UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLI
2003

ULTRAINCREASING DISTRIBUTIONS OF EXPONENTIAL TYPE

by Katarzyna Grasela

Abstract. In this paper Fourier transform images of Gevrey ultradistribution spaces are described. It is proved that such spaces with the strong topology in regard to natural duality are of the M^* type in the sense of Silva. It is also proved that the space of test functions of such images is a locally convex convolution algebra of the LN^* type. The received results complete one known statement of Hörmander.

1. Introduction. The objective of this paper is to study some locally convex topological vector spaces. Namely, we will consider the space which is the image under the Fourier transform of the space of functions defined on $\mathbb{R}^n$ which have compact supports and are ultradifferentiable in the sense of Gevrey. This Fourier transform image is the subspace of the vector space of all entire functions of exponential type. Therefore, our research completes Hörmander's known statement [**2**, V.2, Lemma 12.7.4], which is essentially used in the proof of existence of the solution of a Cauchy problem for the hyperbolic equation (see, [**2**, V.2, 12.7.5]).

The dual space of the considered Fourier transform image is larger than the known space of all analytic functionals on $\mathbb{R}^n$ [**2**, V.1,9.1]. On the other hand, this dual space does not belong to the class of spaces considered in [**6**].

We shall prove that the considered spaces of entire functions of exponential type have the structure of the inductive limit of a sequence of Banach spaces, such that inclusions mappings are compact. It means that the space of entire functions of exponential type belongs to the known class LN^* of the locally convex topological vector spaces investigated by S. di Silva [**4**]. Therefore, its dual space, called the space of ultraincreasing distributions of exponential type

1991 *Mathematics Subject Classification.* Primary 46T30; Secondary 46F12.
Key words and phrases. Gevrey distribution, analytical functionals, Fourier transform.

belongs to the known class M^* (cf. [**4**]), so it has the structure of the projective limit of a sequence of Banach spaces with compact projections.

We shall also prove that the space of entire functions of exponential type is a topological algebra with respect to convolution.

2. Main results. For given real number $\aleph$ such that $1 < \aleph < e$, arbitrarily chosen vector $\nu = (\nu_1, \ldots, \nu_n) \in \operatorname{int} \mathbb{R}^n_+$ and vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ such that $b \succ a$ (i.e. $b_j > a_j$ for $j \in \{1, \ldots n\}$), we define the space of entire functions of exponential type

$$E_{\nu,[a,b]} = \Big\{ \Phi : \mathbb{C}^n \ni \zeta \longmapsto \Phi(\zeta) \in \mathbb{C}, \quad \|\Phi\|_{E_{\nu,[a,b]}} < \infty \Big\}$$

with the norm

$$\|\Phi\|_{E_{\nu,[a,b]}} = \sup_{k \in \mathbb{Z}^n_+} \sup_{\zeta \in \mathbb{C}^n} \frac{\big|\zeta^k \Phi(\zeta)\, e^{-H_{[a,b]}(\eta)}\big|}{\nu^k k^{k\aleph}},$$

where $\zeta = \xi + \mathrm{i}\eta = (\zeta_1, \ldots, \zeta_{\mathrm{n}}) \in \mathbb{C}^{\mathrm{n}}$, $\xi = (\xi_1, \ldots, \xi_n)$ and $\eta = (\eta_1, \ldots, \eta_n)$ are in $\mathbb{R}^n$, $\zeta^k = \zeta_1^{k_1} \ldots \zeta_n^{k_n}$, $\nu^k = {\nu_1}^{k_1} \ldots {\nu_n}^{k_n}$, $k^{k\aleph} = {k_1}^{k_1 \aleph} \ldots {k_n}^{k_n \aleph}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ and

$$H_{[a,b]}(\eta) = \sup_{t \in [a,b]} (t, \eta), \qquad ((t,\eta) = \sum_{j=1}^{n} t_j \eta_j)$$

is the supporting function of n–dimensional cube $[a,b] := \big\{ t = (t_1, \ldots, t_n) \in \mathbb{R}^n : t_j \in [a_j, b_j], \ \forall\, j = 1, \ldots, n \big\}$.

