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CONSERVATION LAWS FOR NON–GLOBAL LAGRANGIANS

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Abstract. In the Lagrangian framework for symmetries and conservation laws of field theories, we investigate globality properties of conserved currents associated with non–global Lagrangians admitting global Euler–Lagrange morphisms. Our approach is based on the recent geometric formulation of the calculus of variations on finite order jets of fibered manifolds in terms of variational sequences.

1. Introduction. In the Lagrangian framework for symmetries and conservation laws of field theories, we investigate globality properties of conserved currents associated with non–global Lagrangians which admit global Euler–Lagrange morphisms (see also [**18**]). Our approach is based on the geometric formulation of the calculus of variations on finite order jets of fibered manifolds in terms of variational sequences [**16**]. It was shown in [**13**] that the Lie derivative operator with respect to fiber–preserving vector fields passes to the quotient, thus yielding a new operator on the sheaves of the variational sequence, which was called the *variational Lie derivative*. Making use of a representation given in [**22**] for the quotient sheaves of the variational sequence as concrete sheaves of forms, some abstract versions of Noether's theorems have been provided, which can be interpreted in terms of conserved currents for Lagrangians and Euler–Lagrange morphisms.

Non–global Lagrangians are here defined as Čech cochains valued into the sheaf of generalized Lagrangians. We relate globality properties to the topology

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of the relevant manifold in terms of the Čech cohomology of the manifold with values in the sheaves of the variational sequence (see also [**5**]). To this aim we provide a slightly modified version of some well known results due to [**3**] (Theorems 3.1 and 3.2). We shall in particular investigate the case of Čech cochains of Lagrangians admitting global Euler–Lagrange morphisms but having non–trivial cohomology class. In this case globality properties still hold true for the conserved quantities associated with the cochain of Lagrangians itself. For analogous results obtained in a different framework we refer the reader to the interesting paper by Aldrovandi [**1**].

In Section 2 we state the main notation and recall some basic facts about sheaves of forms on finite order jets of fibered manifolds, together with some standard results about Čech cohomology. In Subsection 2.3 we recall general results concerning symmetries in variational sequences. Section 3 is concerned with the main results of the paper. We prove the existence of global conserved quantities associated with Lagrangian symmetries and generalized Lagrangian symmetries of Čech cochains of Lagrangians.

2. Preliminaries and notation.

2.1. *Sheaves of forms on jets of fibered manifolds.* Let us consider a fibered manifold $\pi : \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n + m$. For $r \geq 0$ we are concerned with the r–jet space $J_r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold π; in particular, we set $J_0\boldsymbol{Y} \equiv \boldsymbol{Y}$. We recall the natural fiberings $\pi^r_s : J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, and $\pi^r : J_r\boldsymbol{Y} \to \boldsymbol{X}$; among these the fiberings π^r_{r-1} are *affine bundles*.

Greek indices $\lambda, \mu, \ldots$ run from 1 to n and they label basis coordinates, while Latin indices $i, j, \ldots$ run from 1 to m and label fibre coordinates, unless otherwise specified. We denote multi–indices of dimension n by boldface Greek letters such as $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, with $0 \leq \alpha_\mu$, $\mu = 1, \ldots, n$; by an abuse of notation, we denote with λ the multi–index such that $\alpha_\mu = 0$, if $\mu \neq \lambda$, $\alpha_\mu = 1$, if $\mu = \lambda$. We also set $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$. The charts induced on $J_r\boldsymbol{Y}$ are denoted by $(x^\lambda, y^i_{\boldsymbol{\alpha}})$, with $0 \leq |\boldsymbol{\alpha}| \leq r$; in particular, we set $y^i_{\boldsymbol{0}} \equiv y^i$. They are fibered charts, so that the choice of different letters (x for the basis and y for the fibers) stresses different transformation laws: in fact, fibered transformation laws of the kind $x' = x'(x)$ and $y' = y'(x, y)$. The local bases of vector fields and 1–forms on $J_r\boldsymbol{Y}$ induced by the above coordinates are denoted by $(\partial_\lambda, \partial^{\boldsymbol{\alpha}}_i)$ and $(d^\lambda, d^i_{\boldsymbol{\alpha}})$, respectively.

