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FIRST NEIGHBOURHOOD OF THE DIAGONAL, AND GEOMETRIC DISTRIBUTIONS

by Anders Kock

Abstract. For any manifold, we describe the notion of geometric distribution on it, in terms of its first neighbourhood of the diagonal. In these "combinatorial" terms, we state the Frobenius Integrability Theorem, and use it to give a combinatorial proof of the Ambrose–Singer Theorem on connections in principal bundles.

The consideration of the k'th neighbourhood of the diagonal of a manifold M, $M_{(k)} \subseteq M \times M$, was initiated by Grothendieck to import notions from differential geometry into the realm of algebraic geometry. These notions were reimported into differential geometry by Malgrange [**14**], Kumpera and Spencer [**12**], ... They utilized the notion of *ringed* space (a space equipped with a *structure sheaf* of functions). The only points of (the underlying space of) $M_{(k)}$ are the diagonal points (x, x) with $x \in M$. But it is worthwhile to describe mappings to and from $M_{(k)}$ *as if* $M_{(k)}$ consisted of "pairs of k-neighbour points (x, y)" (write $x \sim_k y$ for such a pair; such x and y are "points proches" in the terminology of A. Weil). The introduction of topos theoretic methods has put this "synthetic" way of speaking onto a rigourous basis, and we shall freely use it.

We shall be only interested in the case $k = 1$, so we are considering the *first* neighbourhood of the diagonal, $M_{(1)} \subseteq M \times M$, and we shall write $x \sim y$ instead of $x \sim_1 y$ whenever $(x, y) \in M \times M$ belongs to the subspace $M_{(1)}$. The relation $\sim$ is reflexive and symmetric, but unlike the notion of "neighbour" relation in Non Standard Analysis, it is *not* transitive. In some previous writings, [**6**], [**7**], [**8**], [**10**], [**11**], we have discussed the paraphrasing of several differential–geometric notions in terms of the combinatorics of the neighbour relation $\sim$. We shall remind the reader about some of them concerning differential forms and connections in Section 5 below.

The new content in the present note is a description of the differential-geometric notion of *distribution* and *involutive* distribution in combinatorial terms (Section 3, notably Theorem 10), and an application of these notions to a proof of a version of the Ambrose–Singer holonomy theorem (Section 5).

The proofs will depend on a correspondence between the classical Grassmann algebra, and the algebra of "combinatorial differential forms", which has been considered in the series of texts by the author, mentioned above, and by Breen and Messing, [**2**]. We summarize, without proofs, the relevant part of this theory in the first two sections. (We hope that this summary may have also an independent interest.)

1. The algebra of combinatorial differential forms. In the smooth category, a manifold M can be recovered from the ring $C^\infty(M)$ of smooth functions on it, (– unlike in the analytic category, or the category of schemes in algebraic geometry, where there are not enough *global* functions on M, so that one here needs the *sheaf* of (algebraic) functions to recover the space). In any case, "once the seed of algebra is sown it grows fast" (Wall, [**17**, p. 185]): if we let a geometric object like a manifold be represented by a ring (its "ring of functions"), we may conversely consider (suitable) rings to represent "virtual" geometric objects. The ring in question (better: **R**–algebra, or even "C^∞–algebra", cf. e.g. [**15**]) is then considered as the ring of smooth real valued functions on the virtual object it represents. In particular, from Grothendieck, Malgrange and others, as alluded to in the introduction, we get the following geometric object $M_{(1)}$, where M is a manifold:

In the ring $C^\infty(M \times M)$, we have the ideal I of functions $f(x, y)$ vanishing on the diagonal, $f(x, x) = 0$, in other words I is the kernel of the restriction map $C^\infty(M \times M) \to C^\infty(M)$ (restrict along the diagonal $M \to M \times M$). Since this restriction map is a surjective ring homomorphism, we have $C^\infty(M) \cong C^\infty(M \times M)/I$. Now I contains I^2, the ideal of functions vanishing to the *second* order on the diagonal. The ring $C^\infty(M \times M)/I^2$ represents a new, virtual, geometric object $M_{(1)}$, and the surjection $C^\infty(M \times M)/I^2 \to C^\infty(M \times M)/I \cong C^\infty(M)$, we may see as the restriction map for a "diagonal" inclusion $M \to M_{(1)}$.

