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SOME VERSION OF GOWERS' DICHOTOMY FOR BANACH SPACES

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Abstract. In this paper another version of Gowers' dichotomy for Banach spaces, involving topologies of special type on the Cartesian product of Banach spaces, is presented. These topologies are closely related to the game used by W. T. Gowers in his proof of the dichotomy.

1. Introduction. In this paper we will present some version of an important special case of general Gowers' dichotomy theorem, namely

Theorem 1.1. *A Banach space contains either an unconditional basic sequence or a HI (hereditarily indecomposable) subspace.*

First we will recall basic notions and introduce some notation.

Given a real Banach space E, by $B = B_E$ (resp. $S = S_E$) we denote the closed unit ball (resp. the unit sphere) in E. Given $x \in E$ and $r > 0$ we put $B(x, r) = x + rB$ and $S(x, r) = x + rS$.

Let E be a real Banach space with a basis $\{e_n\}_{n=1}^{\infty}$. Given a vector $x = \sum_{n=1}^{\infty} x_n e_n$, the support of x, written $\operatorname{supp} x$, is the set $\{n \in \mathbf{N} : x_n \neq 0\}$. Given two non-zero vectors $x, y \in E$ we write $x < y$, if $\max(\operatorname{supp} x) < \min(\operatorname{supp} y)$. We will also write $0 < x$ for any non-zero vector $x \in E$. A block basis is a sequence $a_1 < a_2 < \ldots$ of non-zero vectors, a block subspace – a closed subspace spanned by a block basis.

A basis $\{e_n\}$ is called unconditional if there exists a constant C such that for any sequence $\{a_n\}$ of scalars and any sequence $\{\varepsilon_n\}$ with each ε_n of absolute value one we have

$$\left\| \sum_{n=1}^{\infty} \varepsilon_n a_n e_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n e_n \right\|.$$

A space E is decomposable if it can be written as a direct sum $F+G$ with F and G infinite dimensional.

A space E is HI (hereditarily indecomposable) if no subspace G of E is decomposable.

All spaces considered here will be closed and infinite dimensional.

2. Some topologies on the Cartesian product of a Banach space. Let T be a topology on a Banach space E. By T° we denote a topology on $E\times E$ given by the formula

$$T^\circ = (T\times T_d)\cap(T_d\times T),$$

where T_d is the discrete topology on E.

Notice that the family of sets of the form $V=\bigcup_{n=1}^\infty V_n$, where for $n\geq 1$

$$V_n = \bigcup_{(x,y)\in V_{n-1}} \big(\{x\}\times(y+Z_{x,y})\big)\cup\left((x+Z_{x,y})\times\{y\}\right),$$

with $\{Z_{x,y}\}$ – neighbourhoods of the origin in (E,T), forms a basis of neighbourhoods of the origin in the topology T°.

Fix now a filtering family $\mathcal{L}$ of subspaces of E. Given a vector $x\in E$ put $\mathcal{L}_x=\{x+L,\ L\in\mathcal{L}\}$. Then the family $\{\mathcal{L}_x,\ x\in E\}$ forms a basis of neighbourhoods for some topology on E. Denote this topology by $T_{\mathcal{L}}$. Transforming slightly the form of the neighbourhood of the origin discussed above we get a basis of neighbourhoods of the origin in $T^\circ_{\mathcal{L}}$ consisting of the sets of the form

$$W=\bigcup_{n=1}^{\infty} W_n,\ \text{where } W_1=L_{0,0},\ \text{ and for } n\geq 1$$

$$(\mathbf{W})\qquad W_{2n}=\bigcup_{(x,y)\in W_{2n-1}}\left\{(x+z,y):\ z\in L_{x,y}\right\},$$

$$W_{2n+1}=\bigcup_{(x,y)\in W_{2n}}\{(x,y+z):\ z\in L_{x,y}\}$$

for some $L_{x,y}\in\mathcal{L}$.

