open and connected, then by [**13**] the ring S of all C^∞ functions $f : U \to \mathbb{R}$ belonging to $\mathfrak{S}$ is *quasianalytic*, i.e. for each nonzero $f \in S$ and $x \in U$, the Taylor series at x of f is not zero. Let us recall some examples of o–minimal structures that are polynomially bounded (*cf* [**5**]).

(2.1) Semialgebraic sets (*see* [**2**]);

(2.2) The structure $\mathfrak{S}(\mathbb{R}_{an})$ with $\mathbb{R}_{an} := (\mathbb{R}, +, \cdot, (f))$ where f ranges over all functions $f : \mathbb{R}^n \to \mathbb{R}$ ($n \in \mathbb{N}$) that vanish identically off $[-1, 1]^n$ and that are germs on $[-1, 1]^n$ of analytic functions. This structure consists of the so-called *finitely* (or *globally*) *subanalytic functions* (*see* [**4**], [**8**]);

(2.3) The structure $\mathfrak{S}(\mathbb{R}_{an}^{\mathbb{R}})$ with $\mathbb{R}_{an}^{\mathbb{R}} := (\mathbb{R}, +, \cdot, (f), (x^r)_{r \in \mathbb{R}})$ (introduced by Miller [**12**]), where f ranges over all restricted analytic functions as in (2.2), and the function $x^r : \mathbb{R} \to \mathbb{R}$ is given by $t \to t^r$ for $t > 0$ and 0 for $t \le 0$.

(2.4) Let $\mathbb{R}_{an^*} = (\mathbb{R}, +, \cdot, (f))$, where f ranges over all functions $f : \mathbb{R}^n \to \mathbb{R}$ (for all $n \in \mathbb{N}$) that are 0 outside $[0,1]^n$ and are given on $[0,1]^n$ by a generalized power series $F = \sum c_\alpha x^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in [0, \infty)^n$, the coefficients c_α are real, x^α denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the set $\{\alpha \in [0, \infty)^n : c_\alpha \neq 0\}$ is countable and $\sum |c_\alpha| r^\alpha < \infty$ for some polyradius $r = (r_1, \ldots, r_n)$ with $r_1 > 1, \ldots, r_n > 1$. Then the structure $\mathfrak{S}(\mathbb{R}_{an^*})$ is o–minimal and polynomially bounded (*see* [**6**]).

By a recent result of J.-P. Rolin, P. Speissegger and A. Wilkie [**22**] we also have the following example.

(2.5) Let (g) be the family of all functions $g : \mathbb{R}^n \to \mathbb{R}$ (for all $n \in \mathbb{N}$) defined by $g(x) := f(x)$ if $x \in I^n := [-1, 1]^n$ and $g(x) := 0$ if $x \notin I^n$, where f ranges over a *Denjoy–Carleman class* of C^∞ functions on I^n that satisfy

$$|f^{(\alpha)}(x)| \leq A(f)^{|\alpha|} M_{|\alpha|} \quad \text{for all } x \in I^n \text{ and } \alpha \in \mathbb{N}_0^n.$$

Suppose that the sequence $1 \leq M_0 \leq M_1 \leq \ldots$ is *strongly logarithmically convex*, that is, for each $p \geq 1$,

$$\left(\frac{M_p}{p!} \right)^2 \leq \frac{M_{p+1}}{(p+1)!} \cdot \frac{M_{p-1}}{(p-1)!},$$

and *quasianalytic* in the sense of Denjoy–Carleman (*see e.g.* [**23**]), that is

$$\sum_{p=0}^{\infty} \frac{M_p}{M_{p+1}} = \infty.$$

Then the structure $\mathfrak{S}(\mathbb{R}, +, \cdot, (g))$ is o–minimal and polynomially bounded.

Let us note that in [**22**] the o–minimality of $\mathfrak{S}$ is established under the condition for (M_p) to be *residually logarithmically convex*. By a remark of Vincent Thilliez [**29**] this condition can be replaced by a simpler one of (M_p) to be strongly logarithmically convex.