We also define the space of ultradifferentiable functions in the sense of Gevrey

$$G_{\nu,[a,b]} = \Big\{ \phi(t) \in C^\infty(\mathbb{R}^n) : \operatorname{supp} \phi \subset [a,b], \ \|\phi\|_{G_{\nu,[a,b]}} < \infty \Big\}$$

with the norm

$$\|\phi\|_{G_{\nu,[a,b]}} = \sup_{k \in \mathbb{Z}^n_+} \sup_{t \in [a,b]} \frac{|D^k \phi(t)|}{\nu^k k^{k\aleph}},$$

where $D^k = D_1^{k_1} \ldots D_n^{k_n}$, $D_j^{k_j} = (-\mathrm{i})^{k_j} \dfrac{\partial^{k_j}}{\partial {t_j}^{k_j}}$. One can prove that $G_{\nu,[a,b]}$ is a Banach space.

Now let us consider the inductive limit of spaces $E_{\nu,[a,b]}$; we will denote it by $E(\mathbb{C}^n)$

$$E(\mathbb{C}^n) = \bigcup_{\nu \succ 0} \bigcup_{b \succ a} E_{\nu,[a,b]} = \liminf_{\nu,[a,b]} E_{\nu,[a,b]},$$

where all injections $E_{\nu,[a,b]} \hookrightarrow E_{\nu',[a',b']}$ $\big(\nu' \succ \nu;\ [a,b] \subset [a',b']\big)$ are continuous. In the same way we define

$$G(\mathbb{R}^n) = \bigcup_{\nu\succ 0} \bigcup_{b\succ a} G_{\nu,[a,b]} = \liminf_{\nu,[a,b]} G_{\nu,[a,b]}.$$

For such spaces we can write the Fourier transform

$$\mathcal{F} :\ G(\mathbb{R}^n) \ni \phi \longmapsto \widehat{\phi}(\zeta) := \int\limits_{\mathbb{R}^n} \phi(t)\, e^{-\mathrm{i}(\mathrm{t},\zeta)}\, dt.$$

It will be shown later that $\mathcal{F}\big(G(\mathbb{R}^n)\big) = E(\mathbb{C}^n)$. Therefore, we can also consider the dual Fourier transform

$$\mathcal{F}' :\ E'(\mathbb{C}^n) \longmapsto G'(\mathbb{R}^n),$$

where $G'(\mathbb{R}^n)$ and $E'(\mathbb{C}^n)$ denote spaces of linear continuous functionals on $G(\mathbb{R}^n)$ and $E(\mathbb{C}^n)$, respectively. In the dual spaces $G'(\mathbb{R}^n)$ and $E'(\mathbb{C}^n)$, we consider the strong topology. We shall prove the following statement.

Theorem 1. *The following topological isomorphisms*

$$\mathcal{F}\big(G(\mathbb{R}^n)\big) \simeq E(\mathbb{C}^n), \qquad \mathcal{F}'\big(E'(\mathbb{C}^n)\big) \simeq G'(\mathbb{R}^n)$$

are valid. Moreover, $E(\mathbb{C}^n)$ is an LN^*–space and $E'(\mathbb{C}^n)$ is an M^*–space in the sense of Silva.

First we shall prove the following auxiliary statement. Let us construct the locally convex inductive limits of Banach spaces

$$E[a,b] = \bigcup_{\nu\succ 0} E_{\nu,[a,b]} = \liminf_{\nu\succ 0} E_{\nu,[a,b]},$$

where injections $E_{\nu,[a,b]} \hookrightarrow E_{\nu',[a,b]}$ are continuous and

$$G[a,b] = \bigcup_{\nu\succ 0} G_{\nu,[a,b]} = \liminf_{\nu\succ 0} G_{\nu,[a,b]}$$

with continuous injections $G_{\nu,[a,b]} \hookrightarrow G_{\nu',[a,b]}$ for any ordered pair $\nu' \succ \nu$. From the Denjoy–Carleman theorem [**2**, Theorem 1.3.8] it follows that the space $G[a,b]$ is not trivial.

Lemma 1. $\mathcal{F}\big(G_{[a,b]}\big) = E_{[a,b]}$.

