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HARMONIC BERGMAN KERNEL FOR SOME BALLS

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Abstract. We treat the complex harmonic function on the N_p–ball which is defined by the N_p–norm related to the Lie norm. As a subspace, we treat Hardy spaces and consider the Bergman kernel on those spaces. Then, we try to construct the Bergman kernel in a concrete form in 2–dimensional Euclidean space.

Introduction. In [**2**], [**4**], [**6**] and [**7**], we studied holomorphic functions and analytic functionals on the N_p–ball in the complex Euclidean space $\mathbf{C}^{n+1}$, $n \geq 2$, and in [**2**], we expressed the Bergman kernel for a Hardy space on the N_p–ball by a double series by using of homogeneous harmonic extended Legendre polynomials. The closed form is known only for $p = 2$ and ∞. In the 2–dimensional case, we can calculate the coefficients of the double series expansion ([**3**]). However, even if we restrict our consideration to the 2–dimensional case, it is hard to express the Bergman kernel in a closed form.

In this paper, we mainly treat complex harmonic functions on the N_p–balls and determine the "harmonic" Bergman kernel by an infinite sum (Theorem 3.1). Then we represent the harmonic Bergman kernel more explicitly for the 2–dimensional N_p–ball (Theorem 3.2) and represent it in a concrete form for $p = 1, 2$ and ∞.

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1. N_p–ball.

1.1. *N_p–norm.*
First we review the definition of the N_p–balls in $\mathbf{C}^{n+1}, n = 0, 1, 2, \cdots$.

For $z = (z_1, z_2, \cdots, z_{n+1})$, let

$$L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}}$$

be the Lie norm on $\mathbf{C}^{n+1}$, where $z\cdot w = z_1w_1 + z_2w_2 + \cdots + z_{n+1}w_{n+1}$, $z^2 = z\cdot z$ and $\|z\|^2 = z\cdot \overline{z}$. For $p \in \mathbf{R}$, consider the function

$$N_p(z) = \left(\frac{1}{2}\left(L(z)^p + (|z^2|/L(z))^p\right)\right)^{1/p}.$$

If $p \geq 1$, then $N_p(z)$ is a norm on $\mathbf{C}^{n+1}$ (see [**1**] or [**8**]). Note that $N_2(z) = \|z\|$ is the complex Euclidean norm, $N_1(z) = \sqrt{(\|z\|^2 + |z^2|)/2} = L^*(z)$ is the dual Lie norm and $L(z) = \lim_{p\to\infty} N_p(z)$.

In $\mathbf{C}$, $N_p(z) = |z|$ for all $p \in \mathbf{R}$. In $\mathbf{C}^2$, $L(z)$ and $M(z) = |z^2|/L(z)$ are reduced to

$$L(z) = \max\{|z_1 \pm iz_2|\}, \quad M(z) = \min\{|z_1 \pm iz_2|\}, \tag{1}$$

and we have

$$N_p(z) = \left(\frac{|z_1 + iz_2|^p + |z_1 - iz_2|^p}{2}\right)^{1/p}.$$

Thus the N_p–norm is equivalent to the L_p–norm $\|\cdot\|_p$, and the Lie norm $L(z)$ to the supremum norm $\|\cdot\|_\infty$ in $\mathbf{C}^2$: Noting that

$$\begin{aligned}&\|w\|_p = (|w_1|^p + |w_2|^p + \cdots + |w_{n+1}|^p)^{1/p}, \quad p \geq 1,\\ &\|w\|_\infty = \sup\{|w_j|; j = 1, \cdots, n+1\},\end{aligned}$$

the N_p–norm (resp., the Lie norm) is another generalization of the 2–dimensional L_p–norm (resp., the supremum norm).

