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A CONSTRUCTION OF TRANSVERSE SUBMANIFOLDS

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Abstract. In case of Riemannian manifolds isometric actions admitting submanifolds which intersect each orbit orthogonally have nice geometric properties which generalize those of adjoint actions of compact semi-simple Lie groups as given by their Cartan–Weyl theory [**1**], [**4**], [**5**]. In case of isometric actions on Lorentz manifolds degenerate orbits may occur and this fact renders the very definition of orthogonally transverse submanifolds problematic, since orthogonality then does not imply transversality. Furthermore, simple examples show that it would be too restrictive to require that all orbits of an action should be intersected orthogonally by a single submanifold as in the Riemannian case. For the above reasons, it seems justified to reconsider the problem in more general affine settings. A construction is proposed below which in the case of an affine action under some assumptions yields a set of submanifolds intersecting generic orbits of the highest dimension transversally. The results thus obtained are then applied to isometric actions on Lorentz manifolds.

1. Some general facts.

Definitions. Let M be a smooth manifold and $\Phi : G \times M \to M$ effective smooth action of a connected Lie group G. Elements of the Lie algebra $\mathfrak{g}$ of G will be identified with the corresponding infinitesimal generators of Φ. Accordingly, if $X \in \mathfrak{g}$, then

$$\mathcal{Z}(X) = \{ z \in M \mid X(z) = 0_z \}$$

is called the *zero set* of X. Moreover, if $\mathfrak{h} < \mathfrak{g}$ is an arbitrary subalgebra, then the closed set

$$\mathcal{Z}(\mathfrak{h}) = \cap\{ \mathcal{Z}(X) \mid X \in \mathfrak{h} \}$$

is called the *zero set of the subalgebra.* If $z \in M$, then by the *isotropy subalgebra of the action* at z the subalgebra

$$\mathfrak{g}_z = \{ X \in \mathfrak{g} \mid X(z) = 0_z \}$$

is meant. Accordingly, if $\mathfrak{h} < \mathfrak{g}$ is an arbitrary subalgebra, then

$$\mathcal{I}(\mathfrak{h}) = \{ z \in M \mid \mathfrak{g}_z = \mathfrak{h} \}$$

is called the *isotropy set of the subalgebra* $\mathfrak{h}$, which is non-empty if and only if $\mathfrak{h}$ is an isotropy subalgebra. The inclusion $\mathcal{I}(\mathfrak{h}) \subset \mathcal{Z}(\mathfrak{h})$ is an obvious consequence of the preceding definitions.

PROPOSITION 1.1. *If* $\mathfrak{h} < \mathfrak{g}$ *is a subalgebra, then* $\mathcal{I}(\mathfrak{h})$ *is open in the closed set* $\mathcal{Z}(\mathfrak{h})$.

PROOF. A vector space homomorphism $\Phi_z : \mathfrak{g} \to T_zM$ is given for any $z \in M$ by

$$\Phi_z : \mathfrak{g} \ni X \mapsto X(z) \in T_zM.$$

Thus $z \in \mathcal{I}(\mathfrak{h})$ is valid if and only if the kernel of Φ_z is the subalgebra $\mathfrak{h}$. Assume now that $\mathcal{I}(\mathfrak{h})$ is not empty, fix a subspace $\mathfrak{m} \subset \mathfrak{g}$ which is a complement of the subalgebra $\mathfrak{h}$, and fix also a base $(X_1, \dots, X_k)$ of the subspace $\mathfrak{m}$. If $z \in \mathcal{I}(\mathfrak{h})$, then $(X_1(z), \dots, X_k(z))$ is a base of of the subspace $T_zG(z) \subset T_zM$ which is the tangent space of the orbit $G(z)$. But then there is a neighbourhood U of z in M such that the system $(X_1(x), \dots, X_k(x))$ is independent if $x \in U$ and consequently

$$\dim T_zG(z) \leq \dim T_xG(x), \ x \in U.$$

On the other hand, $x \in U \cap \mathcal{Z}(\mathfrak{h})$ implies that $\mathfrak{g}_z < \mathfrak{g}_x$ holds and therefore