(2.6) (*see* [**22**, Example 3.1. (2)]) Let $\mathcal{R}$ be a polynomially bounded o–minimal structure on the real field. For compact $K \subset \mathbb{R}^n$, let $\mathcal{C}_K$ denote the collection of all C^∞–functions on K that are definable in $\mathcal{R}$. Now fix, for each $n \geq 1$, an arbitrary subcollection $\mathcal{D}_n$ of $\mathcal{C}_{I^n}$, where $I^n = [-1, 1]^n$, which is closed with respect to taking partial derivatives and contains all polynomials. Let $\mathcal{R}_\mathcal{D}$ be the smallest structure on the real field containing the graphs of all functions $f : \mathbb{R}^n \to \mathbb{R}$ (for any n) that are the restrictions to I^n of functions from $\mathcal{C}_{I^n}$ and $f(x) = 0$ if $x \notin I^n$. Then by [**13**] and [**22**, Theorems 5.2 and 5.4] the structure $\mathcal{R}_\mathcal{D}$ is o–minimal and polynomially bounded.

Let $\mathfrak{S}$ be an o–minimal structure on $(\mathbb{R}, +, \cdot)$ that is polynomially bounded. It is known (*see e.g.* [**10**]) that

LEMMA 2.7. *If $A \in \mathfrak{S}_n$ then the distance function* $\mathrm{dist}(\cdot, A) : \mathbb{R}^n \to \mathbb{R}$ *defined by* $\mathrm{dist}(x, A) = \inf_{y \in A} |x - y|$ *belongs to* $\mathfrak{S}$.

By 4.14 (2) of [**5**] we have

LEMMA 2.8. *[ŁOJASIEWICZ INEQUALITY] If $A \in \mathfrak{S}_n$ is compact and if $f : A \to \mathbb{R}$ is a continuous function belonging to $\mathfrak{S}$ then for every continuous function $g : A \to \mathbb{R}$ belonging to $\mathfrak{S}$ with $f^{-1}\{0\} \subset g^{-1}\{0\}$ there exist $N > 0$ and $C > 0$ such that $|g(x)|^N \leq C|f(x)|$ for all $x \in A$.*

3. Pluriregularity of sets in o–minimal structures. The nice geometric properties of polynomially bounded o–minimal structures $\mathfrak{S}$ that have been listed in Section 2 will permit one to prove pluriregularity of sets belonging to such structures. In the sequel we shall consider subsets $A \subset \mathbb{R}^n$ definable in $\mathfrak{S}$ that satisfy at some point $a \in \overline{\mathrm{int}\,A}$ the following

C^∞ CURVE SELECTING ASSUMPTION. *There is a C^∞–function $\gamma : [0, \epsilon] \to \mathbb{R}^n$ belonging to $\mathfrak{S}$ such that $a = \gamma(0)$ and $\gamma((0, \epsilon]) \subset \mathrm{int}\,A$.*

We have

PROPOSITION 3.1. *Let $\mathfrak{S}$ be a polynomially bounded o–minimal structure. Let $A \subset \mathbb{R}^n$ belong to $\mathfrak{S}$ and suppose that $a \in \overline{\mathrm{int}\,A}$. Suppose moreover that A satisfies at a the above C^∞ Curve Selecting Assumption. Then A is pluriregular at the point a.*

PROOF. Since $A \in \mathfrak{S}$, then also $\mathrm{int}\,A \in \mathfrak{S}$ (*see* [**5**]). Let $\gamma : [0, \epsilon] \to \mathbb{R}^n$ be a function fulfiling at a the C^∞ Curve Selecting Assumption. Since the boundary ∂A of A belongs to $\mathfrak{S}$ and since the composition of $\mathfrak{S}$–functions is a $\mathfrak{S}$–function (*see* [**5**]), by Lemma 2.7 the function

$$f : [0, \epsilon] \ni t \to \mathrm{dist}(\gamma(t), \mathbb{R}^n \setminus A) \in \mathbb{R}$$

belongs to $\mathfrak{S}$. By (S2), the function $g : [0, \epsilon] \ni t \to t \in \mathbb{R}$ also belongs to $\mathfrak{S}$. Clearly we have $f^{-1}\{0\} \subset g^{-1}\{0\}$. Hence by Łojasiewicz's inequality (Lemma 2.8), one can find a constant $C > 0$ and a positive integer N such that