The points of M may be recovered from the algebra $C^\infty(M)$ as the algebra maps $C^\infty(M) \to \mathbf{R}$. The "virtual" character of $M_{(1)}$ is the fact that the *real* points of it are the same as the real points of M, or put differently, a real point (x, y) in $M \times M$ belongs to $M_{(1)}$ precisely when $x = y$. So we cannot get a full picture of $M_{(1)}$ by looking at its real points. The synthetic way of speaking, and reasoning, talks about objects like $M_{(1)}$ in terms of the virtual points (x, y), and this mode of speaking can be fully justified by interpretation via categorical logic in suitable "well adapted toposes", see e.g. [**3**], [**6**]. To illustrate the synthetic language concretely, let us consider the following definition.

Definition 1. A *combinatorial differential 1–form* ω on a manifold M is a function $M_{(1)} \to \mathbf{R}$ which vanishes on the diagonal $M \subseteq M_{(1)}$.

Let us analyze this notion in non-virtual terms. A *function* $\omega : M_{(1)} \to \mathbf{R}$ is an element of the ring $C^\infty(M \times M)/I^2$, by the very definition. When does such ω satisfy the clause that "it vanishes on the diagonal"? precisely when it belongs to I (– the functions vanishing on the diagonal), or more precisely, when it belongs to the image of I under the homomorphism $C^\infty(M \times M) \to C^\infty(M \times M)/I^2$. So a combinatorial differential 1–form on M is an element of I/I^2; but this is the module of *Kaehler differentials* on M. Kaehler observed that this module was isomorphic to the classical module of differential 1–forms on M, i.e. the module of fibrewise linear maps $\overline{\omega} : T(M) \to \mathbf{R}$. This results thus can be formulated:

Theorem 2. *(Kaehler) There is a bijective correspondence between the module of classical differential 1–forms on M, and the module of functions $M_{(1)} \to \mathbf{R}$ which vanish on the diagonal. Or, in the terminology of Definition 1: a 1-1 correspondence between classical differential 1–forms $\overline{\omega}$ and combinatorial differential 1–forms ω.*

Note that for combinatorial 1–forms, there is *no* linearity requirement. Furthermore, combinatorial 1–forms are automatically *alternating* in the sense that $\omega(x,y) = -\omega(y,x)$ for all $x \sim y$. This is essentially because for any function $f(x,y)$ vanishing on the diagonal, the function $f(x,y) + f(y,x)$ also vanishes on the diagonal and is furthermore symmetric in x, y, hence vanishes to the second order on the diagonal.

There are also combinatorial versions of the notion of differential k–form, $k \geq 2$, [**6**], [**8**], [**10**], [**2**], and a comparison result with classical (multilinear alternating) differential forms. We summarize the definition and comparison here, using synthetic language. A $k+1$–tuple of elements $(x_0, \dots, x_k)$ of (virtual!) points of M is called an *infinitesimal k–simplex* if $x_i \sim x_j$ for all i, j, and is called *degenerate* if $x_i = x_j$ for some $i \neq j$. The "set" of infinitesimal k–simplices in M form an object

$$M_{[k]} \subseteq M^{k+1}$$

(which may be described in ring theoretic terms, in the spirit of I/I^2, but more complicated; cf. [**2**]). The following definition from [**6**] is then an extension of the previous definition:

Definition 3. A combinatorial differential k–form ω on a manifold M is a function $M_{[k]} \to \mathbf{R}$ which vanishes on all degenerate k–simplices.

In the rest of this text, we shall often just say (combinatorial) k–form instead of combinatorial differential k–form.

Note again that in the Definition, there is no (multi-)linearity requirement; and ω can be proved to be alternating in the sense that the interchange of x_i and x_j results in a change of sign in the value of ω.

We proceed to describe the exterior derivative of combinatorial forms, and wedge product; both these structures are analogous to structures (coboundary and cup product) on the singular cochain complex of a topological space (see [**10**]).