In other words, an element of such a neighbourhood is a pair of finite sums of vectors taken from suitable subspaces:

$$(z_1+z_3+\cdots+z_{2n-1}, z_2+z_4+\cdots+z_{2n}),\ \text{ where } z_1\in L_{0,0},$$

$$z_{2i}\in L_{z_1+\cdots+z_{2i-1},z_2+\cdots+z_{2i-2}},\ z_{2i+1}\in L_{z_1+\cdots+z_{2i-1},z_2+\cdots+z_{2i}}$$

where all subspaces are taken from the family $\mathcal{L}$ and $n\in\mathbf{N}$.

In the next section we shall need a topology on $E\times E$ of a form similar to T°, but comparable to the norm topology $T_{\|\cdot\|}\times T_{\|\cdot\|}$. To achieve this we shall "thicken" $T^\circ_{\mathcal{L}}$ by adding balls of an arbitrarily small diameter on both axes of $E \times E$. We discuss this in detail below.

Given a vector topology T on E we define a topology $T^\bullet$ on $E \times E$ (examined in [**3**]) by the formula:

$$T^\bullet = (T \times T_{\|\cdot\|}) \cap (T_{\|\cdot\|} \times T).$$

An argumentation similar to the previous one shows that there is a basis of neighbourhoods of the origin in the topology $T^\bullet$ given by the family of sets of the form $V = \bigcup_{n=1}^\infty V_n$, where for $n \geq 1$

$$V_n = \bigcup_{(x,y) \in V_{n-1}} \big(B(x, \varepsilon_{x,y}) \times (y + Z_{x,y})\big) \cup \big((x + Z_{x,y}) \times B(y, \varepsilon_{x,y})\big),$$

where $\varepsilon_{x,y} > 0$, and $\{Z_{x,y}\}$ are neighbourhoods of the origin in (E, T).

As before, fix a filtering family $\mathcal{L}$ of subspaces of E. By $\mathcal{C}(\mathcal{L})$ denote the family of cylinders with axes from the family $\mathcal{L}$:

$$\mathcal{C}(\mathcal{L}) = \{C(L, \varepsilon),\ L \in \mathcal{L},\ \varepsilon > 0\}, \quad \text{where} \quad C(L, \varepsilon) = L + B(0, \varepsilon).$$

Let $T_{\mathcal{C}(\mathcal{L})}$ be the topology given by $\mathcal{C}(\mathcal{L})$ as a basis of neighbourhoods of the origin.

As in the previous case, there is a basis of neighbourhoods of the origin in $T^\bullet_{\mathcal{L}}$ consisting of sets of the form

$$U = \bigcup_{n=1}^\infty U_n, \quad \text{where } U_1 = C(L_{0,0}, \varepsilon_{0,0}), \quad \text{and for } n \geq 1$$

$$(\mathbf{U}) \qquad U_{2n} = \bigcup_{(x,y) \in U_{2n-1}} \{(x+z, y+w) :\ z \in C(L_{x,y}, \varepsilon_{x,y}),\ w \in B(0, \varepsilon_{x,y})\},$$

$$U_{2n+1} = \bigcup_{(x,y) \in U_{2n}} \{(x+w, y+z) :\ z \in C(L_{x,y}, \varepsilon_{x,y}),\ w \in B(0, \varepsilon_{x,y})\}$$

for some subspaces $L_{x,y} \in \mathcal{L}$ and scalars $\varepsilon_{x,y} > 0$.

In other words, an element of such a neighbourhood is a pair of finite sums of vectors, obviously more complicated than in the case of $T^\circ_{\mathcal{L}}$:

$$(z_1 + w_2 + z_3 + w_4 + \cdots + w_{2n-2} + z_{2n-1} + w_{2n}, w_1 + z_2 + w_3 + z_4 + \cdots + w_{2n-1} + z_{2n}),$$

where z_i are chosen from suitable cylinders, w_i – from suitable balls.

3. Application of the game theory. The definition of the neighbourhoods of the origin in the topologies considered before seemed to be complicated, but the construction of these neighbourhoods appears to be quite natural while using the game-theory language: neighbourhoods of the form (**W**) or (**U**) correspond to strategies of players in some two-player games which we shall define below.

