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CONNECTIONS, LOCAL SUBGROUPOIDS, AND A HOLONOMY LIE GROUPOID OF A LINE BUNDLE GERBE

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Abstract. Our main aim is to associate a holonomy Lie groupoid to the connective structure of an abelian gerbe. The construction has analogies with a procedure for the holonomy Lie groupoid of a foliation, in using a locally Lie groupoid and a globalisation procedure. We show that path connections and 2–holonomy on line bundles may be formulated using the notion of a connection pair on a double category, due to Brown–Spencer, but now formulated in terms of double groupoids using the thin fundamental groupoids introduced by Caetano–Mackaay–Picken. To obtain a locally Lie groupoid to which globalisation applies, we use methods of local subgroupoids as developed by Brown–İçen–Mucuk.

1. Introduction. On investigating the potential applications of double groupoids in homotopy theory, Brown and Spencer in 1976 [**12**] developed the notion of a *connection pair* (Υ, Hol) consisting of *the transport* Υ and *holonomy* 'Hol', which led to an equivalence of crossed modules with edge symmetric double groupoids with special connections. The key 'transport law' for Υ used in this equivalence was abstracted from a law for path connections on principal bundles due to Virsik [**19**], applied to a connection pair on the double category ΛT of Moore paths on a topological category T. The relation of these ideas with the connections of differential geometry has been undeveloped. However, there is now a growing interest in 2-dimensional ideas in holonomy, particularly in those areas of mathematics and mathematical physics where the theory of *gerbes* plays a prominent role (see e.g. [**2**] [**3**] [**4**] [**13**] [**17**]). From a technical point of view, it is useful in the case where T is a Lie groupoid to move from the

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double category ΛT as above to a smooth double groupoid. Our first step is to use the notion of the thin fundamental groupoid $\Lambda_1^1(X)$ of a smooth manifold X (see e.g. [**14**] [**17**]) (§3). A major step is to construct a smooth connection pair from the data of gerbes and 2–holonomy in abelian gerbes (§4,5).

To obtain a locally Lie groupoid we use in §6 the methods of local subgroupoids and their holonomy Lie groupoids as in [**6**] [**7**]. In particular, conditions are given in [**7**] for a connection pair to yield a local subgroupoid and so a locally Lie groupoid $(\mathbb{G}, W)$ with an associated holonomy Lie groupoid $\mathrm{Hol}(\mathbb{G}, W)$, via a *Globalisation Theorem* of Aof–Brown [**1**][1]. Brown–Mucuk [**11**] showed that it recovered as a special case (and so with a new universal property), the holonomy Lie groupoid of a foliation where $\mathbb{G}$ is the equivalence relation determined by the leaves of the foliation. Two important points about this construction are (i) $\mathrm{Hol}(\mathbb{G}, W)$ comes with a universal property, and (ii) the basic method of construction of $\mathrm{Hol}(\mathbb{G}, W)$ involves an algebraic framework for the intuition of 'iteration of local procedures', using Ehresmann's local smooth admissible sections. It is of course this intuition which is behind holonomy constructions in differential geometry.

Thus in this note we establish:

Theorem 1.1. *We can associate to certain abelian gerbe data* $(\mathcal{P}, \mathcal{A}, \mathrm{Geod})$ *over a path connected manifold* X*, a local subgroupoid* $C(\mathcal{P}, \mathcal{A}, \mathrm{Geod})$*. Relative to a strictly regular atlas* $\mathcal{U}(\mathcal{P}, \mathcal{A}, \mathrm{Geod})$*, there exists a holonomy Lie groupoid* $\mathrm{Hol}(\mathcal{P}, \mathcal{A})$ *with base space* X*.*

The universal property satisfied by this holonomy groupoid will be investigated in later work.

2. Transport and holonomy in groupoids–some background.

2.1. *Connection and transport in a double category.*
Firstly, let $T = (H, X, \alpha, \beta, \circ, \epsilon)$ be a topological category in which $\alpha, \beta \colon H \longrightarrow X$ are the initial and final maps, respectively, $\circ$ denotes partial composition, and $\epsilon : X \longrightarrow H$ is the unit function. A double category is specified by four related category structures:

$$\begin{cases} (D, H, \alpha_1, \beta_1, \circ_1, \epsilon_1), & (D, V, \alpha_2, \beta_2, \circ_2, \epsilon_2), \\ (V, X, \alpha, \beta, \circ, e), & (H, X, \alpha, \beta, \circ, f), \end{cases} \tag{2.1}$$

of which each of the structures of the first row is compatible with the other. For more details, see for instance [**10**]. The elements of D are called *squares*, those of H and V *the horizontal and vertical edges*, respectively, while X consists of *points*. A double category can be enhanced by the abstract notion of a *connection* as specified by a pair (Υ, Hol) in which $\mathrm{Hol} : V \longrightarrow H$ (*the holonomy*)

[1]The origin of this theorem as in the work of J. Pradines, is explained in [**1**]

is a functor of categories, and $\Upsilon : V \longrightarrow D$ (*the transport*) is a function, such that in the formalism of the governing (higher dimensional) algebraic rules [**12**] (see also [**5**] [**7**] [**10**]), we have:

(1) The bounding edges of $\mathrm{Hol}(a)$ and $\Upsilon(a)$, for $a : x \to y$ in V, are described by the diagram

$$\begin{array}{ccc} x & \xrightarrow{\mathrm{Hol}(a)} & y \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle e_y} \\ y & \xrightarrow{f_y} & y \end{array} \tag{2.2}$$