For a k–form ω, we let $d\omega$ be the $k+1$–form given by

$$d\omega\ (x_0, \ldots, x_{k+1}) := \sum_{i=0}^{k+1} (-1)^i \omega(x_0, \ldots, \hat{x}_i, \ldots, x_{k+1});$$

for a k–form ω and for an l–form θ, we let $\omega \wedge \theta$ be the $k+l$–form given by

$$(\omega \wedge \theta)(x_0, \ldots, x_{k+l}) := \omega(x_0, \ldots, x_k) \cdot \theta(x_k, x_{k+1}, \ldots, x_{k+l}).$$

(It is not trivial, but true, that one gets the value 0 when $\omega \wedge \theta$ is applied to a $k+l$–simplex which is degenerate by virtue of $x_i = x_j$ with $i < k$ and $j > k$.)

Equipped with these structures d and $\wedge$, together with the evident "pointwise" vector space structure, the combinatorial forms together make up a differential graded algebra $\Omega^\bullet(M)$. Let $\overline{\Omega}^\bullet(M)$ be the classical differential graded algebra of classical differential forms. A part of the content of [**6**], [**8**], [**10**] may be summarized in the following "Comparison" Theorem (see also [**2**] for the generalization to schemes):

Theorem 4. *The differential graded algebras $\Omega^\bullet(M)$ and $\overline{\Omega}^\bullet(M)$ are isomorphic.*

The correspondence is given in [**6**], Corollary 18.2; the compatibility with the differential and product structure is proved in [**10**] p. 259 and 263, resp.

It should be said, though, that the strict validity of the theorem depends on the conventions for defining the classical exterior derivative and wedge; the conventions often differ by a factor $k!$ or $k!l!/(k+l)!$. So with different conventions, the equalities claimed by the theorem hold only modulo such rational factors. The uses we shall make of the theorem, however, have the character "if one thing is zero, then so is the other", so they are independent of these rational factors, and thus indpendent of the conventions.

2. The D–construction; $\log$ and $\exp$. For any finite dimensional vector space E, we let $D(E)$ denote the subset of $x \in E$ with $x \sim 0$. If in particular $E = \mathbf{R}$, we write D for $D(\mathbf{R})$. This is the most basic object in Synthetic

Differential Geometry, and appears as such in [**13**], [**5**]. In algebraic guise, when D is presented in terms of the ring of functions on it, it appears much earlier, namely as the "ring $\mathbf{R}[\epsilon] = \mathbf{R}[X]/(X^2)$ of dual numbers". The object $D \subseteq \mathbf{R}$ is the set of (virtual) elements $d \in \mathbf{R}$ which satisfy $d^2 = 0$. If M is a manifold, a map $t : D \to M$ with $t(0) = x \in M$ is a *tangent vector* at $x \in M$. (This assertion is just the synthetic reformulation of: an algebra map $C^\infty(M) \to \mathbf{R}[\epsilon]$ corresponds to a derivation $C^\infty(M) \to \mathbf{R}$.)

A fundamental fact about "ringed spaces" gets its synthetic formulation in the following "cancellation principle":

(1) If $a \in \mathbf{R}$ satisfies $d \cdot a = 0$ for *all* $d \in D$, then $a = 0$.

(Verbally: *universally quantified d's may be cancelled"*. It is part of what sometimes goes under the name "Kock–Lawvere axiom", see e.g. [**15**].)

From this cancellation principle, it is immediate to deduce the following

Proposition 5. *Let E be a vector space, and let U and V be linear subspaces. If V is the kernel of a linear map $\phi : E \to \mathbf{R}^k$, and if $d \cdot U \subseteq V$ for all $d \in D$, then $U \subseteq V$.*

Note that not every linear subspace V is a kernel; for instance, the "set" $D_\infty \subseteq \mathbf{R}$ of nilpotent elements is a linear subspace of a 1–dimensional vector space; it is not a kernel, and it does not qualify as finite dimensional.)