$$\dim T_xG(x) \leq \dim T_zG(z).$$

The preceding inequalities imply that the equality

$$\dim T_zG(z) = \dim T_xG(x), \ x \in U \cap \mathcal{Z}(\mathfrak{h})$$

holds. But then the inclusion $\mathfrak{g}_z < \mathfrak{g}_x$ implies that $\mathfrak{g}_x = \mathfrak{g}_z$ for $x \in U \cap \mathcal{Z}(\mathfrak{h})$. Therefore $U \cap \mathcal{Z}(\mathfrak{h}) \subset \mathcal{I}(\mathfrak{h})$. □

DEFINITIONS. An isotropy subalgebra $\mathfrak{h} < \mathfrak{g}$ of the smooth effective action Φ is said to be *minimal* if there is no isotropy subalgebra $\mathfrak{h}' < \mathfrak{g}$ of Φ such that

$$\mathfrak{h}' \underset{\neq}{<} \mathfrak{h}.$$

An isotropy subalgebra $\mathfrak{h} < \mathfrak{g}$ of Φ is said to be of *minimal dimension* if there is no isotropy subalgebra $\mathfrak{h}' < \mathfrak{g}$ of Φ with

$$\dim \mathfrak{h}' < \dim \mathfrak{h}.$$

An isotropy subalgebra of minimal dimension is obviously a minimal one. Isotropy subalgebras $\mathfrak{h}, \mathfrak{h}' < \mathfrak{g}$ of the action Φ are said *to be of the same type* if they are conjugate, i.e. if $\mathfrak{h}' = Ad(g)\mathfrak{h}$ holds with some $g \in G$. If $\mathfrak{h}$ is an isotropy subalgebra of Φ, then by its *stratum* the set $\mathcal{S}(\mathfrak{h}) \subset M$ of those points of M is meant which have isotropy subalgebras of the same type as the given one $\mathfrak{h}$. If $G(z) \subset M$ is an orbit of Φ and $x = \Phi(g, z)$, then $\mathfrak{g}_x = Ad(g)\mathfrak{g}_z$, which means that $\mathfrak{g}_z$ and $\mathfrak{g}_x$ are of the same type. Therefore, the isotropy subalgebras corresponding to points of $G(z)$ form a complete conjugacy class of subalgebras in $\mathfrak{g}$. Consequently,

$$\mathcal{S}(\mathfrak{h}) = G(\mathcal{I}(\mathfrak{h})).$$

For the above reason, two orbits of Φ are said to have *the same type* if they yield the same conjugacy class of subalgebras. Therefore the union of those orbits which have the same type as a given one $G(z)$ is called the *stratum of the orbit* $G(z)$ and denoted by $\mathcal{S}(G(z))$. Therefore, if $\mathfrak{h} < \mathfrak{g}$ is an isotropy subalgebra and $\mathfrak{g}_z = \mathfrak{h}$ for some $z \in M$, then $\mathcal{S}(\mathfrak{h}) = \mathcal{S}(G(z))$.

PROPOSITION 1.2. *Let M be a smooth manifold and $\Phi : G \times M \to M$ effective smooth action of a connected Lie group G. Then the following hold:*

1. The union of the strata of minimal dimensional isotropy subalgebras of Φ is an open set.

2. If all the minimal isotropy subalgebras of Φ are of minimal dimension, then any orbit of Φ is intersected by the zero set of a minimal dimensional isotropy subalgebra of Φ.

PROOF. Let M' be the union of the strata of minimal dimensional isotropy subalgebras of Φ. Consider $\mathfrak{h} < \mathfrak{g}$ an isotropy subalgebra of minimal dimension, fix a subspace $\mathfrak{m} \subset \mathfrak{g}$, which is a complement of $\mathfrak{h}$, and also a base $(X_1, \ldots, X_k)$ of $\mathfrak{m}$. Let now $z \in \mathcal{S}(\mathfrak{h})$, then $(X_1(z), \ldots, X_k(z))$ is a base of $T_z G(z)$. Moreover, there is a neighbourhood $U \subset M$ of z such that the system $(X_1(x), \ldots, X_k(x))$ is independent for $x \in U$ and consequently,

$$\dim G(x) \geq \dim G(z)$$

for $x \in U$. But then $\dim \mathfrak{g}_x \leq \dim \mathfrak{g}_z$ for $x \in U$ and therefore $x \in M'$. Thus $U \subset \mathcal{S}(\mathfrak{h})$ is obtained.