PROOF. Let $\phi \in G_{\nu,[a,b]}$ and $\widehat{\phi} = \Phi$. Hence $\widehat{D^k\phi}(\zeta) = \zeta^k\Phi(\zeta)$ and for all ζ, k there is

$$
\begin{aligned}
|\zeta^k\Phi(\zeta)| &\leq e^{H_{[a,b]}(\eta)} \int\limits_{[a,b]} |D^k\phi(t)|\,dt \\
&\leq \nu^k k^{k\aleph} e^{H_{[a,b]}(\eta)} \|\phi\|_{G_{\nu,[a,b]}} \prod_{j=1}^{n}(b_j - a_j).
\end{aligned}
\tag{2.1}
$$

Hence the inclusion $\mathcal{F}\big(G_{\nu,[a,b]}\big) \subset E_{\nu,[a,b]}$ follows.

Now we take $\Phi \in E_{\nu,[a,b]}$. We will prove that $\xi^l\Phi(\xi)$ is summable on $\mathbb{R}^n$ for all $l \in \mathbb{Z}^n_+$. In $\mathbb{R}^n$ we consider the following sets

$$
\begin{aligned}
&\Omega_0 = \big\{\xi : \nu e \succcurlyeq \xi\big\},\\
&\Omega_1 = \big\{\xi : |\xi_1| > \nu_1 e,\ |\xi_2| \leq \nu_2 e, \ldots, |\xi_n| \leq \nu_n e\big\},\\
&\Omega_2 = \big\{\xi : |\xi_1| \leq \nu_1 e,\ |\xi_2| > \nu_2 e, \ldots, |\xi_n| > \nu_n e\big\},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\Omega_{2^n} = \big\{\xi : \xi \succ \nu e\big\},
\end{aligned}
$$

where $\nu e \succcurlyeq \xi$ means that $\nu_j e \geq \xi_j$ for each $j \in \big\{1, \ldots, n\big\}$. It is obvious that $\mathbb{R}^n = \Omega_0 \cup \Omega_1 \cup \ldots \cup \Omega_{2^n}$. For $\xi \in \Omega_0$ and all $\eta \in \mathbb{R}^n$, there is

$$
|\Phi(\zeta)| \leq \nu^k k^{k\aleph} |\xi|^{-k} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}}, \text{ for all } k \in \mathbb{Z}^n_+.
$$

Therefore, for $k = (0, \ldots, 0)$ we obtain

$$
|\Phi(\zeta)| \leq C_0\, e^{-\sum_1^n \left|\frac{\xi_j}{\nu_j e}\right|^{\frac{1}{\aleph}}} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}},
$$

where $C_0 = \max\limits_{\xi \in \Omega_0} e^{\sum_1^n \left|\frac{\xi_j}{\nu_j e}\right|^{\frac{1}{\aleph}}}$. If $\xi \in \Omega_1$, there exists $k = (k_1, 0, \ldots, 0) \in \mathbb{Z}^n_+$ such that

$$
\left(\frac{|\xi_1|}{\nu_1 e}\right)^{\frac{1}{\aleph}} - 1 < k_1 < \left(\frac{|\xi_1|}{\nu_1 e}\right)^{\frac{1}{\aleph}} \quad \text{in particular} \quad \frac{\nu_1 {k_1}^{\aleph}}{|\xi_1|} < \frac{1}{e}
$$

and for all $\eta \in \mathbb{R}^n$ the following estimation holds:

$$
\begin{aligned}
|\Phi(\zeta)| &\leq |\xi^{-k_1}| {\nu_1}^{k_1} k_1^{k_1\aleph}\, e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}} \\
&\leq e^{-k_1}\, e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}} \leq e^{1-\left|\frac{\xi_1}{\nu_1 e}\right|^{\frac{1}{\aleph}}} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}} \\
&\leq C_1\, e^{-\sum_1^n \left|\frac{\xi_j}{\nu_j e}\right|^{\frac{1}{\aleph}}} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}},
\end{aligned}
$$

where $C_1 = e \max\limits_{|\xi_2|\le \nu_2 e} \dots \max\limits_{|\xi_n|\le \nu_n e} e^{\sum_2^n \left|\frac{\xi_j}{\nu_j e}\right|^{\frac{1}{\aleph}}}$. Similarly, for any $\xi \in \Omega_2$ the following inequality holds:

$$|\Phi(\zeta)| \le C_2\, e^{-\sum_1^n \left|\frac{\xi_j}{\nu_j\, e}\right|^{\frac{1}{\aleph}}} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}}$$

for all $\eta \in \mathbb{R}^n$ and $C_2 = e^{n-1} \max\limits_{|\xi_1|\le \nu_1} e^{\left|\frac{\xi_1}{\nu_1 e}\right|^{\frac{1}{\aleph}}}$. Now we proceed by induction. For $\xi \in \Omega_{2^n}$ there exists $k = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$ such that

$$\left(\frac{|\xi_j|}{\nu_j e}\right)^{\frac{1}{\aleph}} - 1 < k_j < \left(\frac{|\xi_j|}{\nu_j e}\right)^{\frac{1}{\aleph}} \quad \big(\text{in particular} \quad \frac{\nu_j {k_j}^{\aleph}}{|\xi_j|} < \frac{1}{e}\big)$$

for all $j \in \{1, \dots, n\}$. Thence, for all $\eta \in \mathbb{R}^n$, there is

$$\begin{aligned}|\Phi(\zeta)| &\le |\xi^{-k}|\, \nu^k k^{k\aleph}\, e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}} \\ &\le e^{-|k|}\, e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}} \le e^{n-\sum_1^n \left|\frac{\xi_j}{\nu_j\, e}\right|^{\frac{1}{\aleph}}} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}},\end{aligned}$$

where $|k| = \sum_1^n k_j$. Thus, combining the inequalities received above and taking $C = \max\{e^n, \dots, C_2, C_1, C_0\}$, we obtain

$$\forall \zeta \in \mathbb{C}^n \quad |\Phi(\zeta)| \le C\, e^{-\sum_1^n \left|\frac{\xi_j}{\nu_j\, e}\right|^{\frac{1}{\aleph}}} e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}}. \tag{2.2}$$

If we take $n = 1$ then from the de l'Hospital formula, for each number $m \in \mathbb{N}$ there is

$$\lim_{\xi_j \to +\infty} (1+\xi_j)^m e^{-(\frac{\xi_j}{\nu+j\, e})^{\frac{1}{\aleph}}} = \lim_{\xi_j \to +\infty} m!\aleph^m (e\nu_j)^m e^{-(\frac{\xi_j}{\nu_j\, e})^{\frac{1}{\aleph}}}. \tag{2.3}$$

Therefore, for ξ_j $(j = 1, \dots, n)$ sufficiently large and for each $m \in \mathbb{N}$, there exists constant $C^j_{m,\nu}$ such that $e^{-\left|\frac{\xi_j}{\nu_j e}\right|^{\frac{1}{\aleph}}} \le \frac{C^j_{m,\nu}}{(1+|\xi_j|)^m}$. Since $\prod\limits_1^n (1 + |\xi_j|) \ge 1 + \sum\limits_1^n |\xi_j|$ there is

$$e^{-\sum_1^n \left|\frac{\xi_j}{\nu_j\, e}\right|^{\frac{1}{\aleph}}} \le \frac{\widetilde{C}_{m,\nu}}{\prod_1^n (1+|\xi_j|)^m} \le \frac{\widetilde{C}_{m,\nu}}{(1+|\xi|)^m}, \quad \widetilde{C}_{m,\nu} = \prod_1^n C^j_{m,\nu}. \tag{2.4}$$

Therefore, for each $m \in \mathbb{N}$, there exists constant $C_{m,\nu} = C \cdot \widetilde{C}_{m,\nu}$ such that the following inequality

$$\forall \zeta \in \mathbb{C}^n\,, \quad |\Phi(\zeta)| \leq \frac{C_{m,\nu}}{(1+|\xi|)^m}\, e^{H_{[a,b]}(\eta)}\, \|\Phi\|_{E_{\nu,[a,b]}} \tag{2.5}$$

is valid. If we take $\eta = 0$ and $m = l + n + 1$, than for some constant $C_{l,\nu}$ we get

$$\begin{aligned} |\xi^l \Phi(\xi)| &\leq \frac{|\xi|^l\, C_{m,\nu}}{(1+|\xi|)^m}\, \|\Phi\|_{E_{\nu,[a,b]}} = \frac{|\xi|^l\, C_{m,\nu}}{(1+|\xi|)^l (1+|\xi|)^{n+1}} \\ &\leq \frac{C_{l,\nu}}{(1+|\xi|)^{n+1}}\, \|\Phi\|_{E_{\nu,[a,b]}}\,. \end{aligned}$$