We consider again a general finite dimensional vector space E. Let $\tilde{D}^p(E) \subseteq (D(E))^p$ denote the set of p–tuples $(x_1, \ldots, x_p)$ with $x_i \in D(E)$ and which satisfy $x_i \sim x_j$ for all $i, j = 1, \ldots, p$. So we have inclusions

$$\tilde{D}^p(E) \subseteq D(E)^p \subseteq E^p. \tag{2}$$

Let us for short call a (partial) $\mathbf{R}$–valued function θ on E^p *normalized* if its value is 0 as soon as one of the input arguments is $0 \in E$. The following auxiliary result is proved (for $E = \mathbf{R}^n$) in [**6**, I.16], for the bijection between 1) and 3); the bijection between 1) and 2) is proved in a similar way, but is easier.

Proposition 6. *The restriction along the inclusions in (2) establish bijections between*

1. *multilinear alternating maps $E^p \to \mathbf{R}$,*
2. *normalized alternating maps $D(E)^p \to \mathbf{R}$,*
3. *normalized maps $\tilde{D}^p(E) \to \mathbf{R}$.*

Now consider a manifold M, then its tangent bundle TM is likewise a manifold, and we may talk about when two tangents are neighbours. In particular, we may ask when a tangent vector at $x \in M$ is neighbour to the zero tangent vector at x. The set of all such tangents is denoted $DM \subseteq TM$. Alternatively DM is formed by applying the D construction for vector spaces, as given above, to each fibre T_xM individually.

From [**4**], we have that there is a canonical bijection $\exp : DM \to M_{(1)}$. Its inverse, which we of course have to call log, may be described explicitly as follows: if $(x, y) \in M_{(1)}$, then $\log(x, y) \in TM$ is the tangent vector at $x \in M$ in M given by

$$d \mapsto (1-d)x + dy \quad \text{for} \quad d \in D. \tag{3}$$

Here we are forming an *affine combination* $(1-d)x + dy$ of two points x and y in a manifold M; this can be done (for *any* $d \in \mathbf{R}$, in fact) by choosing a coordinate chart around x and y, and making the affine combination in coordinates. It is proved in [**9**, Theorem 1] that when $x \sim y$, the result does not depend on the chosen coordinate chart.

For $s \in \mathbf{R}$ and $t \in TM$, $s \cdot t$ denotes the tangent vector given by $\delta \mapsto t(s \cdot \delta)$. For $d \in D$ and $t \in TM$, $d \cdot t \in DM$, and also $t(d) \sim t(0)$ $(= x$, say). We have the equations

$$\exp(d \cdot t) = t(d); \text{ and } \log(x, t(d)) = d \cdot t \tag{4}$$

In terms of log, we can be more explicit about the correspondence of Theorem 4: the combinatorial p–form θ corresponding to a classical p–form $\overline{\theta}$ is given by

$$\theta(x_0, \ldots, x_p) = \overline{\theta}(\log(x_0, x_1), \ldots, \log(x_0, x_p)). \tag{5}$$

Note that the left hand side vanishes if $x_i = x_0$, because $\overline{\theta}$ is normalized (being multilinear), and vanishes if $x_i = x_j$ $(i, j \geq 1)$ because $\overline{\theta}$ is alternating.

Note also that the right hand side is defined on more $p+1$–tuples than the left hand side, since on the right hand side, no assumption $x_i \sim x_j$ for $i, j \geq 1$ enters.

3. Geometric distributions.

DEFINITION 7. A *predistribution* on a manifold M is a reflexive symmetric refinement $\approx$ of the relation $\sim$.

(That $\approx$ is a *refinement* of $\sim$ is taken in the sense: $x \approx y$ implies $x \sim y$, for all x and y.) For instance, if $f : M \to N$ is a submersion, we get a predistribution $\approx$ by putting $x \approx y$ when $x \sim y$ and $f(x) = f(y)$. A predistribution arising in this way is clearly *involutive* in the following sense:

DEFINITION 8. A predistribution $\approx$ on M is called (combinatorially) *involutive* if $x \approx y$, $x \approx z$, and $y \sim z$ imply $y \approx z$.