1.2. *A relation between the N_p–norms and the Tchebycheff polynomials.* The Tchebycheff polynomial $T_k(x)$ of degree k is defined by

$$T_k(x) = \frac{(x + i\sqrt{1-x^2})^k + (x - i\sqrt{1-x^2})^k}{2}.$$

We define the homogeneous extended Tchebycheff polynomial of degree $2k$ in $\mathbf{C}^{n+1}$ by

$$\begin{aligned}\tilde{T}_{k,n}(z,w) &= (\sqrt{z^2})^k(\sqrt{w^2})^k T_k\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right)\\ &= \frac{(z\cdot w + i\sqrt{z^2w^2 - (z\cdot w)^2})^k + (z\cdot w - i\sqrt{z^2w^2 - (z\cdot w)^2})^k}{2}.\end{aligned}$$

Further extend the parameter k to $\alpha \in \mathbf{R}$ and consider the function

$$\tilde{T}_{\alpha,n}(z,w) = \frac{(z\cdot w + i\sqrt{z^2w^2 - (z\cdot w)^2})^\alpha + (z\cdot w - i\sqrt{z^2w^2 - (z\cdot w)^2})^\alpha}{2}.$$

Then $\tilde{T}_{\alpha,n}(z,z) = (z^2)^\alpha$, $\tilde{T}_{\alpha,n}(z,w) = (z\cdot w)^\alpha$ if $z^2 = 0$ or $w^2 = 0$,

$$\tilde{T}_{\alpha,n}(z,\overline{z}) = \frac{1}{2}\left(L(z)^{2\alpha} + \left(\frac{|z^2|}{L(z)}\right)^{2\alpha}\right).$$

Thus the function $N_p(z)$ is represented by

$$N_p(z) = \left(\tilde{T}_{p/2,n}(z, \overline{z})\right)^{\frac{1}{p}}, \quad z \in \mathbf{C}^{n+1}.$$

Therefore, if $p = 2k$ is a positive even natural number, the last formula expresses the norm N_p in terms of the Tchebycheff polynomial T_p of degree p. Hence N_{2k}^{2k} is a homogenous polynomial of $2(n+1)$ real variables of degree $2k$.

1.3. *Hardy space on the N_p–ball.*

We define the N_p–ball $\tilde{B}_p^{n+1}(r)$ by

$$\tilde{B}_p^{n+1}(r) = \left\{z \in \mathbf{C}^{n+1}; N_p(z) < r\right\}, \quad p \geq 1,$$

$$\tilde{B}^{n+1}(r) = \bigcap_{p \geq 1} \tilde{B}_p^{n+1}(r) = \left\{z \in \mathbf{C}^{n+1}; L(z) < r\right\}.$$

Note that $\tilde{B}_1^{n+1}(r)$ is the dual Lie ball, $\tilde{B}_2^{n+1}(r)$ is the complex Euclidean ball, and $\tilde{B}^{n+1}(r)$ is the Lie ball of radius r in $\mathbf{C}^{n+1}$.

We denote by $\mathcal{O}(\tilde{B}_p^{n+1}(r))$ the space of holomorphic functions on $\tilde{B}_p^{n+1}(r)$ equipped with the topology of uniform convergence on compact sets. Put

$$H\mathcal{O}(\tilde{B}_p^{n+1}(r)) = \left\{ f \in \mathcal{O}(\tilde{B}_p^{n+1}(r)); \int_{\tilde{B}_p^{n+1}[r]} |f(w)|^2 dV_{p,r}(w) < \infty \right\},$$

where $dV_{p,r}(w)$ is the normalized Lebesgue measure on $\tilde{B}_p^{n+1}(r)$.

Let Δ be the complex Laplacian:

$$\Delta_z \equiv \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \cdots + \partial^2/\partial z_{n+1}^2.$$

Put

$$H\mathcal{O}_\Delta(\tilde{B}_p^{n+1}(r)) = \left\{ f \in H\mathcal{O}(\tilde{B}_p^{n+1}(r))\,;\, \Delta_z f(z) = 0 \right\}.$$

When a function f satisfies $\Delta_z f(z) = 0$, f is called a complex harmonic function. In the following, we call the Bergman kernel on $H\mathcal{O}_\Delta(\tilde{B}_p^{n+1}(r))$ the "harmonic" Bergman kernel.