(2) *The transport law* holds. That is, if $a, b \in V$ and $a \circ b$ is defined, then

$$\Upsilon(a \circ b) = \begin{bmatrix} \Upsilon(a) & \epsilon_2(\mathrm{Hol}(b)) \\ & \\ \epsilon_1(b) & \Upsilon(b) \end{bmatrix} \tag{2.3}$$

A notable example of this construction (following [**12**]) is *the double category of Moore paths* $\Lambda T = (\Lambda H, H, \Lambda X, X)$ on the topological category $T = (H, X)$. Here the squares are elements of ΛH, the horizontal edges are H, the vertical edges are ΛX, and the set of points is X. Accordingly, a connection pair (Υ, Hol) for ΛT consists of (1), the transport $\Upsilon : \Lambda X \longrightarrow \Lambda H$, and (2), the holonomy $\mathrm{Hol} : \Lambda X \longrightarrow H$. One aim is to realise [**12**] for double groupoids and connection pairs (Υ, Hol) in terms of the geometric data available. In view of the growing interest in 2-dimensional structures in differential geometry [**2**] [**17**], we note the following:

Theorem 2.1. [**9**] [**10**] [**12**] *The following categories are equivalent: 1) Crossed modules of groupoids, 2)* 2*–Groupoids, and 3) edge symmetric double groupoids with special connection.*

Structures 1) and 3) have been used extensively in homotopical investigations (see the survey [**5**]) while crossed modules of Lie groups are used in [**17**], and are called Lie 2-groups.

2.2. *The groupoid* $\mathbb{G}(P, X)$ *associated to a principal G–bundle.* Let X be a smooth connected manifold and consider a principal G–bundle $\pi : P \longrightarrow X$, where G is a Lie group. There is an associated locally trivial groupoid over X given by

$$\mathbb{G}(P, X) = P \times_G P \rightrightarrows X, \tag{2.4}$$

where for $u_1, u_2, \in P,\ g \in G$, we have $(u_1, u_2)g = (u_1 \cdot g, u_2 \cdot g)$, with equivalence classes satisfying the multiplication rule $[u_1, u_2][u_3, u_4] = [u_1 \cdot g, u_4]$, for which $u_3 = u_2 \cdot g$ in the fibre over $\pi(u_2) = \pi(u_3)$. Furthermore, if $u_0 \in P,\ x_0 = \pi(u_0)$, then there are homeomorphisms and isomorphisms respectively, given by

$$P \longrightarrow P \times_G P|_{x_0},\ u \mapsto [u, u_0],$$
$$G \longrightarrow P \times_G P|_{x_0}^{x_0},\ g \mapsto [u_0, g \cdot u_0], \tag{2.5}$$

(see [**18**, Ch. II]). The groupoid $\mathbb{G}(P, X)$ (sometimes called the *Ehresmann symmetry groupoid*) will play a significant role in all that follows.

3. The thin path groupoid.

3.1. *Thin higher homotopy groups.* Our development here follows [**14**] [**17**] to which we refer for further details. Let the set of all smooth n–loops in X be denoted by $\Omega_n^\infty(X, *)$, where for $n = 1$, we shall just write $\Omega^\infty(X, *)$. The product $\ell_1 \star \ell_2$ of two n–loops and the inverse of an n–loop are well–defined.

Definition 3.1. Two loops ℓ_1, ℓ_2 are called *rank–n homotopic* or *thin homotopic*, denoted by $\ell_1 \overset{n}{\sim} \ell_2$, if there exists an $\epsilon > 0$, and a homotopy $H : [0,1] \times I^n \longrightarrow X$, satisfying:

(1) $t_i \in [0, \epsilon) \cup (1 - \epsilon, 1] \Rightarrow\ H(s, t_1, \ldots, t_n) = *,\ 1 \leqslant i \leqslant n.$

(2) $s \in [0, \epsilon) \Rightarrow\ H(s, t_1, \ldots, t_n) = \ell_1(t_1, \ldots, t_n).$

(3) $s \in (1 - \epsilon, 1] \Rightarrow\ H(s, t_1, \ldots, t_n) = \ell_2(t_1, \ldots, t_n).$

(4) H is smooth throughout its domain.

(5) rank $DH_{(s,t_1,\ldots,t_n)} \leqslant n$, throughout its domain.

We denote by $\pi_n^n(X, *)$ the set of equivalences classes under $\overset{n}{\sim}$ of (thin) n–loops in X. Observe that $\pi_n^n(X, *)$ is abelian for $n \geqslant 2$. Also, for $\dim X \leqslant n$, we have $\pi_n^n(X, *) = \pi_n(X, *)$, and for $\dim X > n$, the group $\pi_n^n(X, *)$ is infinite dimensional.

3.2. *The smooth thin path groupoid* $\Lambda_1^1(X)$. Here we will set $n = 1$. For a smooth path $\lambda \in \Lambda_1^1(X)$, a point $t_0 \in I$ is a *sitting instant* if there exists $\epsilon > 0$ such that λ is constant on $[0,1] \cap (t_0 - \epsilon, t_0 + \epsilon)$. It is shown in [**14**] that there is always a re–parametrization of a smooth path such that it sits this way at its endpoints. In this case $\pi_1^1(X, *) = \Omega^\infty(X, *)/\overset{1}{\sim}$. Likewise, there is the *smooth thin path groupoid* $\Lambda_1^1(X) \subset \Lambda(X)$ consisting of smooth paths $\lambda : I \longrightarrow X$ which are constant in a neighbourhood of $t = 0,\ t = 1$, identified up to rank 1 homotopy, with

$$0 \leqslant t \leqslant \epsilon \Rightarrow H(s,t) = \lambda(0),\ 1 - \epsilon \leqslant t \leqslant 1 \Rightarrow H(s,t) = \lambda(1), \tag{3.1}$$

and with multiplication $\star$. Henceforth, relevant path spaces will be considered as smooth thin path groupoids.