In analogy with the object $M_{[k]}$ of infinitesimal k–simplices, we may in the presence of a distribution define $M_{[[k]]} \subseteq M_{[k]}$ to consist of $k+1$–tuples $(x_0, \ldots, x_k)$ satisfying $x_i \approx x_j$ for all i and j. If we call such infinitesimal simplices *flat*, we can reformulate the notion of an involutive distribution as follows: *if two of the faces of an infinitesimal 2–simplex are flat, then so is the third*; or equivalently: *if two of the faces of an infinitesimal 2–simplex are flat, then so is the 2–simplex itself.*

We need to explain in which sense a (geometric) distribution on M in the classical sense gives rise to a predistribution. Recall that a k–dimensional distribution on an n-dimensional manifold M is a k–dimensional sub-bundle E of the tangent bundle $TM \to M$. For $x \in M$, $E_x \subseteq T_xM$ is thus a k–dimensional linear subspace of the tangent vector space to M at x.

If $E \subseteq TM$ is a distribution in the classical sense, we can for $x \sim y$ define a predistribution $\approx$ (or $\approx_E$) by

$$x \approx y \text{ iff } \log(x,y) \in E_x, \tag{6}$$

or equivalently, the set of y ($\sim x$) which satisfy $x \approx y$ is the image of $E_x \cap DM$ under exp. It is clearly a reflexive relation, since $\log(x,x)$ is the zero tangent vector at x. The symmetry of $\approx$ is less evident, but is proved in Proposition 9 below. A predistribution which comes about in this way from a classical distribution E, we call simply a *distribution*. (This is justified, since one can prove (using Proposition 5) that two classical distributions, giving rise to the same predistribution, must be equal.)

Proposition 9. *Let $\approx$ be derived from a classical distribution, as in (6). Then $x \approx y$ implies $y \approx x$.*

Proof. Recall that a classical distribution E, of dimension k, say, may be presented locally by $n-k$ non-singular differential 1–forms on M, $\overline{\omega}_1, \ldots, \overline{\omega}_{n-k}$, with $t \in E_x$ iff t is annihilated by all the $\overline{\omega}_i$'s. So $x \approx y$, iff $\overline{\omega}_i(\log(x,y)) = 0$ for $i = 1, \ldots, (n-k)$, iff $\omega_i(x,y) = 0$ for $i = 1, \ldots, (n-k)$. But since each ω_i is alternating, the relation $\omega_i(x,y) = 0$ is symmetric. □

The justification for our use of the phrase "involutive" is contained in

Theorem 10. *Let the classical distribution $E \subseteq TM$ be given. Then it is involutive in the classical sense if and only if $\approx_E$ is involutive in the combinatorial sense of Definition 8.*

Proof. Represent, as above, the classical distribution E by $n-k$ differential 1–forms on M $\overline{\omega}_1, \ldots, \overline{\omega}_{n-k}$. If $\overline{I}$ denotes the ideal in the Grassman (exterior) algebra $\overline{\Omega}^{\bullet}(M)$ generated by the $\overline{\omega}_i$'s, then by the classical definition, the distribution is involutive precisely when $\overline{I}$ is closed under exterior differentiation. (It suffices that each $d\overline{\omega}_i$ is in $\overline{I}$.) (The alternative, equivalent, definition of involutiveness in terms of vector fields along E is less convenient for the comparison we are about to make.)

Now assume that E is classically involutive. Let $x \approx y$, $x \approx z$ and $y \sim z$. We need to prove $y \approx z$. It suffices to prove for each i that $\overline{\omega}_i(\log(y,z)) = 0$, or equivalently that $\omega_i(y,z) = 0$, where the ω_i corresponds to $\overline{\omega}_i$ under the correspondence of Theorem 4. But by assumption $d\overline{\omega}_i \in \overline{I}$, hence $d\omega_i \in I$, and so

$$0 = d\omega_i(x,y,z) = \omega_i(x,y) - \omega_i(x,z) + \omega_i(y,z).$$

Since the two first terms here are 0 by $x \approx y$ and $x \approx z$, we conclude $\omega_i(y,z) = 0$. So $\approx$ is combinatorially involutive.