Note that for the complex plane, every N_p–ball of radius r is equal to the disk $D(r) = \{z; |z| < r\}$, and the Bergman kernel $B_r^1(z, w)$ for $H\mathcal{O}(D(r))$ is given by

$$B_r^1(z, w) = \sum_{k=0}^{\infty} (k+1) \left(\frac{z}{r}\right)^k \left(\frac{\overline{w}}{r}\right)^k = \frac{r^4}{(r^2 - z\overline{w})^2}.$$

Since $H\mathcal{O}_\Delta(D(r)) = \{a + bz\}$, the Bergman kernel $B_{r,\Delta}^1(z, w)$ for $H\mathcal{O}_\Delta(D(r))$ is given by

$$B_{r,\Delta}^1(z, w) = 1 + 2\frac{z}{r}\frac{\overline{w}}{r}.$$

For $n \geq 1$, in general, the exsistance of the Bergman kernel $B^{n+1}_{p,r}(z,w)$ on $H\mathcal{O}(\tilde{B}^{n+1}_p(r))$ is known, but it is difficult to express it in a concrete form. However for $H\mathcal{O}(\tilde{B}^{n+1}_2(r))$ and for $H\mathcal{O}(\tilde{B}^{n+1}(r))$, the following are known:

$$B^{n+1}_{2,r}(z,\overline{w}) = \frac{r^{2n+4}}{(r^2 - z\cdot w)^{n+2}},\ n \geq 0,$$

$$B^{n+1}_r(z,\overline{w}) \equiv B^{n+1}_{\infty,r}(z,\overline{w}) = \frac{r^{4n+4}}{(r^4 - 2r^2 z\cdot w + z^2 w^2)^{n+1}},\ n \geq 0. \tag{2}$$

For a proof of (2) for $n \geq 2$, see e.g. [**5**].

Put

$$Z_1 = z_1 + iz_2, Z_2 = z_1 - iz_2, \quad D_2(r) = \{(Z_1, Z_2); |Z_1| < r, |Z_2| < r\}.$$

Since $\tilde{B}^2(r) \cong D_2(r)$, we can find the Bergman kernels $B^2_r(z,w)$ and $B^2_{r,\Delta}(z,w)$ by using the results in 1–dimensional case. Let us check them here.

Since the volume of $D_2(r)$ is $V(D_2(r)) = \pi^2 r^4$, the Bergman kernel for $H\mathcal{O}(D_2(r))$ is as follows:

$$\begin{aligned} B^2_r(Z,W) &= \sum_{k_1,k_2} (k_1+1)(\frac{Z_1}{r}\frac{\overline{W_1}}{r})^{k_1}(k_2+1)(\frac{Z_2}{r}\frac{\overline{W_2}}{r})^{k_2} \\ &= \frac{r^4}{(r^2 - Z_1\overline{W_1})^2}\frac{r^4}{(r^2 - Z_2\overline{W_2})^2}. \end{aligned}$$

Note that this is equivalent to

$$B^2_r(z,w) = \frac{r^8}{(r^4 - 2r^2 z\cdot\overline{w} + z^2\overline{w}^2)^2}. \tag{3}$$

Therefore, since we have (1) and $\tilde{B}^2(r) \cong D_2(r)$, (3) is the Bergman kernel for $H\mathcal{O}(\tilde{B}^2(r))$.