3.3. *The transport law.* Following [**19**], we outline several properties of the (smooth) path connection

$$\Upsilon : \Lambda_1^1(X) \longrightarrow \Lambda_1^1 \mathbb{G}(P,X), \qquad \lambda \mapsto \Upsilon^{\lambda}, \tag{3.2}$$

which for $t \in [0,1]$, satisfies $\alpha\Upsilon^{\lambda}(t) = \lambda(0)$, and $\beta\Upsilon^{\lambda}(t) = \lambda(t)$.

If $\psi : [0,1] \longrightarrow [t_0, t_1] \subset [0,1]$ is a diffeomorphism, we have the relationship

$$\Upsilon^{\lambda} \cdot \psi = \Upsilon^{\lambda\psi} \circ_2 \Upsilon^{\lambda}(\psi(0)), \tag{3.3}$$

which leads to $\Upsilon^{\lambda}(0) = \widetilde{(\lambda(0))}$. Further, if $\lambda, \bar{\lambda} \in \Lambda_1^1(X)$ satisfy $\lambda(1) = \bar{\lambda}(0)$, so that $\lambda \circ \bar{\lambda}$ is defined and is smooth, then

$$\lambda = (\lambda \circ \bar{\lambda}) \cdot \psi_1, \;\; \bar{\lambda} = (\lambda \circ \bar{\lambda}) \cdot \psi_2, \tag{3.4}$$

where $\psi_1(t) = \frac{1}{2}t$ and $\psi_2(t) = \frac{1}{2}t + \frac{1}{2}$. Given $\lambda \circ \bar{\lambda} \in \Lambda_1^1(X)$, then on applying (3.3) to $\lambda \circ \bar{\lambda}$, along with either ψ_1 or ψ_2, leads to an explicit statement of the transport law for this case:

$$\Upsilon^{\lambda \circ \bar{\lambda}}(t) = \begin{cases} \Upsilon^{\lambda}(2t), \;\; 0 \leqslant t \leqslant \frac{1}{2}, \\ \Upsilon^{\bar{\lambda}}(2t-1) \circ_2 \Upsilon^{\lambda}(1), \;\; \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \tag{3.5}$$

In particular, we have $\Upsilon^{\lambda \circ \bar{\lambda}}(1) = \Upsilon^{\bar{\lambda}}(1) \circ_2 \Upsilon^{\lambda}(1)$, and $\Upsilon^{\lambda^{-1}}(1) = [\Upsilon^{\lambda}(1)]^{-1}$.

Now suppose that ω_P denotes a given connection 1–form on $P \longrightarrow X$. We refer to e.g. [**16**] [**18**] for the usual concept of horizontal path lifting and parallel transport induced by ω_P (which defines a 'simple infinitesimal connection' in the sense of [**19**]).

Lemma 3.1. [**19**] [**18**, Theorem 7.3] *Given a (smooth) principal G–bundle $P \longrightarrow X$, a path connection $\Upsilon : \Lambda_1^1(X) \longrightarrow \Lambda_1^1 \mathbb{G}(P,X)$ is determined uniquely by a choice of connection 1–form ω_P. Specifically, there exists a one–to–one correspondence between Υ and ω_P, such that for $\gamma \in \Lambda_1^1(X)$, $\hat{\gamma} = \Upsilon(\gamma)$ and $t_0 \in [0,1]$, we have*

$$\frac{d}{dt}\, \hat{\gamma}(t_0) = (R_{\hat{\gamma}(t_0)})_* \; \omega_P(\frac{d}{dt}\gamma(t_0)). \tag{3.6}$$

Thanks to thin homotopies, we can replace the topological double category of Moore paths by a double Lie groupoid. Given a smooth principal G–bundle $P \longrightarrow X$ with connection 1–form ω_P, we can use $\Lambda_1^1(X)$ together with the data provided by ω_P, to produce a double groupoid:

$$\begin{cases} (\Lambda_1^1\mathbb{G}(P,X), \mathbb{G}(P,X), \alpha_1, \beta_1, \circ_1, \epsilon_1), & (\Lambda_1^1\mathbb{G}(P,X), \Lambda_1^1(X), \alpha_2, \beta_2, \circ_2, \epsilon_2), \\ (\Lambda_1^1(X), X, \alpha, \beta, \circ, e), & (\mathbb{G}(P,X), X, \alpha, \beta, \circ, e). \end{cases} \tag{3.7}$$

In the context of [**12**], the existence of the connection pair (Υ, Hol) thus specializes to (1), the parallel transport as a smooth function on smooth groupoids $\Upsilon : \Lambda_1^1(X) \longrightarrow \Lambda_1^1 \mathbb{G}(P, X)$ satisfying the transport law, and (2), the holonomy $\mathrm{Hol} : \Lambda_1^1(X) \longrightarrow \mathbb{G}(P, X)$.

4. Gerbes and 2–holonomy.

4.1. *Abelian gerbes.* The references for this section are [**4**] [**13**] [**15**] [**17**]. Let X be a smooth (finite dimensional) connected manifold and let $\mathcal{U} = \{U_i : i \in J\}$ be 'good' open cover of X meaning that all p–fold ($p \geqslant 1$) intersections $U_{i_1 \cdots i_p} = U_{i_1} \cap \cdots \cap U_{i_p}$, are contractible. The data for a *line bundle* ($G = \mathrm{U}(1)$) *gerbe* $\mathcal{P} \longrightarrow X$, is given as follows:

- On each U_{ij}, there is a line bundle $L_{ij} \longrightarrow U_{ij}$, satisfying $L_{ij} \cong L_{ij}^{-1}$.
- There are trivializations θ_{ijk} of $L_{ij} L_{jk} L_{ki}$ on U_{ijk} that satisfy the cocycle condition $\delta\theta \equiv 1$ on U_{ijkl}.