Given a family $\mathcal{L}$ of closed infinite dimensional subspaces of E and a set σ of finite sequences of vectors of E we define a two-player game of type I as follows: in the first step, the first player S chooses a subspace $M_1 \in \mathcal{L}$ and then the second player P picks a vector $z_1 \in M_1$. In the second step, the player S chooses a subspace $M_2 \in \mathcal{L}$, then the player P – a vector $z_2 \in M_2$. They continue in this way, alternately choosing subspaces (the player S) and vectors (the player P) in them.

The player P wins the game if $(z_1, \ldots, z_n) \in \sigma$ for some $n \in \mathbf{N}$. If it does not happen for any $n \in \mathbf{N}$, the player S wins.

A strategy of the player S is a method of choosing subspaces, defined for all possible moves of the player P, ie. a strategy is given by a function that for every $n \geq 1$ associates with every possible sequence of subspaces and vectors $M_1, z_1, M_2, z_2, \ldots, M_n, z_n$ chosen by players S and P in the first n steps of some game, a subspace M_{n+1} to be chosen by the player S in the $(n+1)-$th step. A strategy of the player P is defined analogously.

A strategy of the player S (resp. P) is winning if applying it the player S (resp. P) wins every game.

Assume now that the family $\mathcal{L}$ is filtering. We will explain now how strategies of the player S correspond to the $(\mathbf{W})$–form neighbourhoods of the origin in $T^{\circ}_{\mathcal{L}}$.

Given a strategy $\mathcal{S}$ of the player S define the neighbourhood of the form $(\mathbf{W})$ by induction. Put $W_0 = M_1$. With $W_0, \ldots, W_n$ defined, construct W_{n+1} by defining for every $(x, y) \in W_n$ the subspace $L_{x,y}$ as follows. Fix a vector $(x, y) \in W_n$. Then for some finite sequence $\{z_i\}$ of vectors we have $x = \sum z_{2i-1}$, $y = \sum z_{2i}$, where $z_1 \in L_{0,0}$, $z_{2i} \in L_{z_1+\cdots+z_{2i-1}, z_2+\cdots+z_{2i-2}}$, $z_{2i+1} \in L_{z_1+\cdots+z_{2i-1}, z_2+\cdots+z_{2i}}$ with subspaces defined by the form of $W_1, \ldots, W_n$. Now let $L_{x,y}$ be the subspace chosen by the player S according to the strategy $\mathcal{S}$ in $(n+1)-$th step in the game where in the previous steps the following objects were chosen: $L_{0,0} \in \mathcal{L}$, $z_1 \in L_{0,0}$, $L_{z_1,0} \in \mathcal{L}$, $z_2 \in L_{z_1,0}$, $L_{z_1,z_2} \in \mathcal{L}$, $z_3 \in L_{z_1,z_2}$, $L_{z_1+z_3,z_2}$, $z_4 \in L_{z_1+z_3,z_2}$,$\ldots$, and in $n-$th step $z_n \in L_{z_1+\cdots+z_{n-2},y}$ (if $n \in 2\mathbf{N}+1$) or $z_n \in L_{x,z_2+\cdots+z_{n-2}}$ (if $n \in 2\mathbf{N}$).

And the other way round, given a $(\mathbf{W})$–form neighbourhood W of the origin in $E \times E$ define a strategy $\mathcal{S}$ of the player S by induction. Put $M_1 = L_{0,0}$. Fix a rule $\mathcal{S}'$: if a sequence $z_1, \ldots, z_n$ of vectors chosen by the player P during some game satisfies $(\sum z_{2i}, \sum z_{2i-1}) \in W$, then the player S chooses $M_{n+1} = L_{\sum z_{2i}, \sum z_{2i-1}}$. Using the form $(\mathbf{W})$ of the neighbourhood W, one can prove by induction that if during the first n steps of the game the player S was chosing subspaces according to the rule $\mathcal{S}'$, then for the player P's any choice of z_{n+1}, the sequence $z_1, \ldots, z_{n+1}$ also satisfies $(\sum z_{2i}, \sum z_{2i+1}) \in W$. Hence the rule $\mathcal{S}'$ covers all possible choices of the player P and gives a strategy $\mathcal{S}$ of the player S.