Since $\Delta_z = 4\ \frac{\partial^2}{\partial Z_1 \partial Z_2}$, we have $H\mathcal{O}_\Delta(D_2(r)) = \{a + \sum b_1 Z_1^k + \sum c_k Z_2^k\}$. Thus the Bergman kernel $B^2_{r,\Delta}(Z,W)$ for $H\mathcal{O}_\Delta(D_2(r))$ is as follows:

$$\begin{aligned} B^2_{r,\Delta}(Z,W) &= 1 + \sum_{k_1=1}^{\infty}(k_1+1)(\frac{Z_1\overline{W_1}}{r^2})^{k_1} + \sum_{k_2=1}^{\infty}(k_2+1)(\frac{Z_2\overline{W_2}}{r^2})^{k_2} \\ &= \left(1 - \frac{Z_1\overline{W_1}Z_2\overline{W_2}}{r^4}(2 - \frac{Z_1\overline{W_1}}{r^2})(2 - \frac{Z_2\overline{W_2}}{r^2})\right) B^2_r(Z,W) \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
B^2_{r,\Delta}(z,w) &= \left(1-\frac{z^2\overline{w}^2}{r^4}(4-4\frac{z\cdot\overline{w}}{r^2}+\frac{z^2\overline{w}^2}{r^4})\right)B^2_r(z,w), \quad z,w\in \mathbf{C}^2 \\
&= \frac{r^8-z^2\overline{w}^2(4r^4-4r^2z\cdot\overline{w}+z^2\overline{w}^2)}{(r^4-2r^2z\cdot\overline{w}+z^2\overline{w}^2)^2}, \quad z,w\in\mathbf{C}^2. \qquad (4)
\end{aligned}
$$

Since $\tilde{B}^2(r)\cong D_2(r)$, (4) is the harmonic Bergman kernel for $H\mathcal{O}_\Delta(\tilde{B}^2(r))$. After we review some results on spherical harmonic functions, we treat harmonic Bergman kernel again in the last section.

2. Spherical harmonic functions.

From now on, we consider $n\ge 1$.

Let $P_{k,n}(t)$ be the orthogonal polynomial of degree k whose highest coefficient is positive and determined by

$$\int_{-1}^1 P_{k,n}(t)P_{l,n}(t)(1-t^2)^{(n-2)/2}dt = \frac{\sqrt{\pi}\Gamma(n/2)}{\Gamma((n+1)/2)N(k,n)}\delta_{kl},$$

where $N(k,n)$ is the dimension of the space of homogeneous harmonic polynomials of degree k in $\mathbf{C}^{n+1}$:

$$N(0,n)=1,\quad N(k,n)=\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!},\ k=1,2,\cdots.$$

We call $P_{k,n}(t)$ the Legendre polynomial of degree k and of dimension $n+1$. We define the homogeneous harmonic extended Legendre polynomial $\tilde{P}_{k,n}(z,w)$ of degree k and of dimension $n+1$ by

$$\tilde{P}_{k,n}(z,w)=(\sqrt{z^2})^k(\sqrt{w^2})^kP_{k,n}(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}).$$

Note that $\tilde{P}_{k,n}(z,w)=\tilde{P}_{k,n}(w,z)$ and $\Delta_z\tilde{P}_{k,n}(z,w)=0$.

When $n=1$, $\tilde{P}_{k,1}(z,w)=\tilde{T}_{k,1}(z,w)$ and we denote it by $\tilde{T}_k(z,w)$.

For the Bergman kernel for $H\mathcal{O}(\tilde{B}^{n+1}_p(r))$, we proved the following theorem for $n\ge 2$ in [**2**] and for $n=1$ in [**3**]:

Theorem 2.1. *The Bergman kernel $B^{n+1}_{p,r}(z,w)$ for $H\mathcal{O}(\tilde{B}^{n+1}_p(r))$ is given as follows:*

$$B^{n+1}_{p,r}(z,w)=\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(\beta^{n+1,p}_{k,l,r})^{-1}(z^2)^l(\overline{w}^2)^l\tilde{P}_{k-2l,n}(z,\overline{w}),\quad z,w\in\tilde{B}^{n+1}_p(r)$$

where

$$\beta^{n+1,p}_{k,l,r}=\int_{\tilde{B}^{n+1}_p[r]}|(\zeta^2)^l\tilde{P}_{k-2l,n}(\zeta,\omega)|^2dV_{p,r}(\zeta),\quad \omega\in S^n,$$

and $dV_{p,r}(\zeta)$ *denotes the normalized Lebesgue measure on* $\tilde{B}_p^{n+1}(r)$ *and* S^n *is the real unit sphere in* $\mathbf{R}^{n+1}$.