The corresponding data for a connective structure on $\mathcal{P} \longrightarrow X$, is given as follows:

(1) A 0–*object connection* is a covariant derivative ∇_{ij} on L_{ij}, such that for each $i, j, k \in J$, it satisfies the condition:

$$\nabla_{ijk} \theta_{ijk} = 0. \tag{4.1}$$

In terms of the corresponding 1–forms $A_{ij} \in A^1(U_{ij})$, there is the equivalent relationship

$$\iota(A_{ij} + A_{jk} + A_{ki}) = -d \log g_{ijk}, \ (A_{ij} = -A_{ji}), \tag{4.2}$$

where g is a Čech 2–cocycle, $\delta g \equiv 1$.

(2) A 1–*connection* is defined by local 2–forms $F_i \in A^2(U_i)$ such that on U_{ij}, it satisfies

$$F_i - F_j = F_{ij} = \sigma^*(K(\nabla_{ij})), \tag{4.3}$$

where $\sigma_{ij} \in \Gamma(U_{ij}, L_{ij})$, and K denotes the usual curvature. The latter is equivalent to the condition $F_i - F_j = F_{ij} = dA_{ij}$.

The abelian gerbe with its connective structure is denoted by $(\mathcal{P}, \mathcal{A})$.

4.2. *The holonomy of $(\mathcal{P}, \mathcal{A})$.* Suppose that $s : I^2 \longrightarrow X$ is a 2–loop. The pull–back gerbe $s^*(\mathcal{P})$ is then a trivial gerbe and we can choose some trivialization such that an object is given in terms of line bundles trivialized by sections σ_i over $V_i = s^{-1}(U_i)$, with an object connection ∇_i. A global 2–form ε (the error 2–form) is defined on I^2 by

$$\varepsilon | V_i = s^*(F_i) - \sigma_i^*(K(\nabla_i)). \tag{4.4}$$

The holonomy $\mathrm{Hol}(s)$ of $(\mathcal{P}, \mathcal{A})$ around the 2–loop s is then given by

$$\exp(\iota \int_{I^2} \varepsilon). \tag{4.5}$$

It is independent of the choice of object and the connection on the object, and is constant on thin homotopy classes. Furthermore, Hol defines the 2–*holonomy* of $(\mathcal{P}, \mathcal{A})$ in terms of a group homomorphism

$$\mathrm{Hol} \ : \ \pi_2^2(X, *) \longrightarrow \mathrm{U}(1), \tag{4.6}$$

depending on $(\mathcal{P}, \mathcal{A})$ up to equivalence, and which is smooth on families of 2–loops in $\Omega_2^\infty(X)$ when projected to $\pi_2^2(X, *)$ (see [**17**]).

5. Parallel transport and holonomy in abelian gerbes.

5.1. *Transport of the gerbe data.* We have already noted the (thin) parallel transport

$$\Upsilon \ : \ \Lambda_1^1(X) \longrightarrow \Lambda_1^1 \mathbb{G}(P, X), \tag{5.1}$$

determined by the connection 1–form ω_P on a principal G–bundle $P \longrightarrow X$. Now we look for the analogous functor in the case of the gerbe connection. The idea is that the gerbe data $(\mathcal{P}, \mathcal{A})$ determines a groupoid on the thin loop groupoid $L_1^1(X) \subset \Lambda_1^1(X)$. But if we assume X is path connected and fix a base point, as we will henceforth, then the gerbe data will readily lead us to the relevant transport groupoids over their space of objects $\pi_1^1(X, *)$.

Using $(\mathcal{P}, \mathcal{A})$, it is shown in [**13**] that there corresponds a smooth line bundle $L^{\mathcal{P}} \longrightarrow \Omega^\infty(X, *)$ with connection. Here one considers a quadruple of the type (γ, F, ∇, z) where $\gamma \in \Omega^\infty(X, *)$, F is an object for $\gamma^* \mathcal{P}$ on S^1, ∇ is an object connection in F, and $z \in \mathbb{C}^*$. These are defined up to a certain equivalence relation. Now consider a homotopy $H : I^2 \longrightarrow X$ between loops γ, μ and let F, ∇ denote the object and object connections respectively, for the pull–back gerbe $H^*(\mathcal{P}) \longrightarrow I^2$. The parallel transport Υ along the homotopy H is given explicitly by:

$$\Upsilon(H)(\gamma, \gamma^* F, \gamma^* \nabla, 1) = (\mu, \mu^* F, \mu^* \nabla, 1) \exp(\iota \int_{I^2} \varepsilon). \tag{5.2}$$

Note that the smooth line bundle with connection

$$p : (L^{\mathcal{P}}, D) \longrightarrow \Omega^\infty(X, *), \tag{5.3}$$

is representable as a principal U(1)–bundle

$$\mathrm{U}(1) \hookrightarrow (L^{\mathcal{P}}, \omega_P) \longrightarrow \Omega^\infty(X, *), \tag{5.4}$$

with connection 1–form ω_P. Parallel transport under ω_P is defined along the cylindrical groupoid

$$\mathbb{C}(X, *) = \Lambda_1^1 \Omega^\infty(X, *) \rightrightarrows \Omega^\infty(X, *), \tag{5.5}$$

where elements of $\mathbb{C}(X,*)$ are regarded as homotopies between loops and whose morphisms are thin homotopy classes of homotopies between loops (via based loops). At the same time (5.4) as determined by the gerbe data leads to the groupoid

$$\mathbb{G}(\mathcal{P}) = \mathbb{G}(\mathcal{P},\mathcal{A},X) = L^{\mathcal{P}} \times_{\mathrm{U}(1)} L^{\mathcal{P}} \rightrightarrows \Omega^{\infty}(X,*). \tag{5.6}$$