Consider now an arbitrary point $x \in M$ and its isotropy subalgebra $\mathfrak{g}_x < \mathfrak{g}$. Then there is a minimal isotropy subalgebra $\mathfrak{h}$ of Φ with

$$\mathfrak{h} < \mathfrak{g}_x.$$

But then $x \in \mathcal{Z}(\mathfrak{h})$. On the other hand, by *assumption 2* above, $\mathfrak{h}$ is of minimal dimension. $\square$

2. Transverse submanifolds of affine actions.

DEFINITION. Let (M, ∇) be an affine manifold where M is a smooth manifold and ∇ a covariant derivation defined on M and

$$\Phi : G \times M \to M$$

an *affine action* on M, i.e. a smooth effective action such that the diffeomorphisms $\Phi_g : M \to M$ defined by

$$\Phi_g(z) = \Phi(g, z), \ \ z \in M$$

are affine transformations with respect to the covariant derivation ∇.

The following facts play the fundametal role in a subsequent construction: *Let* (M, ∇) *be an affine manifold and* $\Phi : G\times M \to M$ *an affine action, if* $\mathfrak{h} < \mathfrak{g}$ *is a subalgebra, then the connected components of the zero set* $\mathcal{Z}(\mathfrak{h})$ *are closed totally geodesic submanifolds of the affine manifold* ([**2**, II. p 61]). Consider now the isotropy set $\mathcal{I}(\mathfrak{h}) \subset \mathcal{Z}(\mathfrak{h})$; since it is open in $\mathcal{Z}(\mathfrak{h})$ by *Proposition 1.1*, it is also a totally geodesic submanifold of M and therefore it carries a canonical smooth manifold structure.

Lemma 2.1. *If* (M, ∇) *is an affine manifold,* $\Phi : G \times M \to M$ *affine action of a connected Lie group* G*, and* $\mathfrak{h} < \mathfrak{g}$ *an isotropy subalgebra of minimal dimension, then let*

$$\{ \ \mathfrak{h}_i \mid i \in I \ \}$$

be the set of those isotropy subalgebras which satisfy the following conditions:

1. $\mathfrak{h} \underset{\neq}{<} \mathfrak{h}_i \underset{\neq}{<} \mathfrak{g}$.

2. There is no isotropy subalgebra $\mathfrak{h}'$ *with* $\mathfrak{h} \underset{\neq}{<} \mathfrak{h}' \underset{\neq}{<} \mathfrak{h}_i$ *for any fixed* $i \in I$.

Then the following is true:

$$\mathcal{Z}(\mathfrak{h}) - \mathcal{I}(\mathfrak{h}) = \cup\{ \ \mathcal{Z}(\mathfrak{h}_i) \mid i \in I \ \}.$$

Proof. If $x \in \mathcal{Z}(\mathfrak{h}) - \mathcal{I}(\mathfrak{h})$ is an arbitrary point, then for the corresponding isotropy subalgebra $\mathfrak{g}_x$ the following hold:

$$\mathfrak{h} \underset{\neq}{<} \mathfrak{g}_x, \ \mathcal{Z}(\mathfrak{g}_x) \underset{\neq}{\subset} \mathcal{Z}(\mathfrak{h}).$$