For the Bergman kernel $B_r^{n+1}(z,w)$, the following formula is known:

Formula 2.2. *We have*

$$\frac{r^{4n+4}}{(r^4-2r^2z\cdot w+z^2w^2)^{n+1}}=\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}a_{k,k-2l}^n(z^2/r^2)^l(w^2/r^2)^l\tilde{P}_{k-2l,n}(z/r,w/r)$$

where

$$a_{k,k-2l}^n=\frac{2\Gamma(l+\frac{n+3}{2})\Gamma(k+n-l+1)N(k-2l,n)}{(n+1)!l!\Gamma(k+\frac{n+1}{2}-l)}.$$

Further in the 2–dimensional case, we calculated the coefficients $\beta_{k,l,r}^{2,p}$ in [**3**], and Theorem 2.1 is restated as follows:

Theorem 2.3. *The Bergman kernel* $B_{p,r}^2(z,w)$ *for* $H\mathcal{O}(\tilde{B}_p^2(r))$ *is as follows*:

$$\begin{aligned}B_{p,r}^2(z,w)&=\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}\frac{N(k-2l,1)\Gamma(\frac{2}{p})^2\Gamma(\frac{2k+4}{p}+1)}{\Gamma(\frac{4}{p}+1)\Gamma(\frac{2k-2l+2}{p})\Gamma(\frac{2l+2}{p})2^{\frac{2k}{p}}r^{2k}}(z^2)^l(\overline{w}^2)^l\tilde{T}_{k-2l}(z,\overline{w}),\\&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{\Gamma(\frac{2}{p})^2\Gamma(\frac{2k+4}{p}+1)}{\Gamma(\frac{4}{p}+1)\Gamma(\frac{2k-2l+2}{p})\Gamma(\frac{2l+2}{p})2^{\frac{2k}{p}}r^{2k}}(X_1)^{k-l}(X_2)^l,\end{aligned}$$

where

$$X_1=z\cdot\overline{w}+i\sqrt{z^2\overline{w}^2-(z\cdot\overline{w})^2},X_2=z\cdot\overline{w}-i\sqrt{z^2\overline{w}^2-(z\cdot\overline{w})^2}.$$

Note that

$$\begin{aligned}\lim_{p\to\infty}\beta_{k,l,r}^{2,p}&=\lim_{p\to\infty}\frac{\Gamma(\frac{4}{p}+1)}{N(k-2l,1)\Gamma(\frac{2}{p})^2}\frac{\Gamma(\frac{2k-2l+2}{p})\Gamma(\frac{2l+2}{p})}{\Gamma(\frac{2k+4}{p}+1)}2^{\frac{2k}{p}}r^{2k}\\&=\frac{r^{2k}}{N(k-2l,1)(k-l+1)(l+1)}.\end{aligned}$$

For convenience, we introduce the following notation:

$$X_1=z\cdot\overline{w}+i\sqrt{z^2\overline{w}^2-(z\cdot\overline{w})^2},X_2=z\cdot\overline{w}-i\sqrt{z^2\overline{w}^2-(z\cdot\overline{w})^2}. \tag{5}$$