As a consequence of this inequality, the function $\xi^l \Phi(\xi)$ is summable on $\mathbb{R}^n$. There exists $\mathcal{F}^{-1}\Phi = \phi$ and

$$D^k \phi(t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \xi^k \Phi(\xi)\, e^{\mathrm{i}(t,\xi)}\, d\xi, \qquad k \in \mathbb{Z}_+^n. \tag{2.6}$$

From inequality (2.5) for $m = n + 1$ there follows that the following integral

$$\phi(t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \Phi(\zeta)\, e^{\mathrm{i}(t,\zeta)}\, d\xi, \qquad \zeta \in \mathbb{C}^n$$

converges and, since $\mathrm{i}(t,\zeta) = \mathrm{i}(t,\xi) - (t,\eta)$ and inequality (2.5) holds for $m = n + 1$ the following inequality holds:

$$|\phi(t)| \leq C_{\nu,a,b} \exp\big[-(t,\eta) + H_{[a,b]}(\eta)\big] \int\limits_{\mathbb{R}^n} \frac{d\xi}{(1+|\xi|)^{n+1}} \tag{2.7}$$

for all $\eta \in \mathbb{R}^n$ and constant $C_{\nu,a,b} = C_{m,\nu}\|\Phi\|_{E_{\nu,[a,b]}}$. By replacing η with $r\eta$, where $r \to \infty$ in inequality (2.7), we imply that $\phi(t) \neq 0$, provided $(t,\eta) \leq H_{[a,b]}(\eta)$ for all $\eta \in \mathbb{R}^n$, hence $t \in [a,b]$ $\big($see [**2**, Theorem 4.3.2]$\big)$. It means that $\operatorname{supp}\phi \subset [a,b]$.
From (2.6) and (2.2) we obtain the following estimate

$$|D^k \phi(t)| \leq C_{a,b,\nu} \int\limits_{\mathbb{R}^n} |\xi^k|\, e^{-\sum\limits_1^n \left|\frac{\xi}{e\nu}\right|^{\frac{1}{\aleph}}}\, d\xi, \qquad k \in \mathbb{Z}_+^n. \tag{2.8}$$

By calculating the previous integral we obtain:

$$\int\limits_{\mathbb{R}^n} |\xi^k|\, e^{-\sum\limits_1^n \left|\frac{\xi}{e\nu}\right|^{\frac{1}{\aleph}}}\, d\xi = 2(e\nu)^{k+1} \aleph^n\, \Gamma\big(\aleph(k_1+1)\big) \cdots \Gamma\big(\aleph(k_n+1)\big)$$

for all $k \in \mathbb{Z}_+^n$. Using the following asymptotic equality

$$\Gamma\big(\aleph(k_j+1)\big) \approx (k_j+1)^{k_j\aleph} \tag{2.9}$$

we obtain

$$|D^k\phi(t)| \le 2C_{a,b,\nu}\aleph^n\,(e\nu)^{k+1}\,(k+1)^{k\aleph}, \qquad k \in \mathbb{Z}_+^n, \tag{2.10}$$

and

$$\frac{|D^k\phi(t)|}{(e\nu)^k k^{k\aleph}} \le \tilde{C}\Big(\frac{k+1}{k}\Big)^{k\aleph}. \tag{2.11}$$

Since $\operatorname{supp}\phi \subset [a,b]$ and inequality (2.11) holds, there is $\phi \in G_{e\nu,[a,b]}$. Hence the inclusion $E_{\nu,[a,b]} \subset \mathcal{F}\big(G_{e\nu,[a,b]}\big)$ follows.

Taking into account the randomness of the vector ν and the properties of inductive limit, we receive: $E_{[a,b]} = \mathcal{F}\big(G_{[a,b]}\big)$.

Now let us explain relation (2.9).