Let then $T = (\mathbb{G}(\mathcal{P}), \Omega^{\infty}(X,*), \alpha, \beta, \circ, e)$ whose associated double category of paths $\Lambda_1^1 T$ contains the horizontal and vertical groupoids $H = \mathbb{G}(\mathcal{P}) \rightrightarrows \Omega^{\infty}(X,*)$, and $V = \mathbb{C}(X,*) \rightrightarrows \Omega^{\infty}(X,*)$, respectively. As a result $\Lambda_1^1 T$ is specified by the four related groupoids:

$$\begin{cases}(\Lambda_1^1\mathbb{G}(\mathcal{P}), \mathbb{G}(\mathcal{P}), \alpha_1, \beta_1, \circ_1, \epsilon_1), & (\Lambda_1^1\mathbb{G}(\mathcal{P}), \mathbb{C}(X,*), \alpha_2, \beta_2, \circ_2, \epsilon_2),\\ (\mathbb{C}(X,*), \Omega^{\infty}(X,*), \alpha,\beta, \circ, e), & (\mathbb{G}(\mathcal{P}), \Omega^{\infty}(X,*), \alpha,\beta, \circ, f).\end{cases} \tag{5.7}$$

To proceed, let $s_\gamma, s_\mu \in \Gamma(\Omega^{\infty}(X,*), L^{\mathcal{P}})$ be smooth sections and set $g_\varepsilon = \exp(\iota \int_{I^2} \varepsilon) \in \mathbb{C}^*$, so that (5.2) can be expressed as $\Upsilon(H)(s_\gamma) = s_\mu \cdot g_\varepsilon$. In this way, we can reduce matters to considering the usual parallel transport in the U(1)–bundle $(L^{\mathcal{P}}, \omega_P) \longrightarrow \Omega^{\infty}(X,*)$ as induced by ω_P. Following Lemma 3.1, ω_P uniquely determines a smooth (thin) parallel transport

$$\Upsilon : \mathbb{C}(X,*) \longrightarrow \Lambda_1^1\mathbb{G}(\mathcal{P}), \tag{5.8}$$

via the homotopy $H = \circ_2$ (horizontal structure) satisfying the transport law (3.5). As for the holonomy, we see from (2.5) that the assignment $g \mapsto [u_0, u_0 \cdot g]$ for $u_0 \in L^{\mathcal{P}}$, $p(u_0) = \gamma$, induces just as in (2.5) an isomorphism

$$\mathrm{U}(1) \cong \mathbb{G}(\mathcal{P})|_\gamma^\gamma. \tag{5.9}$$

Since X is (path) connected, this leads to the holonomy $\mathrm{Hol} : \mathbb{C}(X,*) \longrightarrow \mathrm{U}(1)$, and in the context of the double category of paths, the holonomy functor $\mathrm{Hol} : \mathbb{C}(X,*) \longrightarrow \mathbb{G}(\mathcal{P})$. It is straightforward to check that $\mathrm{Hol}(a)$ satisfies the relations showing that it is a bounding edge of the square $\Upsilon(a)$. We can summarize matters as follows:

Proposition 5.1. *Given the* U(1)*–gerbe data* $(\mathcal{P},\mathcal{A})$ *over a path–connected space* X*, there is an associated double groupoid of thin paths*

$$\begin{cases}(\Lambda_1^1\mathbb{G}(\mathcal{P}), \mathbb{G}(\mathcal{P}), \alpha_1, \beta_1, \circ_1, \epsilon_1), & (\Lambda_1^1\mathbb{G}(\mathcal{P}), \mathbb{C}(X,*), \alpha_2, \beta_2, \circ_2, \epsilon_2),\\ (\mathbb{C}(X,*), \Omega^{\infty}(X,*), \alpha,\beta, \circ, e), & (\mathbb{G}(\mathcal{P}), \Omega^{\infty}(X,*), \alpha,\beta, \circ, f),\end{cases} \tag{5.10}$$

and a connection pair (Υ, Hol) *given by* (1)*, the transport* $\Upsilon : \mathbb{C}(X,*) \longrightarrow \Lambda_1^1\mathbb{G}(\mathcal{P})$*, and* (2)*, the holonomy* $\mathrm{Hol} : \mathbb{C}(X,*) \longrightarrow \mathrm{U}(1)$*.*

5.2. *Thin homotopies again.* If a pair of homotopies $H_1, H_2 : I^2 \longrightarrow X$, between a given pair of paths, are themselves homotopic via a homotopy $Q : I^3 \longrightarrow X$, then the parallel transport around $H_1 H_2^{-1}$ is expressed by

$$\int_{I^3} Q^* B. \tag{5.11}$$

Accordingly, the parallel transport along H_1 and H_2 is the same if the latter are thin homotopic since Q may be chosen to have rank $\leqslant 2$ everywhere. In this way we actually achieve a line bundle descending to $\pi_1^1(X, *)$:

$$p : (L^{\mathcal{P}}, D) \longrightarrow \pi_1^1(X, *). \tag{5.12}$$

It will be convenient to express this in terms of the principal U(1)–bundle with connection 1–form ω_P,

$$\mathrm{U}(1) \hookrightarrow (L^{\mathcal{P}}, \omega_P) \longrightarrow \pi_1^1(X, *), \tag{5.13}$$

together with the groupoid

$$\mathbb{G}(\mathcal{P}) = \mathbb{G}(\mathcal{P}, \mathcal{A}, X) = L^{\mathcal{P}} \times_{\mathrm{U}(1)} L^{\mathcal{P}} \rightrightarrows \pi_1^1(X, *). \tag{5.14}$$