Then the set of those isotropy subalgebras $\tilde{\mathfrak{h}}$ which satisfy the condition

$$\mathfrak{h} \underset{\neq}{<} \tilde{\mathfrak{h}} < \mathfrak{g}_x$$

is not empty. Let now $\mathfrak{h}_i$ be any minimal element of the above set. Then

$$x \in \mathcal{Z}(\mathfrak{g}_x) \subset \mathcal{Z}(\mathfrak{h}_i)$$

obviously holds and thus the assertion of the *lemma* follows. □

Corollary. *Let* $\Phi : G\times M \to M$ *be an affine action such that the set of its isotropy algebra types is countable and the number of the connected components of the zero set of every isotropy subalgebra is also countable. If* $\mathfrak{h} < \mathfrak{g}$ *is an isotropy subalgebra of minimal dimension, then the set* $\mathcal{Z}(\mathfrak{h}) - \mathcal{I}(\mathfrak{h})$ *has an empty interior.*

Proof. In order to prove the *corollary* by an indirect argument assume $\mathcal{Z}(\mathfrak{h}) - \mathcal{I}(\mathfrak{h})$ has a non-empty interior. Then by the preceding *lemma* the set

$$\cup\{ \ \mathcal{Z}(\mathfrak{h}_i) \mid i \in I \ \}$$

has a non-empty interior in $\mathcal{Z}(\mathfrak{h})$. But the set I is countable by assumptions above therefore by *Baire's theorem* at least one of the sets $\mathcal{Z}(\mathfrak{h}_i)$ has a non-empty interior in $\mathcal{Z}(\mathfrak{h})$. But then as the zero sets are totally geodesic submanifolds, the equality

$$\mathcal{Z}(\mathfrak{h}_i) = \mathcal{Z}(\mathfrak{h})$$

follows which yields a contradiction with the definition of $\mathfrak{h}_i$. □

Theorem 2.2. *Let (M, ∇) be an affine manifold, $\Phi : G \times M \to M$ effective affine action of a connected Lie group G such that all the minimal isotropy subalgebras of Φ are of minimal dimension. Then the following holds:*

1. Each orbit of Φ is intersected by the zero set $\mathcal{Z}(\mathfrak{h})$ of some minimal dimensional isotropy subalgebra $\mathfrak{h}$.

2. If the set of isotropy algebra types of Φ is countable and the number of the connected components of the zero set of every isotropy subalgebra is also countable, then M' the union of maximal dimensional orbits is dense in M.

Proof. The *first assertion* of the *theorem* that each orbit is intersected by the zero set of a minimal dimensional isotropy subalgebra is a direct consequence of *Proposition 1.2.* Let now $\mathcal{H}$ be the set of the minimal dimensional isotropy subalgebras of Φ. Then by the preceding assertion

$$M = \cup\{G(\mathcal{Z}(\mathfrak{h})) \mid \mathfrak{h} \in \mathcal{H} \} = \cup\{ \mathcal{Z}(\mathfrak{h}) \mid \mathfrak{h} \in \mathcal{H} \};$$

in fact, $\Phi(g, \mathcal{Z}(\mathfrak{h})) = \mathcal{Z}(Ad(g)\mathfrak{h})$ for $g \in G,\ \mathfrak{h} \in \mathcal{H}$ and obviously $Ad(g)\mathfrak{h} \in \mathcal{H}$.

In order to prove the *second assertion* by an indirect argument assume that M' is not dense in M, in other words the open set

$$M^\diamond = M - \overline{M'}$$

is not empty. Then by the equality above there is an $\mathfrak{h} \in \mathcal{H}$ such that

$$M^\diamond \cap \mathcal{Z}(\mathfrak{h}) = (M - \overline{M'}) \cap \mathcal{Z}(\mathfrak{h})$$

is a non-empty open subset of the totally geodesic submanifold $\mathcal{Z}(\mathfrak{h})$. On the other hand

$$(M - \overline{M'}) \cap \mathcal{Z}(\mathfrak{h}) \subset (M - M') \cap \mathcal{Z}(\mathfrak{h}) = \cup\{ \mathcal{Z}(\mathfrak{h}_i) \mid i \in I \}$$

holds by *Lemma 2.1.* But by the preceding *corollary* the above set has an empty interior in $\mathcal{Z}(\mathfrak{h})$ which yields a contradiction. □

Proposition 2.3. *Let (M, ∇) be an affine manifold, $\Phi : G \times M \to M$ affine action of a connected Lie group G and $\mathfrak{h}$ an isotropy subalgebra of minimal dimension such that the following holds:*

1. The stratum $\mathcal{S}(\mathfrak{h}) \subset M$ of $\mathfrak{h}$ is an open set.

2. $T_z\mathcal{I}(\mathfrak{h}) \cap T_zG(z) = \{0_z\}$ for $z \in \mathcal{I}(\mathfrak{h})$.