Conversely, assume that $\approx_E$ is combinatorially involutive. Let $\overline{\omega}$ be one of the 1–forms generating $\overline{I}$. Being generated by 1–forms, it follows from classical multilinear algebra that the ideal $\overline{I}$ has the property: for a p–form $\overline{\theta}$, if

$$\overline{\theta} : TM \times_M \ldots \times_M TM \to \mathbf{R}$$

annihilates E^p, then $\overline{\theta} \in \overline{I}$. So to prove $d\overline{\omega} \in \overline{I}$, it suffices to prove that if $t_1, t_2 \in E_x \subseteq T_xM$, then

$$d\overline{\omega}(t_1, t_2).$$

By bilinearity, and by the cancellation principle (1) applied twice, it suffices to prove, for all d_1 and d_2 in D, that $0 = d\overline{\omega}(d_1 \cdot t_1, d_2 \cdot t_2)$. We calculate this expression, using (4):

$$d\overline{\omega}(d_1 \cdot t_1, d_2 \cdot t_2) = d\overline{\omega}(\log(x, t_1(d_1)), \log(x, t_2(d_2))$$

$$= \overline{d\omega}(\log(x, t_1(d_1)), \log(x, t_2(d_2))$$

where $d\omega$ is the combinatorial exterior derivative of ω, and $\overline{d\omega}$ the corresponding classical form (using the fact that the bijection $\theta \leftrightarrow \overline{\theta}$ of Theorem 4 commutes with exterior derivative). Unfortunately, we do not necessarily have $t_1(d_1) \sim t_2(d_2)$, so that we cannot rewrite this as $d\omega(x, t_1(d_1), t_2(d_2))$; we need first an auxiliary consideration for the 2–form $\theta = d\omega$.

For a combinatorial 2–form θ, we consider a certain function $\tilde{\theta}$ defined on all "semi-infinitesimal 2–simplices", meaning triples x, y, z with $x \sim y$ and $x \sim z$ (but not necessarily $y \sim z$). The function $\tilde{\theta}$ is defined by

$$\tilde{\theta}(x,y,z) := \overline{\theta}(\log(x,y), \log(x,z)).$$

If $y \sim z$, this value is just $\theta(x,y,z)$, so $\tilde{\theta}$ is an extension of θ.

LEMMA 11. *If θ annihilates all infinitesimal 2–simplices (x,y,z) with $x \approx y$ and $x \approx z$, then $\tilde{\theta}$ annihilates all semi-infinitesimal 2–simplices (x,y,z) with $x \approx y$ and $x \approx z$.*

Proof. Let the semi-infinitesimal 2–simplex (x, y, z) be given. Let V denote T_xM and let E denote $E_x \subseteq T_xM$. Under the log / exp–identification, we identify the set of $\{y \mid y \sim x\}$ with $D(V)$, and the set of $\{y \mid y \approx x\}$ with $D(E)$. Then $\theta(x, -, -)$ and $\tilde{\theta}(x, -, -)$ are functions $\theta : \tilde{D}^2(V) \to \mathbf{R}$ and $\tilde{\theta} : D(V)^2 \to \mathbf{R}$, respectively. The function θ is normalized, $\tilde{\theta}$ is normalized and alternating. We have similarly the restrictions of θ and $\tilde{\theta}$ to $\tilde{D}^2(E)$ and $D(E)^2$. We may see $\tilde{\theta} : D(E)^2 \to \mathbf{R}$ as arising *either* by first extending $\theta : \tilde{D}^2(V) \to \mathbf{R}$ to $D(V)^2 \to \mathbf{R}$, and then restricting it to $D(E)^2$, *or* as arising by first restricting to $\tilde{D}^2(E)$ and then extending. Using the correspondence between items 2) and 3) in Proposition 6, we conclude that this must give the same result. The assumptions made on θ guarantee that the "first restricting, then extending" process yields 0, hence so does the other process, so $\tilde{\theta}$ restricts to 0 on $D(E)^2$; but this is, under the log / exp identification, precisely the assertion that $\tilde{\theta}(x, y, z) = 0$ for all $y \approx x$ and $z \approx x$. □