The *contact maps* on jet spaces induce the natural complementary fibered morphisms over the affine fiber bundle $J_r\boldsymbol{Y} \to J_{r-1}\boldsymbol{Y}$

$$
\begin{aligned}
&\text{Д}_r : J_r\boldsymbol{Y} \underset{\boldsymbol{X}}{\times} T\boldsymbol{X} \to TJ_{r-1}\boldsymbol{Y} \\
&\vartheta_r : J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} TJ_{r-1}\boldsymbol{Y} \to VJ_{r-1}\boldsymbol{Y}\,, \qquad r \geq 1\,,
\end{aligned}
\tag{1}
$$

with coordinate expressions, for $0 \leq |\boldsymbol{\alpha}| \leq r-1$, given by

$$
\begin{aligned}
\mathit{\Pi}_r &= d^\lambda \otimes \mathit{\Pi}_\lambda = d^\lambda \otimes (\partial_\lambda + y^j_{\boldsymbol{\alpha}+\lambda} \partial^{\boldsymbol{\alpha}}_j)\,, \\
\vartheta_r &= \vartheta^j_{\boldsymbol{\alpha}} \otimes \partial^{\boldsymbol{\alpha}}_j = (d^j_{\boldsymbol{\alpha}} - y^j_{\boldsymbol{\alpha}+\lambda} d^\lambda) \otimes \partial^{\boldsymbol{\alpha}}_j\,,
\end{aligned} \tag{2}
$$

and the natural fibered splitting [**16**]

$$
J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^* J_{r-1}\boldsymbol{Y} = (J_r\boldsymbol{Y} \underset{J_{r-1}\boldsymbol{Y}}{\times} T^*\boldsymbol{X}) \oplus \operatorname{im} \vartheta^*_r\,. \tag{3}
$$

The above splitting induces also a decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$.

A *projectable vector field* on $\boldsymbol{Y}$ is defined to be a pair (Ξ, ξ), where the vector field $\Xi : \boldsymbol{Y} \to T\boldsymbol{Y}$ is a fibered morphism over the vector field $\xi : \boldsymbol{X} \to T\boldsymbol{X}$. By $(j_r\Xi, \xi)$ we denote the jet prolongation of (Ξ, ξ), and by $j_r\Xi_H$ and $j_r\Xi_V$, respectively, the horizontal and the vertical part of $j_r\Xi$ with respect to the splitting (3).

i. For $r \geq 0$, we consider the standard sheaves $\overset{p}{\Lambda}_r$ of p–forms on $J_r\boldsymbol{Y}$.

ii. For $0 \leq s \leq r$, we consider the sheaves $\overset{p}{\mathcal{H}}_{(r,s)}$ and $\overset{p}{\mathcal{H}}_r$ of *horizontal forms*, *i.e.* of local fibered morphisms over π^r_s and π^r of the type $\alpha : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^* J_s\boldsymbol{Y}$ and $\beta : J_r\boldsymbol{Y} \to \overset{p}{\wedge} T^*\boldsymbol{X}$, respectively.

iii. For $0 \leq s < r$, we consider the subsheaf $\overset{p}{\mathcal{C}}_{(r,s)} \subset \overset{p}{\mathcal{H}}_{(r,s)}$ of *contact forms*, *i.e.* of sections $\alpha \in \overset{p}{\mathcal{H}}_{(r,s)}$ with values into $\overset{p}{\wedge} \operatorname{im} \vartheta^*_{s-1}$. There is a distinguished subsheaf $\overset{p}{\mathcal{C}}_r \subset \overset{p}{\mathcal{C}}_{(r+1,r)}$ of local fibered morphisms $\alpha \in \overset{p}{\mathcal{C}}_{(r+1,r)}$ such that $\alpha = \overset{p}{\wedge} \vartheta^*_{r+1} \circ \tilde{\alpha}$, where $\tilde{\alpha}$ is a section of the fibration $J_{r+1}\boldsymbol{Y} \underset{J_r\boldsymbol{Y}}{\times} \overset{p}{\wedge} V^* J_r\boldsymbol{Y} \to J_{r+1}\boldsymbol{Y}$ which projects down onto $J_r\boldsymbol{Y}$.