Fix a set $A \subset E \times E$. We define the set $\rho(A)$ by the formula

$$\rho(A) = \left\{(z_1, \dots z_n) \in E^n,\ \left(\sum z_{2i-1}, \sum z_{2i}\right) \in A,\ n \geq 1\right\}.$$

Consider the game of the type I with the set $\sigma = \rho(A)$ and a filtering family $\mathcal{L}$. Then by the previous argument the player S has a winning strategy if and only if there exists a neighbourhood of the origin in $T^\circ_{\mathcal{L}}$ disjoint from A, and the player P has a winning strategy if and only if in every neighbourhood of the origin in $T^\circ_{\mathcal{L}}$ there is a vector belonging to A. Hence the following lemma holds true:

Lemma 3.1. *Consider a game of type I with a filtering family $\mathcal{L}$ of subspaces and a set $\sigma = \rho(A)$ for some $A \subset E \times E$. Then*

1. *the player S has a winning strategy* $\iff (0,0) \notin \overline{A}^{T^\circ_{\mathcal{L}}}$
2. *the player P has a winning strategy* $\iff (0,0) \in \overline{A}^{T^\circ_{\mathcal{L}}}$

In a similar way one can construct a neighbourhood of the origin in the topology $T^\bullet_{\mathcal{C}(\mathcal{L})}$. Given a filtering family $\mathcal{L}$ of subspaces of E and a set $\widetilde{\sigma}$ of finite sequences of pairs of vectors of E we define a two-player game of type II as follows: in the first step, the first player S chooses a subspace $M_1 \in \mathcal{L}$ and $\varepsilon_1 > 0$, then the second player P picks two vectors $z_1 \in C(M_1, \varepsilon_1)$ and $w_1 \in B(0, \varepsilon_1)$. In the second step, the player S chooses a subspace $M_2 \in \mathcal{L}$ and $\varepsilon_2 > 0$, then the player P chooses vectors $z_2 \in C(M_2, \varepsilon_2)$ and $w_2 \in B(0, \varepsilon_2)$. They continue in this way, choosing alternately subspaces and scalars (the player S) and pairs of vectors (the player P) in cylinders and balls defined by the player S's choice. As before, the player P wins if $((z_1, w_1), \dots, (z_n, w_n)) \in \widetilde{\sigma}$ for some $n \in \mathbf{N}$. If it never happens, the player S wins.

Assume the family of $\mathcal{L}$ is filtering. As in the previous case, $(\mathbf{U})$–form neighbourhoods of the origin in $T^\bullet_{\mathcal{C}(\mathcal{L})}$ correspond to strategies of the player S in the type II game.

Fix a set $A \subset E \times E$. By $\widetilde{\rho}(A)$ we denote the set

$$\widetilde{\rho}(A) = \left\{ \begin{array}{c} ((z_1, w_1), \dots, (z_n, w_n)) \in (E \times E)^n, \\ (\sum z_{2i-1} + \sum w_{2i}, \sum z_{2i} + \sum w_{2i-1}) \in A,\ n \geq 1 \end{array} \right\}$$

As in the previous case, we get the following

Lemma 3.2. *Consider a game of type II with a filtering family $\mathcal{L}$ of subspaces of E and a set $\widetilde{\sigma} = \widetilde{\rho}(A)$ for some $A \subset E \times E$. Then*

1. *the player S has a winning strategy* $\iff (0,0) \notin \overline{A}^{T^\bullet_{\mathcal{C}(\mathcal{L})}}$
2. *the player P has a winning strategy* $\iff (0,0) \in \overline{A}^{T^\bullet_{\mathcal{C}(\mathcal{L})}}$

4. "Topological" version of Gowers' dichotomy. In this section we show a close relationship between properties of topologies defined previously on the one hand and notions of unconditional basic sequences and HI spaces on the other. Afterwards we will be ready to present Gowers' dichotomy in terms of topologies on $E \times E$ of the form $T^{\bullet}$. At this point, one should also notice that in his proof of the dichotomy W. T. Gowers used a game of type I with the family of block subspaces and some set of sequences of blocks.