Next we recall the cylindrical groupoid $\mathbb{C}(X, *) \rightrightarrows \Omega^\infty(X, *)$ and its morphisms regarded as thin homotopy classes of homotopies constituting the vertical structure of $\Lambda_1^1 T$. As explained in [**17**], the horizontal composition $\circ = \star$ determines the monoidal composition of homotopies and structure defined via the composition of loops $\circ_1$, as well as the corresponding composition of vertical homotopies $H : I^2 \longrightarrow X$ between concatenated loops λ, μ, say. The descent to $\pi_1^1(X, *)$ induces the *thin cylindrical groupoid* $\mathbb{C}_2^2(X, *) \rightrightarrows \pi_1^1(X, *)$. Intuitively, we can view the latter as given by

$$\mathbb{C}_2^2(X, *) = \Lambda_1^1(\pi_1^1(X, *)) = \Lambda(\Omega^\infty(X, *)/\overset{1}{\sim})/\overset{1}{\sim}, \tag{5.15}$$

which encapsulates the 2–holonomy. In order to simplify the notation, let us set $Y = \pi_1^1(X, *)$. Consequently under the relation $\overset{1}{\sim}$, the double category $\Lambda_1^1 T$ specializes to a double groupoid

$$\Lambda_1^1 T = (\Lambda_1^1 \mathbb{G}(\mathcal{P}), \mathbb{G}(\mathcal{P}), \mathbb{C}_2^2(X, *), Y), \tag{5.16}$$

for which the squares are elements of $\Lambda_1^1 \mathbb{G}(\mathcal{P})$, the horizontal edges are $\mathbb{G}(\mathcal{P})$, the vertical edges are $\mathbb{C}_2^2(X, *)$, and the set of points is $Y = \pi_1^1(X, *)$.

Proposition 5.2. (cf [**17**]) *With respect to the principal* U(1)*–bundle* $(L^{\mathcal{P}}, \omega_P) \longrightarrow \pi_1^1(X, *)$ *determined by the gerbe data* $(\mathcal{P}, \mathcal{A})$*, we have a double groupoid*

$$\begin{cases} (\Lambda_1^1 \mathbb{G}(\mathcal{P}), \mathbb{G}(\mathcal{P}), \alpha_1, \beta_1, \circ_1, \epsilon_1), & (\Lambda_1^1 \mathbb{G}(\mathcal{P}), \mathbb{C}_2^2(X, *), \alpha_2, \beta_2, \circ_2, \epsilon_2), \\ (\mathbb{C}_2^2(X, *), Y, \alpha, \beta, \circ, e), & (\mathbb{G}(\mathcal{P}), Y, \alpha, \beta, \circ, f), \end{cases} \tag{5.17}$$

and a connection pair (Υ, Hol) *given by* (1), *the transport* $\Upsilon : \mathbb{C}_2^2(X,*) \longrightarrow \Lambda_1^1 \mathbb{G}(\mathcal{P})$, *and* (2), *the holonomy* $\mathrm{Hol} : \mathbb{C}_2^2(X,*) \longrightarrow \mathbb{G}(\mathcal{P})$. *In particular,* $\mathbb{G}(\mathcal{P})$ *and* $\mathbb{C}_2^2(X,*)$ *each admit the structure of a Lie 2–groupoid.*

Effectively, this is a special case of Proposition 5.1 when restricted to $Y = \pi_1^1(X,*)$. The main point is the existence of a certain normal monoidal subgroupoid $N(X,*)$ of $\mathbb{C}(X,*)$ [**17**] such that on factoring–out by $\overset{1}{\sim}$, the horizontal arrows in the diagram below, represent well–defined morphisms of groupoids:

$$\begin{array}{ccc} \mathbb{C}(X,*) & \xrightarrow{\overset{1}{\sim}} & \mathbb{C}_2^2(X,*) = \mathbb{C}(X,*)/N(X,*) \\ \downdownarrows \Big\downarrow & & \Big\downarrow \downdownarrows \\ \Omega^\infty(X,*) & \xrightarrow{\overset{1}{\sim}} & Y = \Omega^\infty(X,*)/\overset{1}{\sim} = \pi_1^1(X,*). \end{array} \tag{5.18}$$

REMARK 5.1. It is shown in [**17**] that the Lie 2–groupoids

$$\begin{aligned} &\mathbb{D}_1 = (\mathbb{C}_2^2(X,*), H_1, V_1, Y, \circ, id), \\ &\mathbb{D}_2 = (\mathbb{G}(\mathcal{P}), H_2, V_2, Y, \star, id), \ (V_1, V_2 \text{ discrete}), \end{aligned} \tag{5.19}$$

together with their respective monoidal structures, actually reduce to Lie 2–groups, and when X is simply–connected, the gerbe data $(\mathcal{P}, \mathcal{A})$ can be constructed directly from the finer 2–holonomy $\mathrm{Hol} : \pi_2^2(X,*) \longrightarrow \mathrm{U}(1)$, and conversely.