Then the following direct sum decomposition

$$T_zM = T_z\mathcal{I}(\mathfrak{h}) \oplus T_zG(z)$$

is true at any point z of the isotropy set $\mathcal{I}(\mathfrak{h})$.

Proof. For a fixed $z \in \mathcal{I}(\mathfrak{h})$, let $H = G_z^0 < G$ be the identity component of the isotropy subgroup G_z. Since $\mathfrak{g}_x = \mathfrak{g}_z$ for $x \in \mathcal{I}(\mathfrak{h})$, therefore $G_x^0 = H$ for the identity component of the correponding isotropy subgroup G_x. Consequently, there is a smooth covering map

$$\omega_x : G/H \ni gH \mapsto gG_x \in G/G_x$$

of the corresponding smooth quotient manifolds. Moreover, there is a canonical map

$$\chi_x : G/G_x \to G(x) \subset M$$

which is an equivariant bijection onto $G(x)$ and a smooth injective immersion into M for $x \in \mathcal{I}(\mathfrak{h})$.

Consider now the smooth product manifold $\mathcal{I}(\mathfrak{h}) \times G/H$ and also its map Θ given by

$$\Theta : \mathcal{I}(\mathfrak{h}) \times G/H \ni (x, gH) \mapsto \chi_x \circ \omega_x(gH) \in G(\mathcal{I}(\mathfrak{h})) = \mathcal{S}(\mathfrak{h}) \subset M.$$

The map Θ is related to the action Φ by the following obviously valid equality

$$\Theta(x, gH) = \chi_x(gG_x) = \Phi(g, x),\ (x, gH) \in \mathcal{I}(\mathfrak{h}) \times G/H,$$

which implies that the image of Θ is the open set $\mathcal{S}(\mathfrak{h})$. Moreover, the above equality implies the smoothness of Θ as well: Let $\pi : G \to G/H$ be the canonical projection and $U \subset G/H$ a neighbourhood of gH, then there is a smooth submanifold $\tilde{U} \subset G$ such that $g \in \tilde{U}$ and $\pi\lceil\tilde{U}$ is a diffeomorphism onto U. Then

$$(x, g'H) \mapsto \Phi(\pi^{-1}(g'H), x) = \Theta(x, g'H),\ (x, g'H) \in \mathcal{I}(\mathfrak{h}) \times U$$

holds and shows that Θ is smooth.

By the usual decomposition of the tangent space of the product manifold the definition of Θ yields that

$$\begin{aligned}
&T_{(x,gH)}\Theta(T_{(x,gH)}(\mathcal{I}(\mathfrak{h}) \times G/H)) \\
&\quad = T_{(x,gH)}\Theta(T_{(x,gH)}(\mathcal{I}(\mathfrak{h}) \times \{gH\}) \oplus T_{(x,gH)}(\{x\} \times G/H)) \\
&\quad = T_{(x,gH)}\Theta(T_{(x,gH)}(\mathcal{I}(\mathfrak{h}) \times \{gH\}) + T_{(x,gH)}\Theta(T_{(x,gH)}(\{x\} \times G/H) \\
&\quad = T_{x'}\mathcal{I}(Ad(g)\mathfrak{h}) + T_{x'}G(x),
\end{aligned}$$

where $x' = \Phi(g, x)$. Moreover, the restrictions of $T_{(x,gH)}\Theta$ to $T_{(x,gH)}(\mathcal{I}(\mathfrak{h}) \times \{gH\}$ and $T_{(x,gH)}(\{x\} \times G/H)$, respectively, are injective, therefore by *assumption 2*, the map $T_{(x,gH)}\Theta$ is injective as well. Consequently Θ is a smooth immersion. But the image of Θ is $\mathcal{S}(\mathfrak{h})$ which is open by *assumption 1*. Therefore $T_{(x,gH)}\Theta$ has to be an isomorphism onto $T_{\Theta(x,gH)}M$ at each point. Thus

$$T_zM = T_z\mathcal{I}(\mathfrak{h}) \oplus T_zG(z)$$

at every point $z \in \mathcal{I}(\mathfrak{h})$. □

The preceding *proposition* yields the motivation for the following *definition* which is essential for subsequent considerations.