First we will present the connection between the existence of unconditional basic sequences and properties of topologies on $E \times E$ of the form $T^{\bullet}$.

Given a scalar $r > 0$ we put

$$S_r = \{(x, y) \in S \times S : \ \|x - y\| \leq r\}.$$

The following theorem was proved in the paper [**3**]:

THEOREM 4.1. *Let E be a Banach space. Suppose that for some scalar $r > 0$ and some vector topology T weaker than the norm topology holds the following*

$$(0, 0) \notin \overline{S_r}^{T^{\bullet}}.$$

Then there exists an unconditional basic sequence in E with the unconditional basic constant $\leq \frac{2}{r}$.

We will show now that also the inverse implication (however with different ratio between constants) holds true.

PROPOSITION 4.2. *Let E be a Banach space. Suppose there is an unconditional basic sequence with the unconditional basic constant C. Then there exists a metrizable vector topology on E weaker than the norm topology such that:*

$$(*) \quad (0, 0) \notin \overline{S_{\frac{1}{1+C}}}^{T^{\bullet}}.$$

PROOF. Let $\{e_n\}$ be the unconditional basic sequence in E. By L denote the closed subspace spanned by the sequence $\{e_n\}$. Then for every sequence $x_1 < \cdots < x_n$ of vectors in L we have

$$\left\| \sum_{i=1}^{n} x_i \right\| \leq C \left\| \sum_{i=1}^{n} (-1)^i x_i \right\|.$$

Given $n \geq 1$ denote by L_n the closed subspace spanned by the sequence $\{e_n, e_{n+1}, e_{n+2}, \dots\}$. Put

$$\widetilde{S_L} = \left\{ (a, b) \in S_L \times S_L : \ \exists\, a_1 < \ldots < a_n,\ a_i \in L, \ \ a = \sum a_{2i-1},\ b = \sum a_{2i} \right\}.$$

Notice that for any vectors $x, y \in S$ and a scalar $r \in (0, 2]$ the following implication holds:

$$\|x - y\| \le r \implies \frac{2-r}{r}\|x - y\| \le \|x + y\|.$$

Indeed, for $x, y \in S$ we have $\|x - y\| + \|x + y\| \ge 2$, hence $\|x + y\| \ge 2 - r \ge \frac{2-r}{r}\|x - y\|$.

Therefore $\widetilde{S_L} \cap S_{\frac{1,5}{1+C}} = \emptyset$.

Using this property one can easily prove that $(0, 0) \notin \overline{S_{\frac{1,5}{1+C}}}^{T^\circ_{\{L_n\}}}$. However, for the topology of the form $T^\bullet$, which is essentially weaker, we shall need some argument based on taking sufficiently small diameters of balls and cylinders.

We will show that the topology $T = T_{\mathcal{C}(\{L_n\})}$ satisfies $(*)$. By Lemma 3.2 it is enough to prove that in a type II game for the family $\{L_n\}_{n=1}^{\infty}$ and the set $\widetilde{\rho}(S_{\frac{1}{1+C}})$ the player S has a winning strategy.

We define the strategy by induction. First we define scalars to be chosen by the player S: for $n \ge 1$ put $\varepsilon_n = \frac{1}{2^{n+1}}\varepsilon$ for some $\varepsilon > 0$. Put $M_1 = L$. Now let $(z_1, w_1), \ldots, (z_n, w_n)$ be the pairs of vectors chosen by the player P and $M_1, \ldots, M_n$ be the subspaces chosen by the player S in the first n steps of a game. For $i = 1, \ldots, n$ pick some vector $a_i \in B(z_i, \frac{1}{2^i}\varepsilon) \cap M_i$ with a finite support. Put $M_{n+1} = L_{m_{n+1}}$ for m_{n+1} satisfying $\operatorname{supp} a_i < m_{n+1}$ for every $i \le n$.