Since $\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x}\exp(\frac{\theta}{12x})$, $x>0$, $0<\theta<1$ [**5**, 12.33], then for a sufficiently large x the following relation $\Gamma(x) \approx x^{x-\frac{1}{2}}e^{-x}$ is true. Let us take $x = \aleph k$; we obtain $\Gamma(\aleph k) \approx (\aleph k)^{\aleph k-\frac{1}{2}}e^{-\aleph k}$. Now, we should only prove that $\big(\sqrt{\aleph k}\big)^{-1}\aleph^{\aleph k}e^{-\aleph k}k^{\aleph}$ is bounded for each $k \in \mathbb{Z}_+$. Since $\aleph$ is a fixed real number and $\aleph < e$, then

$$\frac{k^{\aleph}\aleph^{\aleph k}}{\sqrt{k\aleph}e^{k\aleph}} = \frac{k^{\aleph}e^{k\aleph(\ln\aleph-1)}}{\sqrt{k\aleph}} \tag{2.12}$$

and $\ln\aleph - 1 < 0$, hence $\dfrac{k^{\aleph}e^{k\aleph(\ln\aleph-1)}}{\sqrt{k\aleph}}$ tends to zero when $k \to +\infty$. Thus we obtain $\Gamma(\aleph k) \approx k^{(k-1)\aleph}$ and relation (2.9) is proved. □

Corollary 1. *The image of* $G(\mathbb{R}^n)$ *under mapping* $\mathcal{F}$ *is equal to* $E(\mathbb{C}^n)$.

Proof. This corollary is a straightforward consequence of Lemma 1 and the properties of inductive limit (cf. [**1**]). □

Now we come back to Theorem 1.

Proof. From inequality (2.1) there follows that

$$\|\Phi(\zeta)\| \le \|\phi\|_{G_{\nu,[a,b]}}\prod_{j=1}^n(b_j-a_j)$$

for all $\phi \in G_{\nu,[a,b]}$ and $\nu \in \operatorname{int}\mathbb{R}_+^n$. Hence, the mapping $G_{[a,b]} \ni \phi \longmapsto \Phi \in E_{[a,b]}$ is continuous. By Lemma 1 this mapping is surjective. Therefore we can apply the Banach theorem about open map in the Grothendieck version [**1**, Theorem 6.7.2], according to which the topological isomorphism $\mathcal{F}\big(G_{[a,b]}\big) \simeq E_{[a,b]}$ is fair. Because of the arbitrary nature of cubes $[a,b]$ and by standard properties of the inductive limits, further topological isomorphisms follow $\mathcal{F}\big(G(\mathbb{R}^n)\big) \simeq E(\mathbb{C}^n)$ and $\mathcal{F}'\big(E'(\mathbb{C}^n)\big) \simeq G'(\mathbb{R}^n)$.

As proved by Lions and Magenes [**3**, Chap.7, Proposition 1.1], for more general spaces all inclusions

$$G_{\nu,[a,b]} \hookrightarrow G_{\mu,[a,b]}, \qquad \text{where} \qquad \mu \succ \nu,$$

$$G_{\nu,[a,b]} \hookrightarrow G_{\nu,[c,d]}, \qquad \text{where} \qquad [a,b] \subset [c,d]$$

are compact. Hence, the inductive limit $G(\mathbb{R}^n) = \liminf\limits_{\nu,[a,b]} G_{\nu,[a,b]}$ belongs to the class of LN^*–spaces in the sense of Silva [**4**]. In view of the topological isomorphism established above, the space $E(\mathbb{C}^n)$ also belongs to the class of LN^*–spaces. Therefore the strong dual space $E'(\mathbb{C}^n)$ belongs to the class of M^*–spaces in the sense of Silva [**4**]. $\square$

Now we consider $E'[a,b]$, the topological dual space of the space $E[a,b]$. Using analogy to the theory of analytical functionals, we shall call the n–dimensional cube $[a,b]$ the determining set for $E'[a,b]$. From Theorem 1 the following important property of determining sets follows directly.