DEFINITION. Let (M, ∇) be an affine manifold, $\Phi : G \times M \to M$ affine action of a connected Lie group G and $\mathfrak{h}$ an isotropy subalgebra of minimal dimension such that

$$T_x\mathcal{Z}(\mathfrak{h}) \cap T_xG(x) = \{0_x\}$$

for $x \in \mathcal{Z}(\mathfrak{h})$. Then the closed totally geodesic submanifold $\mathcal{Z}(\mathfrak{h})$ is said to be *transverse to the action* Φ.

3. The causal character of transverse submanifolds in Lorentz manifolds.

DEFINITIONS. If $(M, <,>)$ is a Lorentz manifold, a smooth submanifold $L \subset M$ is said to be *spacelike, timelike* or *lightlike* provided that all its tangents spaces $T_xL,\ x \in L$, are respectively spacelike, timelike or lightlike. In such cases the submanifold L is said to have a *causal character*. Spacelike and timelike submanifolds are also called *semi-Riemann submanifolds* ([**3**, pp 141–143]).

A slight modification of the standard light cone concept will prove useful subsequently. Namely, by the light cone Λ_z at $z \in M$ the set

$$\{\ v \in T_zM \mid\ < v, v >= 0,\ v \neq 0_z\ \}$$

is meant ([**3**, p 56]). By the *completed light cone* the set

$$\Lambda_z^c = \{\ 0_z\ \} \cup \Lambda_z$$

will be meant subsequently.

PROPOSITION 3.1. *Let $(M, <,>)$ be a Lorentz manifold, then any totally geodesic submanifold $L \subset M$ has a causal character.*

PROOF. The tangent space T_xL at a point $x \in L$ is spacelike, or timelike or lightlike according as its intersection $T_xL \cap \Lambda_x^c$ with the completed lightcone at x is $\{\ 0_x\ \}$ or a set including more than one 1–dimensional subspace or equal to a single 1–dimensional subspace. If z is an arbitrary point of L and C a piecewise smooth curve in L from x to z, then the parallel translation along C maps T_xL onto T_zL since L is totally geodesic; on the other hand, this parallel translation being an isometry of T_xM to T_zM maps Λ_x^c onto Λ_z^c. But then $T_xL \cap \Lambda_x^c$ is mapped onto $T_zL \cap \Lambda_z^c$ by this parallel translation along C. Consequently, T_zL has the same causal character as T_xL. □

COROLLARY. *If L is a lightlike totally geodesic submanifold of a Lorentz manifold, then there is a lightlike* 1*–dimensional distribution on L which is invariant under parallel translations; consequently, it yields a* 1*–dimensional foliation of L by lightlike geodesics.*

PROPOSITION 3.2. *Let (M, ∇) be a Lorentz manifold and $\Phi : G \times M \to M$ isometric action of a connected Lie group G. Then an orbit $G(z) \subset M$ of Φ has a causal character.*

PROOF. An obvious modification of the preceding argument yields the proof. □

DEFINITION. Let $(M, <,>)$ be a Lorentz manifold, ∇ its Levi–Cività covariant derivation and $\Phi : G \times M \to M$ an isometric action. Then Φ is an affine action with respect to ∇ and accordingly the isometric action Φ is said to admit a *transverse submanifold* if it admits one as an affine action.