(3.1) $$f(t) \geq Ct^N \quad \text{for } t \in [0, \epsilon].$$

Let $T_0^N \gamma$ denote the Taylor polynomial of γ at 0 of order N. Then by (3.1) we get

$$\mathrm{dist}(T_0^N \gamma(t), \mathbb{R}^n \setminus A) \geq Ct^N - |\gamma(t) - T_0^N \gamma(t)| \geq Ct^N - o(t^N)$$

as $t \in [0, \epsilon]$ with $t \to 0$. Hence we can find $\delta > 0$ such that

$$\mathrm{dist}(T_0^N \gamma(t), \mathbb{R}^n \setminus A) > 0 \quad \text{for } t \in (0, \delta].$$

It follows that the values of the polynomial map $h(t) := T_0^N \gamma(t)$ lie in $\mathrm{int}\,A$ as $t \in (0, \delta]$. Take now the extremal function V_A associated with the set A. By

definition $V_A = 0$ in int A. It is also known (*see e.g.* [**9**]) that any analytic arc $l : [0, \delta] \ni t \to l(t) \in \mathbb{C}^n$ is not plurithin at any point of its graph. Therefore we get

$$V_A^*(a) = V_A^*(h(0)) = \limsup_{t \to 0, t > 0} V_A^*(h(t)) = \limsup_{t \to 0, t > 0} V_A(h(t)) = 0.$$

In other words, the function V_A is continuous at a as claimed. $\square$

If $\mathfrak{S}$ is the Rolin–Speissegger–Wilkie structure described in (2.5) or else any structure described in (2.6), we have the following (*see* [**22**, Lemma 5.3])

LEMMA 3.2. *[Curve Selection Lemma] Let $A \subset \mathbb{R}^n$ be definable in $\mathfrak{S}$ and let $a \in \overline{\text{int } A}$. Then there exists $\epsilon > 0$ and a C^∞ map $\gamma = (\gamma_1, \ldots, \gamma_n) : [0, \epsilon] \to \mathbb{R}^n$ definable in $\mathfrak{S}$, with γ_j belonging to the quasianalytic class $C_{[-\epsilon,\epsilon]}(M_p)$, such that $a = \gamma(0)$ and $\gamma((0, \epsilon]) \subset \text{int } A$.*

Hence by Proposition 3.1 we get

COROLLARY 3.3. *Let $\mathfrak{S}$ be an o–minimal structure defined in (2.5) or (2.6). Let $A \subset \mathbb{R}^n$ be a definable set in $\mathfrak{S}$. Then A is pluriregular at every point of $\overline{\text{int } A}$.*

REMARK 3.4. If $E \subset \mathbb{C}^n$ is compact, by [**27**, Prop. 2.13] pluriregularity of E at every point of E implies continuity of the extremal function V_E in the whole space $\mathbb{C}^n$.

Similarly to results of [**20**], one can also prove the following

PROPOSITION 3.5. *Let A be a fat ($A \subset \overline{\text{int } A}$) subset of $\mathbb{C}^n$ that is definable (as a subset of $\mathbb{R}^{2n}$) in a structure of Corollary 3.3. Then A is pluriregular (in the sense of $\mathbb{C}^n$) at every point $a \in \overline{\text{int } A}$.*

PROOF. By Corollary 3.3 the set A is pluriregular in $\mathbb{C}^{2n}$ at every point $a \in \overline{\text{int } A} \subset \mathbb{R}^{2n}$. Hence by [**20**, Lemma 7] A is pluriregular at a in $\mathbb{C}^n$. $\square$

EXAMPLE 3.6. Let $A = \{z \in \mathbb{C}^n : |h_1(z)| < 1, \ldots, |h_m(z)| < 1\}$, where $h_j(z) = p_j(z) + iq_j(z)$ with real functions p_j and q_j that belong to a fixed structure of Corollary 3.3 ($j = 1, \ldots, m$), be nonempty. Then A is pluriregular at every point of $\bar{E}$.