We will show now that the strategy defined above is winning. Let (z_1, w_1), $\ldots, (z_n, w_n)$ be the pairs of vectors chosen by the player P in some game, such that vectors

$$x = \sum z_{2i-1} + \sum w_{2i}\ , \quad y = \sum z_{2i} + \sum w_{2i-1}$$

are of the norm one. Take vectors $a_1, \ldots, a_n$ corresponding to $z_1, \ldots, z_n$. By the choice of $\{a_i\}$, we have $a_1 < \cdots < a_n$ and $\|z_i - a_i\| \le \frac{1}{2^i}\varepsilon$ for every $i \le n$. Hence vectors $a = \sum a_{2i-1}$ and $b = \sum a_{2i}$ satisfy $\|a - x\| \le 2\varepsilon$, $\|b - y\| \le 2\varepsilon$ and thus

$$\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \le \|a - b\| + 4\varepsilon \le \|x - y\| \ + 8\varepsilon.$$

Moreover, since $\left(\frac{a}{\|a\|}, \frac{b}{\|b\|}\right) \in \widetilde{S_L}$, we have

$$\left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\| \ge \frac{1,5}{1+C}.$$

If ε is sufficiently small then $\|x - y\| > \frac{1}{1+C}$, which ends the proof of the proposition. $\square$

Now we show a connection between HI property and properties of some topology on $E \times E$ of the form $T^\bullet$.

Given a Banach space E denote by T_E the topology on E given by the family of all open unbounded and absolutely convex sets as a prebasis of neighbourhoods of the origin. We recall

THEOREM 4.3. ([**2**]) *Let E be a Banach space. Then E is a HI space if and only if the topology T_E is weaker than the norm topology on E.*

We have the following

PROPOSITION 4.4. *Let E be a Banach space. Then the following conditions are equivalent:*

1. *E is HI space,*
2. *for any scalar $r > 0$ we have $(**)$ $\quad (0,0) \in \overline{S_r}^{T_E^\bullet}$,*
3. *there exists a scalar $r > 0$ such that $(**)$ $\quad (0,0) \in \overline{S_r}^{T_E^\bullet}$.*

PROOF. 1. $\Longrightarrow$ 2. By Theorem 4.1, $(**)$ holds for every vector topology weaker than the norm topology. By Theorem 4.3, the topology T_E is weaker than the norm topology, hence T_E satisfies $(**)$.

Implication 2. $\Longrightarrow$ 3. is obvious.

3. $\Longrightarrow$ 1. Notice that the set S_r is closed in the topology $T^\bullet_{\|\cdot\|}$, hence by $(**)$ the topology $T^\bullet_E$ is weaker than $T^\bullet_{\|\cdot\|}$. It follows that the topology T_E is weaker than the norm topology $T_{\|\cdot\|}$ which implies HI. $\square$

Now we are ready to present the following version of Gowers' dichotomy (Theorem 1.1):

THEOREM 4.5. *Let E be a Banach space. Then there exists a closed infinite dimensional subspace L of E such that:*

either there exist a scalar $r > 0$ and a vector topology T on L weaker than the norm topology, such that

$$(0,0) \notin \overline{S_r}^{T^\bullet},$$

or for any (equiv. for some) scalar $r > 0$

$$(0,0) \in \overline{S_r}^{T_L^\bullet},$$

where T_L is the topology on L given by the family of all open unbounded and absolutely convex sets in L as a prebasis of neighbourhoods of the origin.

REMARK. One should notice that by contradicting the first condition in Theorem 4.5 we get the following condition:

for any scalar $r > 0$ and for any vector topology T on E weaker than the norm topology there is

$$(0,0) \in \overline{S_r}^{T^\bullet}.$$

What Gowers' dichotomy gives us is the fact that in such a situation for some closed infinite dimensional subspace L of E the supremum of all vector topologies on L weaker than the norm topology in L, ie. T_L (which a priori does not need to be weaker than the norm topology in L) also satisfies

$$(0,0) \in \overline{S_r}^{T_L^\bullet}.$$

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