THEOREM 3.3. *Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ isometric action admitting a transverse submanifold $\mathcal{Z}(\mathfrak{h})$ and $z \in \mathcal{I}(\mathfrak{h}) \subset \mathcal{Z}(\mathfrak{h})$. Then the orbit $G(z)$ is a semi-Riemannian submanifold if and only if $\mathcal{Z}(\mathfrak{h})$ is semi-Riemannian.*

PROOF. Assume first that the subspace $T_z\mathcal{Z}(\mathfrak{h})$ is not lightlike, but $T_zG(z)$ is lightlike. Then the set

$$E_z = \Lambda_z^c \cap T_zG(z)$$

is a 1–dimensional subspace. Moreover, for any $h \in G_z$, the equality

$$T_z\Phi_h E_z = T_z\Phi_h(\Lambda_z^c) \cap T_z\Phi_h(T_zG(z)) = \Lambda_z^c \cap T_zG(z) = E_z$$

holds. Consequently, if $v \in E_z - \{0_z\}$, then

$$T_z\Phi_h v = \lambda(h)v, \ \ h \in G_z,$$

where λ is a function $\lambda : G_z \to \mathbb{R}$. Since the subspace $T_z\mathcal{Z}(\mathfrak{h}) \subset T_zM$ is not lightlike, the orthogonal decomposition

$$T_zM = T_z\mathcal{Z}(\mathfrak{h}) \oplus (T_z\mathcal{Z}(\mathfrak{h}))^\perp$$

exists and yields the corresponding decomposition $v = v' + v''$. But then

$$Tz\Phi_h v' + T_z\Phi_h v'' = T_z\Phi(v' + v'') = \lambda(h)(v + v'') = \lambda(h)v' + \lambda(h)v'', \ \ h \in G_z$$

implies that $T_z\Phi_h v'' = \lambda(h)v''$, and then $\lambda(h) = 1$ as $T_z\Phi_h$ is an isometry and v'' is not lightlike. Therefore

$$\Lambda_z^c \cap T_zG(z) = E_z \subset T_z\mathcal{Z}(\mathfrak{h})$$

follows, since $\mathfrak{h} = \mathfrak{g}_z$. But the above inclusion is in contradiction with the assumption that $\mathcal{Z}(\mathfrak{h})$ is transverse to the action.

Assume secondly that the subspace $T_z\mathcal{Z}(\mathfrak{h})$ is lightlike and $T_zG(z)$ is not. Then the orthogonal decomposition

$$T_zM = T_zG(z) \oplus (T_zG(z))^\perp$$

exists and a 1–dimensional subspace is given by the set

$$F_z = \Lambda_z^c \cap T_z\mathcal{Z}(\mathfrak{h}).$$

If $w \in F_z - \{ 0_z \}$, then the above orthogonal decomposition yields $w = w' + w''$ and for $h \in G_z$ the following holds

$$T_z\Phi_h w' + T_z\Phi_h w'' = T_z\Phi_h w = \kappa(h)w = \kappa(h)w' + \kappa(h)w''.$$

But then $\kappa(h) = 1$ and therefore $w' \in T_z\mathcal{Z}(\mathfrak{h})$ follows in contradiction with the assumption that $\mathcal{Z}(\mathfrak{h})$ is a transverse submanifold. □

Definition. Let M be a smooth manifold and $\Phi : G \times M \to M$ effective smooth action of a connected Lie group G. If $H < G$ is a Lie subgroup, $\mathfrak{h} < \mathfrak{g}$ its Lie subalgebra, then the zero set $\mathcal{Z}(\mathfrak{h})$ is equal to the fixed point set of H. Moreover, simple calculations yield that

$$\Phi_g(\mathcal{Z}(\mathfrak{h})) = \mathcal{Z}(\mathfrak{h})$$

for $g \in G$ if and only if g is element of $N(\mathfrak{h})$, the normalizer of $\mathfrak{h}$ in G. Therefore a maximal restricted action

$$\Phi\lceil(N(\mathfrak{h}) \times \mathcal{Z}(\mathfrak{h}))$$

exists, which in turn induces an effective action

$$\Sigma : (N(\mathfrak{h})/H) \times \mathcal{Z}(\mathfrak{h}) \to \mathcal{Z}(\mathfrak{h}),$$

which is called the *restricted action* subsequently.