Then we have

$$X_1+X_2=2z\cdot\overline{w},\quad X_1X_2=z^2\overline{w}^2,\quad (1-X_1)(1-X_2)=1-2z\cdot\overline{w}+z^2\overline{w}^2, \tag{6}$$

$$\tilde{T}_{k,n}(z,\overline{w})=\frac{1}{2}\left(X_1^k+X_2^k\right).$$

3. Harmonic Bergman kernel.

Any harmonic function f in a neighborhood of 0 can be expanded into the homogenoeus harmonic polynomials:

$$f(z) = \sum_{k=0}^{\infty} f_k(z),$$

for a sufficiently small z, where f_k is the homonenoeous harmonic polynomial of degree k defined by

$$f_k(z) = N(k,n) \int_{S^n} f(\rho\omega)\tilde{P}_{k,n}(z, \omega/\rho) d\omega,$$

for a sufficiently small ρ and $d\omega$ is the normalized invariant measure on S^n. By Proposition 2.1 in [**2**], for $f \in H\mathcal{O}_\Delta(\tilde{B}_p^{n+1}(r))$, we have

$$\int_{\tilde{B}_p^{n+1}[r]} \sum_{k=0}^{\infty} f_k(w) \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l (\overline{w}^2)^l \tilde{P}_{k-2l,n}(\zeta, \overline{w}) dV_{p,r}(w), \quad \zeta \in \tilde{B}_p^{n+1}(r)$$
$$= \sum_{k=0}^{\infty} \int_{\tilde{B}_p^{n+1}[r]} f_k(w) \tilde{P}_{k,n}(\zeta, \overline{w}) dV_{p,r}(w).$$

Thus by Theorem 2.1, we have the following theorem:

Theorem 3.1. *Let $n = 1, 2, \ldots.$ The harmonic Bergman kernel $B^{n+1}_{p,r,\Delta}(z,w)$ on $H\mathcal{O}_\Delta(\tilde{B}_p^{n+1}(r))$ is given as follows:*

$$B^{n+1}_{p,r,\Delta}(z,w) = \sum_{k=0}^{\infty} (\beta^{n+1,p}_{k,0,r})^{-1} \tilde{P}_{k,n}(z, \overline{w}), \quad z, w \in \tilde{B}_p^{n+1}(r).$$

Similary, by Theorem 2.3, we have the following theorem:

Theorem 3.2. *The harmonic Bergman kernel $B^2_{p,r,\Delta}(z,w)$ on $H\mathcal{O}_\Delta(\tilde{B}_p^2(r))$ is given as follows:*

$$\begin{aligned} B^2_{p,r,\Delta}(z,w) &= \sum_{k=0}^{\infty} \frac{N(k,1)\Gamma(\frac{2}{p})\Gamma(\frac{2k+4}{p}+1)}{\Gamma(\frac{4}{p}+1)\Gamma(\frac{2k+2}{p})2^{\frac{2k}{p}} r^{2k}} \tilde{T}_k(z, \overline{w}), \\ &= 1 + \sum_{k=1}^{\infty} \frac{\Gamma(\frac{2}{p})\Gamma(\frac{2k+4}{p}+1)}{\Gamma(\frac{4}{p}+1)\Gamma(\frac{2k+2}{p})2^{\frac{2k}{p}} r^{2k}} \Big((X_1)^k + (X_2)^k\Big) \\ &= F_p(X_1/(2^{2/p}r^2)) + F_p(X_2/(2^{2/p}r^2)) - 1, \end{aligned}$$

where X_1 and X_2 are given by (5) *and $F_p(X)$ is the function defined by*

$$F_p(X) = \sum_{k=0}^{\infty} \frac{\Gamma(\frac{2}{p})\Gamma(\frac{2k+4}{p}+1)}{\Gamma(\frac{4}{p}+1)\Gamma(\frac{2k+2}{p})} X^k. \tag{7}$$

In order to find a closed form of $B^2_{p,r,\Delta}(z,w)$ it is sufficient to find a closed form of $F_p(X)$.

In the following, we set

$$\begin{aligned} a_p &= \frac{X_1}{2^{2/p}} = \frac{z\cdot\overline{w} + i\sqrt{z^2\overline{w}^2 - (z\cdot\overline{w})^2}}{2^{2/p}}, \\ b_p &= \frac{X_2}{2^{2/p}} = \frac{z\cdot\overline{w} - i\sqrt{z^2\overline{w}^2 - (z\cdot\overline{w})^2}}{2^{2/p}}. \end{aligned}$$

3.1. *In case of the Lie ball.*
In Section 1, we have already shown $B^2_{r,\Delta}(z,w)$ is given by (4). Here we derive it as a corollary of Theorem 3.2.