6. Local subgroupoids.

6.1. *The local subgroupoid of a path connection.* In this section we describe how a holonomy Lie groupoid can be associated to a U(1)–gerbe using a local subgroupoid constructed from its local connective structure. To proceed, let X be a topological space and $\alpha, \beta : \mathbb{G} \rightrightarrows X = \mathrm{Ob}(\mathbb{G})$, a topological groupoid. For an open set $U \subset X$, let $\mathbb{G}|U$ be the full subgroupoid of $\mathbb{G}$ on U. Let $L_{\mathbb{G}}(U)$ be the set of all wide subgroupoids of $\mathbb{G}|U$. For $V \subseteq U$, there is a restriction map $L_{UV} : L_{\mathbb{G}}(U) \longrightarrow L_{\mathbb{G}}(V)$ sending $H \mapsto H|V$. Thus $L_{\mathbb{G}}$ has the structure of a presheaf on X.

Consider the sheaf $p_{\mathbb{G}} : \mathcal{L}_{\mathbb{G}} \longrightarrow X$ formed from the presheaf $L_{\mathbb{G}}$. For $x \in X$, the stalk $p_{\mathbb{G}}^{-1}(x)$ of $\mathcal{L}_{\mathbb{G}}$ has elements the germs $[U, H_U]_x$, for U open in X and $x \in U$. $H|U$ is a wide subgroupoid of $\mathbb{G}|U$ and the equivalence relation $\sim_x$ yielding the germ at x, is such that $H_U \sim_x K_V$, where K_V is a wide subgroupoid of $\mathbb{G}|V$ if and only if there exists a neighbourhood W of x such that $W \subseteq U \cap V$ and $H_U|W = K_V|W$. The topology of $\mathcal{L}_{\mathbb{G}}$ is the usual sheaf topology with a sub–base of sets $\{[U, H_U]_x : x \in X\}$, for all open subsets U of X and wide subgroupoids H of $\mathbb{G}|U$.

Definition 6.1. A *local subgroupoid* of $\mathbb{G}$ on the topological space X is a continuous global section S of the sheaf $p_{\mathbb{G}} : \mathcal{L}_{\mathbb{G}} \longrightarrow X$ associated to the presheaf $L_{\mathbb{G}}$.

Associated to a local subgroupoid are a number of technical features such as the type of 'atlas' with which one needs to work. For instance, there is a 'regular atlas' with the 'globally adapted' property; in the case of a Lie local subgroupoid, a 'strictly regular atlas', etc. For an explanation of these terms and further properties we refer to [**6**], [**8**].

Suppose we have a continuous path connection $\Upsilon : \Lambda_1^1(X) \longrightarrow \Lambda_1^1(\mathbb{G})$ with the usual properties as before. We denote by $C_\Upsilon(\mathbb{G})$ the set of all $g \in \mathbb{G}$ such that if $\alpha(g) = x$, then there exists a path λ in X such that $\Upsilon(\lambda)$ joins g to the identity $\mathbf{1}_x$; that is, $\Upsilon(\lambda)(0) = \mathbf{1}_x$ and $\Upsilon(\lambda)(1) = g$. We next state some essential properties of $C_\Upsilon(\mathbb{G})$ following [**8**, §4] :

Proposition 6.1.

(1) *$C_\Upsilon(\mathbb{G})$ is a wide subgroupoid of $\mathbb{G}$.*

(2) *If Υ is a path connection on $\mathbb{G}$ and $\mathcal{U}$ is an open cover of X, then $C_\Upsilon(\mathbb{G})$ is generated by the family $C_\Upsilon(\mathbb{G}|U)$, for all $U \in \mathcal{U}$.*

6.2. *Geodesic and path local property of the atlas.* In order to define a corresponding local subgroupoid $C_\Upsilon(\mathbb{G},\mathcal{U})$, it is necessary to work with an atlas of the following type. Given an open cover $\mathcal{U} = \{U_i : i \in I\}$ for X, we assume for each $i \in I$ there is a collection of paths, denoted by $\mathrm{geod}(U_i)$ in U_i, whereby $\lambda \in \mathrm{geod}(U_i)$ with $\lambda(0) = x,\ \lambda(1) = y$, is called a *geodesic path* from x to y. Further, we assume

(i) If $x, y \in U_i$, then there is a unique geodesic path $\mathrm{geod}_i(x, y)$ from x to y.

(ii) If $x, y \in U_i \cap U_j$, then $\mathrm{geod}_i(x, y) = \mathrm{geod}_j(x, y)$.

(iii) The path connection is flat for this structure, meaning that if $\lambda : x \longrightarrow y$ is any path in U_i, then $\Upsilon(\lambda)(1) = \Upsilon(\mathrm{geod}_i(x, y))(1)$.

For such an atlas it follows from [**8**] (Proposition 4.3) that there exists a local subgroupoid

$$C_\Upsilon(\mathbb{G},\mathcal{U})(x) = [U_i, C_\Upsilon(\mathbb{G}|U_i)]_x. \tag{6.1}$$

We also need to specify the conditions to ensure that (6.1) can be globally adapted. Following [**8**, Proposition and Definition 3.5 and 4.4], the equality $C_\Upsilon(\mathbb{G},\mathcal{U})|U = C_\Upsilon(\mathbb{G}|U,\mathcal{U} \cap U)$ holds if for any $i, j \in I$ and $x \in U_i \cap U_j \cap U$, there is an open set W such that $x \in W \subseteq U_i \cap U_j \cap U$, and $C_\Upsilon(\mathbb{G}|U_i)|W = C_\Upsilon(\mathbb{G}|U_j \cap U)|W$. Let us say that the cover $\mathcal{U}$ is (Υ–)*path local for* $C_\Upsilon(\mathbb{G},\mathcal{U})$ if this condition holds for all open sets U of X. It follows from [**8**, Corollary 7.10] that any (Υ–) path local atlas of the local subgroupoid $C_\Upsilon(\mathbb{G},\mathcal{U})$, is globally adapted. Next let

$$W(\mathcal{U}_S) = \bigcup_{i\in I} H_i, \tag{6.2}$$

where we are given a strictly regular (Υ–) path local atlas $\mathcal{U}_S$ for $\mathbb{G}$, and H_i a Lie subgroupoid of $\mathbb{G}$. Since such an atlas is globally adapted, it follows from [**6**, Theorem 3.7] that there exists a locally Lie groupoid $(\mathrm{glob}(C_\Upsilon(\mathbb{G}|U)), W(\mathcal{U}_S))$. Furthermore, the Globalisation Theorem of [**6**, Theorem 3.8] establishes the existence of the associated holonomy Lie groupoid $\mathrm{Hol}(S,\mathcal{U}_S)$.