Proposition 3.4. *Let $(M, <,>)$ be a Lorentz manifold, $\Phi : G \times M \to M$ an isometric action and $\mathfrak{h}$ an isotropy subalgebra of minimal dimension such that its zero set $\mathcal{Z}(\mathfrak{h})$ is transverse to the action. If the restricted action*

$$\Sigma : (N(\mathfrak{h})/H) \times \mathcal{Z}(\mathfrak{h}) \to \mathcal{Z}(\mathfrak{h})$$

does not leave any 1–dimensional parallel distribution invariant, then $\mathcal{Z}(\mathfrak{h})$ is a semi-Riemannian submanifold.

Proof. In order to prove by an indirect argument assume that $\mathcal{Z}(\mathfrak{h})$ is lightlike. Then there is a 1–dimensional lightlike parallel distribution on $\mathcal{Z}(\mathfrak{h})$ by the Corollary of Proposition 2.1. But this distribution should be invariant under the restricted action Σ which is obviously isometric. □

A simple example is presented at last in order to give the motivation of the definitions proposed above.

Example. Consider the Lorentz manifold $(M, <,>)$ which is obtained from the 4–dimensional Minkowski space $\mathbb{M}^4$. Let (e_1, e_2, e_3, e_4) be the canonical orthonormal base of $\mathbb{M}^4$ and (x^1, x^2, x^3, x^4) the corresponding coordinate system, then the coordinate expression of the semi-euclidean inner product $(\, ,)$ on $\mathbb{M}^4$ is given by

$$(v, w) = -v^1w^1 + \sum_{i=2}^{4} v^iw^i,$$

where $v = \sum_{i=1}^4 v^i e_i,\ w = \sum_{j=1}^4 w^j e_j$. Consider also the Lorentz group $O(1,3)$ and its canonical semi-orthogonal action

$$O(1,m-1) \times \mathbb{M}^4 \to \mathbb{M}^4.$$

As the Lorentz group has 4 connected components, let $G < O(1,3)$ be its identity component and consider the restricted action

$$\Phi : G \times \mathbb{M}^4 \to \mathbb{M}^4.$$

The quadratic form $q(v) = (v,v),\ v \in \mathbb{M}^4$, for $\varrho \in \mathbb{R}$ has the level set

$$S_\varrho = \{\ v \in M \mid q(v) = \varrho\ \}.$$

If $\varrho > 0$, then S_ϱ is a connected hypersurface, which is timelike and it is an orbit of Φ. If $\varrho = 0$, then $S_\varrho - \{0\}$ has 2 connected components, which are lightlike hypersurfaces and also orbits of Φ. If $\varrho < 0$, then S_ϱ has 2 connected components, which are spacelike and are orbits of Φ as well.

If $z \in \mathbb{M}^4 - \{0\}$, then the corresponding isotropy subalgebra $\mathfrak{g}_z < \mathfrak{g}$ of Φ is of minimal dimension. In fact, if $z = a = (1,0,0,0)$, then $\mathfrak{g}_a$ can be identified with $\mathfrak{o}(3)$, which has dimension 3; if $z = b = (0,1,0,0)$, then $\mathfrak{g}_b$ can be identified with $\mathfrak{o}(1,2)$, which has also dimension 3; if $z = c = (1,1,0,0)$, then $\mathfrak{g}_c$ as an obvious simple calculation shows has dimension 3 as well.

Now $\mathcal{Z}(\mathfrak{g}_a)$ is the 1–dimensional timelike subspace spanned by the vector $(1,0,0,0)$, which is transverse to Φ. Furthermore, $\mathcal{Z}(\mathfrak{g}_b)$ is the spacelike 1–dimensional subspace spanned by the vector $(0,1,0,0)$, which is also transverse to Φ. On the other hand, $\mathcal{Z}(\mathfrak{g}_c)$ is the lightlike 1–dimensional subspace spanned by the vector $(1,1,0,0)$, which is not transverse to Φ.

Each orbit of Φ is obviously intersected by one of the zero sets $\mathcal{Z}(\mathfrak{g}_a)$, $\mathcal{Z}(\mathfrak{g}_b)$, $\mathcal{Z}(\mathfrak{g}_c)$.

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