Corollary 2. *Let $[a,b]$ and $[c,d]$ be determining sets for $E'[a,b]$ and $E'[c,d]$ respectively, and $[a,b] \cap [c,d] \neq \emptyset$. Let $T \in E'[a,b] \cap E'[c,d]$. Then $T \in E'\big([a,b] \cap [c,d]\big)$.*

Proof. Actually, according to Theorem 1 it is sufficient to prove the following property:

$$T \in G'[a,b] \cap G'[c,d] \Longrightarrow T \in G'\big([a,b] \cap [c,d]\big),$$

where $G'[a,b]$ is the topological dual space of linear continuous functionals on the space $G[a,b]$. It is easy to observe that $\operatorname{supp} T \subset [a,b] \cap [c,d]$. Therefore $T \in G'([a,b] \cap [c,d])$. $\square$

3. An application. We would also like to present the following theorem.

Theorem 2. *$E(\mathbb{C}^n)$ is a convolution algebra.*

Proof. First we will prove that $G(\mathbb{R}^n)$ is an algebra with respect to multiplication. For fixed ν and $[a,b]$, $[a',b']$ such that $[a,b] \subset [a',b']$ there is

$$\|\phi\|_{G_{\nu,[a,b]}} = \|\phi\|_{G_{\nu,[a',b']}}, \qquad \phi \in G_{\nu,[a,b]}.$$

Further for any vectors $\nu \succ \mu \succ 0$ and fixed $[a,b]$, there is

$$\|\phi\|_{G_{\mu,[a,b]}} \leq \|\phi\|_{G_{\nu,[a,b]}}, \qquad \phi \in G_{\mu,[a,b]}.$$

Let us take $\phi \in G_{\nu,[a,b]}$, $\psi \in G_{\mu,[a',b']}$, where $[a,b] \subset [a',b']$, $\nu \succ \mu \succ 0$, we observe that for all $t \in \operatorname{supp}(\phi\psi) \subset [a,b]$ the following inequalities hold:

$$\begin{aligned}\left|D^k[\phi(t)\psi(t)]\right| &\le \|\phi\|_{G_{\nu,[a,b]}} \|\psi\|_{G_{\mu,[a',b']}} \times \\ &\qquad \times \sum_{|m|=0}^{k} \frac{\nu^m \mu^{(k-m)} m^{m\aleph}((k-m))^{(k-m)\aleph} k!}{m!(k-m)!} \\ &\le \|\phi\|_{G_{\nu,[a,b]}} \|\psi\|_{G_{\mu,[a',b']}} \sum_{|m|=0}^{k} \frac{\nu^m \mu^{(k-m)} k!}{m!(k-m)!} k^{m\aleph} k^{(k-m)\aleph} \\ &\le \|\phi\|_{G_{\nu,[a,b]}} \|\psi\|_{G_{\mu,[a',b']}} (\nu+\mu)^k k^{k\aleph}.\end{aligned}$$

(Note: $k! = k_1! \dots k_n!$.)

Hence there is

$$\|\phi\psi\|_{G_{\nu+\mu,[a,b]}} \le \|\phi\|_{G_{\nu,[a,b]}} \|\psi\|_{G_{\mu,[a',b']}}.$$

Therefore, we in particular conclude that $G[a,b] = \bigcup_{\nu \succ 0} G_{\nu,[a,b]}$ is a locally convex algebra with respect to multiplication. Hence $G(\mathbb{R}^n)$ as an inductive limit is also an algebra with respect to multiplication (cf. [**1**]).

If we now use the known fact that $\widehat{\phi \cdot \psi} = \widehat{\phi} * \widehat{\psi}$ and Theorem 1, we conclude that $E(\mathbb{C}^n)$ is a convolution algebra. $\square$

References

1. Edwards W., *Functional Analysis. Theory and Applications,* Moscow, 1969.
2. Hörmander L., *The Analysis of Linear Partial Differential Operators,* Vol. 1–2, Springer–Verlag, 1983.
3. Lions J.L., Magenes E., *Problemes aux limites non homogenes et applications,* Vol. 3, Dunod, Paris, 1970.
4. di Silva S., *On some classes of locally convex spaces,* Matematika, **1** (1957), 60–67.
5. Whittaker E.T., Watson G.N., *A Course of Modern Analysis,* Vol. 2, Cambrige University Press, 1963.
6. Zarinov V.V., *Compact families of locally convex spaces and FS and DFS spaces,* Uspekhi Mat. Nauk, **34** (1979), 4(208), 97–131 (in Russian).

Received October 7, 2002

Cracow University of Technology
Institute of Mathematics
Warszawska 24
Kraków, Poland
e-mail: `kgrasela@usk.pk.edu.pl`