Since

$$F_p(X) = \sum_{k=0}^{\infty} \frac{(k+1)\Gamma(\frac{2}{p}+1)\Gamma(\frac{2k+4}{p}+1)}{\Gamma(\frac{4}{p}+1)\Gamma(\frac{2k+2}{p}+1)} X^k,$$

by (7), we have

$$F_\infty(X) = \sum_{k=0}^{\infty}(k+1)X^k = \frac{1}{(1-X)^2}.$$

Therefore, we have

$$\begin{aligned} B^2_{r,\Delta}(z,w) &= \sum_{k=0}^{\infty} \frac{(k+1)N(k,1)}{r^{2k}} \tilde{T}_k(z,\overline{w}) \\ &= F_\infty(a_\infty/r^2) + F_\infty(b_\infty/r^2) - 1 \\ &= \frac{1 - \frac{z^2}{r^2}\frac{\overline{w}^2}{r^2}(4 - 4(\frac{z}{r}\cdot\frac{\overline{w}}{r}) + \frac{z^2}{r^2}\frac{\overline{w}^2}{r^2})}{\left(1 - 2\frac{z}{r}\cdot\frac{\overline{w}}{r} + \frac{z^2}{r^2}\frac{\overline{w}^2}{r^2}\right)^2}. \end{aligned}$$

3.2. *In case of the Euclidean ball.*
By Theorem 3.2, we have

$$B^2_{2,r,\Delta}(z,w) = \sum_{k=0}^{\infty} \frac{(k+2)!N(k,1)}{2^k k! 2 r^{2k}} \tilde{T}_k(z,\overline{w}) = F_2(a_2/r^2) + F_2(b_2/r^2) - 1.$$

Since

$$\sum_{k=0}^{\infty}(k+2)(k+1)x^k = \frac{d^2}{dx^2}\left(\frac{x^2}{1-x}\right) = \frac{2}{(1-x)^3},$$

$$F_2(X) = \frac{1}{(1-X)^3}.$$

Thus the Bergman kernel $B^2_{2,r,\Delta}(z,w)$ for $H\mathcal{O}_\Delta(\tilde{B}^2_2(r))$ is given by

$$B^2_{2,r,\Delta}(z,w) = F_2(a_2/r^2) + F_2(b_2/r^2) - 1 = \frac{1}{(1-a_2/r^2)^3} + \frac{1}{(1-b_2/r^2)^3} - 1.$$

Then by (6), we have

$$B^2_{2,r,\Delta}(z,w) = \frac{Q_{2,r}\left(\frac{z}{\sqrt{2}r}\cdot\frac{\overline{w}}{\sqrt{2}r}, \frac{z^2}{2r^2}\frac{\overline{w}^2}{2r^2}\right)}{\left(1-2\frac{z}{\sqrt{2}r}\cdot\frac{\overline{w}}{\sqrt{2}r}+\frac{z^2}{2r^2}\frac{\overline{w}^2}{2r^2}\right)^3},$$

where $Q_{2,r}(\frac{z}{\sqrt{2}r}\cdot\frac{\overline{w}}{\sqrt{2}r}, \frac{z^2}{2r^2}\frac{\overline{w}^2}{2r^2})$ is the polynomial in $s=\frac{z}{\sqrt{2}r}\cdot\frac{\overline{w}}{\sqrt{2}r}=\frac{a_2+b_2}{2r^2}\in\mathbf{C}$ and $t=\frac{z^2}{2r^2}\frac{\overline{w}^2}{2r^2}=\frac{a_2b_2}{r^4}\in\mathbf{C}$ of degree 3 given by

$$\begin{aligned} Q_{2,r}(s,t) &= (1-b_2/r^2)^3+(1-a_2/r^2)^3-(1-a_2/r^2)^3(1-b_2/r^2)^3 \\ &= 1-9t+18ts-3t^2-12ts^2+6t^2s-t^3. \end{aligned}$$