We can now finish the proof. We have that the 2–form $d\omega$ annihilates infinitesimal 2–simplices (x, y, z) with $x \approx y$ and $x \approx z$, because then, by assumption, $y \approx z$. From the Lemma, it follows that $\widetilde{d\omega}$ annihilates semi-infinitesimal 2–simplices x, y, z with $x \approx y$ and $x \approx z$. Since $t_1(d_1) \approx x$ and $t_2(d_2) \approx x$, the simplex $(x, t_1(d_1), t_2(d_2))$ is of this kind, so $\widetilde{d\omega}$ takes value 0 on this simplex. But

$$\widetilde{d\omega}(x, t_1(d_1), t_2(d_2)) = \overline{d\omega}(\log(x, t_1(d_1), \log(x, t_2(d_2)),$$

which thus is 0. This proves the classical involution condition. □

4. Frobenius Theorem. Let $E \subseteq TM$ be a distribution on a manifold M.

Proposition 12. *Let $E \subseteq TM$ be a classical distribution, and $\approx$ the corresponding combinatorial one. Let $Q \subseteq M$ be a submanifold, and $x \in Q$. Then the following are equivalent:*

1) for any $t \in T_xM$, $t \in T_xQ$ implies $t \in E_x$;

2) for any y with $x \sim y$, $y \in Q$ implies $x \approx y$.

Also the following are equivalent:

3) for any $t \in T_xM$, $t \in E_x$ implies $t \in T_xQ$;

4) for any y with $x \sim y$, $x \approx y$ implies $y \in Q$.

Proof. Assume 1). Let $x \sim y$, $y \in Q$. Then $\log(x, y) \in T_xQ$ (as a submanifold, Q is stable under the formation of the affine combinations that make up the values of log), so $\log(x, y) \in E_x$, meaning $x \approx y$.

Conversely, assume 2). To prove $T_xQ \subseteq E_x$, it suffices by Proposition 5 to prove for each $d \in D$ that $d \cdot T_xQ \subseteq E_x$. So consider a tangent vector $d \cdot t$

where $t \in T_xQ$. Then $t(d) \in Q$ and $x \sim t(d)$, whence by assumption $x \approx t(d)$, i.e. $\log(x, t(d)) \in E_x$. But $\log(x, t(d)) = d \cdot t$ by (4), so $d \cdot t \in E_x$.

Assume 3), and assume $x \approx y$, i.e. $\log(x, y) \in E_x$. By assumption $\log(x, y) \in T_xQ$. Applying exp yields $y \in Q$.

Conversely, assume 4). We need to prove $E_x \subseteq T_xQ$. Let $t \in E_x$, then by (4), for each $d \in D$, $x \approx \exp(d \cdot t)$, whence by assumption $\exp(d \cdot t) \in Q$, so $t(d) \in Q$, by (4). Since this holds for all $d \in D$, t is a tangent vector of Q. □

Consider a classical distribution $E \subseteq TM$, and let $\approx$ be the corresponding combinatorial one. A submanifold $Q \subseteq M$ is called *weakly integral* if for each $x \in Q$, the equivalent conditions 1) and 2) of Proposition 12 hold; and it is called (strongly) *integral* if furthermore, the equivalent conditions 3) and 4) hold.

The Frobenius Theorem says that if $E \subseteq TM$ is an involutive distribution, then for every $x \in M$ there exists a (stronly) *integral* submanifold $Q \subseteq M$ containing x; there even exists a unique maximal connected such Q. Maximality here means that if $x \in K \subseteq M$ is any connected submanifold which is weakly integral, then $K \subseteq Q$.