The property of pluriregularity of fat subanalytic subsets E of $\mathbb{R}^n$ established in [**20**] was essentially strengthened in [**14**] where it is shown (with the aid of the Hironaka Rectilinearization Theorem) that if $E = \overline{\text{int } E}$ is compact then V_E is *Hölder continuous* on E, *i.e.* it satisfies the condition

$$(3.2) \qquad V_E(z) \le M(\text{dist}(z, E))^m \quad \text{for } z \in \mathbb{C}^n \quad \text{with } \text{dist}(z, E) \le 1$$

for some positive constants M and m. (Then by Błocki's argument (*see e.g.* [**1**, Remark on p. 213]), V_E has to be Hölder continuous on the whole space $\mathbb{C}^n$.) Since for a compact set $E \subset \mathbb{C}^n$ we have

$$
\begin{aligned}
V_E(z) = \sup\{(1/\deg P)\log|P(z)| : \; &P \in \mathbb{C}[z_1,\ldots,z_n], \\
&\deg P \geq 1, \; \sup|P|(E) \leq 1\}
\end{aligned}
$$
(3.3)

(*see* [**27**, Theorem 4.12]), by Cauchy's Integral Formula, from (3.2) one easily derives that E admits (global) *Markov's Inequality* for the derivatives of polynomials in n variables (*see* Question 3.8 below). Consequently, one can construct (in a relatively easy way) a continuous linear operator extending C^∞ Whitney jets on $\bar{E}$ to C^∞–functions on the whole space $\mathbb{R}^n$ (*see* [**15**]). For other applications of Markov Inequality in differential analysis we refer the reader to [**21**].

It follows from the proof of Proposition 3.1 and Lemma 3.2 (*see* [**14**, Section 4]) that if $A = \overline{\operatorname{int} A}$ is a compact subset of $\mathbb{R}^n$ that is definable in a structure $\mathfrak{S}$ of (2.5) or (2.6) then for every point a of A there exist constants $M > 0$, $m > 0$ and $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$

$$
V_A(z) \leq M\delta^m \quad \text{if } z \in \mathbb{C}^n, \; |z - a| < \delta.
$$
(3.4)

Actually, in order to get (3.4) at a point $a \in \overline{\operatorname{int} A}$ it is sufficient to know that there exists a curve $\gamma \in \mathcal{C}^k([0, \epsilon])$ ($k \geq 1$) in $\mathbb{R}^n$ such that

$(\vee)$ $\gamma(0) = a$, $\gamma((0, \epsilon]) \subset \operatorname{int} A$ *and* $\operatorname{dist}(\gamma(t), \mathbb{R}^n \setminus A) \geq Ct^N$, $0 \leq t \leq \epsilon$,

with some positive constants C and N, where $N \leq k$. We note that a similar sufficient condition for the Hölder continuity of V_E has been given by Siciak [**28**, Prop. 7.6]. The Curve Selection Lemma 3.2, together with Łojasiewicz's Inequality (Lemma 2.8), yields property $(\vee)$ in the o–minimal structures defined in (2.5) and (2.6). It is a problem suggested by the referee of finding other (more general) o–minimal structures that admit property $(\vee)$.

By (3.4) we get the (local) Markov Inequality

COROLLARY 3.7. *If A is a definable set in a structure $\mathfrak{S}$ of (2.5) or (2.6) then for every point $a \in \overline{\operatorname{int} A}$ there exist constants $K > 0$ and $r > 0$ such that we have*

$$
|P^{(\alpha)}(a)| \leq K(\deg P)^{r|\alpha|} \sup|p|(A)
$$

for any polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ and any $\alpha \in \mathbb{N}_0^n$.

As in the case of subanalytic sets, one can put the following

QUESTION 3.8. *If A is a compact definable set of Corollary 3.7, can the constants M, m and δ_0 of property (3.4) be chosen uniformly on A?*

This question has been recently answered in the affirmative by Pierzchała [**16**] (*see also* [**17**]). Consequently, the set A of Corollary 3.7 admits a global Markov Inequality.

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