3.3. *In case of the dual Lie ball.*
By Theorem 3.2, we have

$$\begin{aligned} B^2_{1,r,\Delta}(z,w) &= \sum_{k=0}^{\infty}\frac{N(k,1)(2k+4)(2k+3)(2k+2)}{24\cdot 2^{2k}r^{2k}}\tilde{T}_k(z,\overline{w}) \\ &= F_1(a_1/r^2)+F_1(b_1/r^2)-1. \end{aligned}$$

Since

$$\begin{aligned} \sum_{k=0}^{\infty}(2k+4)(2k+3)(2k+2)x^{2k+1} &= \left(\frac{x^4}{1-x^2}\right)^{(3)} \\ &= 24x(1-x^2)^{-1}+96x^3(1-x^2)^{-2}+120x^5(1-x^2)^{-3}+48x^7(1-x^2)^{-4} \\ &= 24x\frac{1+x^2}{(1-x^2)^4}, \\ F_1(X) = \sum_{k=0}^{\infty}(2k+4)(2k+3)(2k+2)X^k/24 &= \frac{1+X}{(1-X)^4}. \end{aligned}$$

Thus we have

$$B^2_{1,r,\Delta}(z,w) = F_1(a_1/r^2)+F_1(b_1/r^2)-1 = \frac{1+a_1/r^2}{(1-a_1/r^2)^4}+\frac{1+b_1/r^2}{(1-b_1/r^2)^4}-1.$$

By (6)

$$B^2_{1,r,\Delta}(z,w) = \frac{Q_{1,r}(\frac{z}{2r}\cdot\frac{\overline{w}}{2r}, \frac{z^2}{4r^2}\frac{\overline{w}^2}{4r^2})}{\left(1-2\frac{z}{2r}\cdot\frac{\overline{w}}{2r}+\frac{z^2}{4r^2}\frac{\overline{w}^2}{4r^2}\right)^4},$$

where $Q_{1,r}(s,t)$ is a polynomial in two complex variables s, t given by

$$\begin{aligned} Q_{1,r}(s,t) \quad = \quad & 1 + 2s - 24t + 60st + 4t^2 \\ & +18st^2 - 80s^2t - 4t^3 + 48st^3 - 24s^2t^2 + 40s^3t - t^4. \end{aligned}$$

References

1. Baran M., *Conjugate norms in C^n and related geometrical problems,* Dissertationes Mathematicae, **CCCLXXVII** (1998), 1–67.
2. Fujita K., *Bergman transformation for analytic functionals on some balls,* Microlocal Analysis and Complex Fourier Analysis, World Scientific publisher, 2002, 81–98.
3. ______, *Bergman kernel for the 2–dimensional balls,* to appear in Complex Var. Theory Appl.
4. Fujita K., Morimoto M., *On the double series expansion of holomorphic functions,* J. Math. Anal. Appl., **272** (2002), 335–348.
5. Hua L.K., *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains,* Moscow 1959, (in Russian); Translations of Math. Monographs, Vol. **6**, Amer. Math. Soc., Providence, Rhode Island, 1979.
6. Morimoto M., Fujita K., *Analytic functions and analytic functionals on some balls,* Proceedings of the Third ISAAC Congress, Kluwer Academic Publishers, 2003, 150–159.
7. ______, *Holomorphic functions on the Lie ball and related topics,* to appear in Proceedings of the ninth Finite or Infinite Dimensional Complex Analysis and Applications, Kluwer Academic Publisher, 2002, 33–44.
8. ______, *Between Lie norm and dual Lie norm,* Tokyo J. Math., **24** (2001), 499–507.

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