6.3. *Application to abelian gerbes–Proof of Theorem 1.1.* We proceed now to an application in the context of [**13**] Chapter 5 to which we refer for the notions of a *torsor, a connective structure on a sheaf of groupoids* as well as other details. The application is also in the context of thin homotopies as in [**17**].

Let $\mathbb{G} \rightrightarrows X$ be a groupoid and $\mathcal{U} = (U_i)_{i\in I}$ be a good open covering of X. As above, we consider full subgroupoids $\mathbb{G}|U_i$ as well as wide subgroupoids H_i of the latter. We consider principal G–bundles $P_i \longrightarrow U_i$, along with isomorphisms

$$u_{ij} : P_j|U_{ij} \xrightarrow{\cong} P_i|U_{ij}, \tag{6.3}$$

in $\mathbb{G}|U_{ij}$. As previously we assume that G is the abelian group $\mathrm{U}(1)$. We consider a section h_{ijk} (of the band $\underline{\mathbb{C}}^*$) over U_{ijk} by $h_{ijk} = u_{ik}^{-1}u_{ij}u_{jk}$ where the latter is viewed as an equality in $\mathrm{Aut}(P_k)$, noting that this corresponds to a Čech 2–cocycle. Let us decree the full subgroupoids $\mathbb{G}|U_i$ to be $\mathbb{G}(P_i, U_i)$ and the H_i to be locally sectionable wide Lie subgroupoids of the latter.

Just as before let $\mathcal{L}_{\mathbb{G}} \longrightarrow X$ be the sheaf corresponding to the presheaf $L_{\mathbb{G}}$ of wide Lie subgroupoids of $\mathbb{G}$. As a sheaf of groupoids in its own right, we assume that $\mathcal{L}_{\mathbb{G}}$ is equipped with a connective structure Co (in the sense of [**13**]). Next we choose an object ω_i of the torsor $\mathrm{Co}(P_i)$, where we regard ω_i as simply a connection 1–form on $P_i \longrightarrow U_i$, and assume the geodesic–path local property (§6.2) of $\mathcal{U}$ relative to the ω_i. We denote by $(\mathcal{P}, \mathcal{A}, \mathrm{Geod})$ the corresponding abelian gerbe data together with this property. The next step is to apply the techniques of §3 to this situation.

To proceed, we define a 1–form ω_{ij} on U_{ij} by

$$\omega_{ij} = \omega_i - (u_{ij})_*(\omega_j). \tag{6.4}$$

Recalling $h_{ijk} = u_{ik}^{-1}u_{ij}u_{jk}$, it follows that

$$\omega_{ij} + \omega_{jk} - \omega_{ik} = \omega_i - (u_{ij}u_{jk}u_{ki})_*(\omega_i) = h_{ijk}^{-1}\, dh_{ijk}. \tag{6.5}$$

This data, denoted $(\underline{h}, \underline{\omega})$, so defines a Čech 2–cocycle, but with coefficients in the complex of sheaves $\underline{\mathbb{C}}^* \xrightarrow{d\log} \underline{A}^1_{X,\mathbb{C}}$. On restriction to thin path groupoids, the object connections ω_i of $\mathrm{Co}(P_i)$ determine on each U_i a (thin) path connection

$$\Lambda_1^1(U_i)\xrightarrow{\Upsilon_i}\Lambda_1^1(H_i)\subset\Lambda_1^1(\mathbb{G}|U_i), \tag{6.6}$$

satisfying the local flatness property $\Upsilon_i(\lambda)(1) = \Upsilon_i(\mathrm{geod}_i(x, y))(1)$. Also, there is an open set $W \subseteq U_{ij}$, for which $H_i|W = H_j|W = H_{ij}$, and so on the overlaps U_{ij}, we have a (thin) path connection

$$\Lambda_1^1(U_{ij})\xrightarrow{\Upsilon_{ij}}\Lambda_1^1(H_{ij})\subset\Lambda_1^1(\mathbb{G}|U_{ij}). \tag{6.7}$$

Consider the local subgroupoid of the atlas as given by $S(x) = [U_i, H_i]_x$. At the same time there is a wide subgroupoid generated by the family $C_{\Upsilon_i}(\mathbb{G}|U_i)$ for all $U_i \in \mathcal{U}$. As noted earlier, this leads to a local subgroupoid $C(\mathcal{P}, \mathcal{A}, \mathrm{Geod})$ given by

$$C(\mathcal{P}, \mathcal{A}, \mathrm{Geod})(x) = [U_i, C_{\Upsilon_i}(\mathbb{G}|U_i)]_x. \tag{6.8}$$

Relative to a strictly regular path local atlas $\mathcal{U}(\mathcal{P}, \mathcal{A}, \mathrm{Geod})$, we apply the same considerations as before along with the globalisation [**6**, Theorem 3.8], to obtain a holonomy Lie groupoid $\mathrm{Hol}(\mathcal{P}, \mathcal{A})$, thus establishing Theorem 1.1.

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