We shall later use the following "sufficiency" principle

Proposition 13. *Let $H \subseteq G$ be a Lie subgroup of a connected Lie group. Let $\mathcal{M}(e)$ be the set of $g \in G$ with $g \sim e$ (e the neutral element). If $\mathcal{M}(e) \subseteq H$, we have $G = H$.*

Proof. Finite dimensional linear subspaces of a finite dimensional vector space E may be reconstructed from their intersection with $D(E)$, by Proposition 5. Using the log – exp bijection, a finite dimensional linear subspace of $T_e(G)$ may be reconstructed from subsets of $\mathcal{M}(e)$. The assumption of the Proposition therefore gives that $T_eH = T_eG$, and this implies by the classical Lie theory that $H = G$. □

5. Ambrose–Singer Theorem. This Theorem ([**1**], or see [**16**, II.7]), deals with connections in principal bundles. We briefly recall how this gets formulated in synthetic terms; for a more elaborate account, see [**8**] and [**11**]. Let $\pi : P \to M$ be a principal G–bundle, in the smooth category of course; G is a Lie group. The action of G on P is on the right. A *principal connection* ∇ in $P \to M$ is a law which to any $a \sim b$ in M assigns a G–equivariant map $\nabla(a, b) : P_b \to P_a$, with $\nabla(a, a)$ the identity map (this implies that $\nabla(a, b)$ has $\nabla(b, a)$ as the inverse). The *connection form* ω of ∇ is a "G–valued 1–form on P": it is the law which to any $x \sim y \in P$ assigns an element of G, namely the element $\omega(x, y) \in G$ such that

$$x \cdot \omega(x, y) = \nabla(a, b)(y),$$

where $a = \pi(x)$ and $b = \pi(y)$. If one thinks of an infinitesimal 1–simplex of the shape $(\nabla(a,b)(y), y)$ as a *horizontal* 1–simplex, $\omega(x,y)$ measures the lack of horizontality of (x,y); in particular, (x,y) is horizontal precisely when $\omega(x,y) = e$, the neutral element of G. A curve in P is called horizontal if any two neighbour points on it form a horizontal 1–simplex.

The *curvature form* $d\omega$ is the G–valued exterior derivative of ω, namely the law which to an infinitesimal 2–simplex (x,y,z) in P assigns the element $\in G$

$$d\omega(x,y,z) := \omega(x,y) \cdot \omega(y,z) \cdot \omega(z,x). \tag{7}$$

The following is now a (restricted) version of the Ambrose Singer "holonomy" theorem. A principal G–bundle with connection is given, as above; G is assumed connected.

Theorem 14. *Assume that any two points of P can be connected by a horizontal curve, and that $H \subseteq G$ is a Lie subgroup such that for all infinitesimal 2–simplices $(x,y,z) \in P$, we have $d\omega(x,y,z) \in H$. Then $H = G$.*

Proof. We construct a distribution $\approx$ on P, by putting

$$x \approx y \text{ iff } \omega(x,y) \in H.$$

In particular, if x and y are in the same fibre P_a,

$$x \approx y \text{ iff } y = x \cdot h \tag{8}$$

for some $h \in H$ with $h \sim e$, (e the neutral element of G).

This distribution is clearly involutive; for if (x,y,z) is an infinitesimal 2–simplex with $x \approx y$ and $x \approx z$, three of the four factors that occur in the equation (7) are in H, hence so is the fourth factor $\omega(y,z)$, proving $y \approx z$.

Pick a point $x \in P$, with $\pi(x) = a$. By Frobenius, there is a maximal integral submanifold $Q \subseteq P$ for the distribution $\approx$. Now horizontal infinitesimal 1–simplices (x,y) have $x \approx y$, since $\omega(x,y) = e$ (the neutral element of G). Therefore by the Proposition 12 and maximality of Q, any horizontal curve through x must lie entirely in Q. Since any point of P can be connected to x by a horizontal curve, $Q = P$.

Let $\mathcal{M}(x)$ denote the set of neighbours of x. Since $Q = P$, we have

$$P_a \cap \mathcal{M}(x) = Q_a \cap \mathcal{M}(x) = \{y \in P_a \mid y \approx x\}$$

since Q is a integral manifold for $\approx$. But this is $\{y \sim x \mid y = x \cdot h \text{ for some } h \in H\}$; by (8). The left hand side consists of elements $x \cdot g$ with $g \sim e$. So $g \sim e$ in G implies $g \in H$. From Proposition 13, we deduce that $H = G$. $\square$

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