According to [**22**], the fibered splitting (3) naturally yields the sheaf splitting $\overset{p}{\mathcal{H}}_{(r+1,r)} = \bigoplus^p_{t=0} \overset{p-t}{\mathcal{C}}_{(r+1,r)} \wedge \overset{t}{\mathcal{H}}_{r+1}$, which restricts to the inclusion $\overset{p}{\Lambda}_r \subset \bigoplus^p_{t=0} \overset{p-t}{\mathcal{C}}_r \wedge \overset{t}{\mathcal{H}}{}^h_{r+1}$, where $\overset{p}{\mathcal{H}}{}^h_{r+1} := h(\overset{p}{\Lambda}_r)$ for $0 < p \leq n$ and h is defined to be the restriction to $\overset{p}{\Lambda}_r$ of the projection of the above splitting onto the non–trivial summand with the highest value of t.

Let $\alpha \in \overset{1}{\mathcal{C}}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1}$. Then there is a unique pair of sheaf morphisms

$$
E_\alpha \in \overset{1}{\mathcal{C}}_{(2r,0)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1}\,, \qquad F_\alpha \in \overset{1}{\mathcal{C}}_{(2r,r)} \wedge \overset{n}{\mathcal{H}}{}^h_{2r+1}\,, \tag{4}
$$

such that $(\pi^{2r+1}_{r+1})^*\alpha = E_\alpha - F_\alpha$, and F_α is locally of the form $F_\alpha = d_H p_\alpha$, with $p_\alpha \in \overset{1}{\mathcal{C}}_{(2r-1,r-1)} \wedge \overset{n-1}{\mathcal{H}}_{2r}$ (see *e.g.* [**22**]).

Recall (see [**22**]) that if $\beta \in \overset{1}{\mathcal{C}}_s \wedge \overset{1}{\mathcal{C}}_{(s,0)} \wedge \overset{n}{\mathcal{H}}_s$, then, there is a unique $\tilde{H}_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \otimes \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}$ such that, for all $\Xi : \boldsymbol{Y} \to V\boldsymbol{Y}$, $E_{\hat{\beta}} = C^1_1(j_{2s}\Xi \otimes \tilde{H}_\beta)$, where $\hat{\beta} := j_s\Xi \,\rfloor\, \beta$, $\rfloor$ denotes the inner product and C^1_1 stands for tensor contraction. Then there is a unique pair of sheaf morphisms

$$H_\beta \in \overset{1}{\mathcal{C}}_{(2s,s)} \wedge \overset{1}{\mathcal{C}}_{(2s,0)} \wedge \overset{n}{\mathcal{H}}_{2s}\,, \quad G_\beta \in \overset{2}{\mathcal{C}}_{(2s,s)} \wedge \overset{n}{\mathcal{H}}_{2s}\,, \tag{5}$$

such that $\pi^{2s\,*}_s \beta = H_\beta - G_\beta$ and $H_\beta = \frac{1}{2} A(\tilde{H}_\beta)$, where A stands for antisymmetrisation. Moreover, G_β is locally of the type $G_\beta = d_H q_\beta$, where $q_\beta \in \overset{2}{\mathcal{C}}_{2s-1} \wedge \overset{n-1}{\mathcal{H}}_{2s-1}$, hence $[\beta] = [H_\beta]$. Coordinate expressions of the morphisms E_α and H_β can be found in [**22**]. The morphism $\tilde{H}$ is called the Helmhotz–Sonin morphism associated with an Euler–Lagrange type morphism. It is a global morphism the kernel of which expresses the Helmholtz conditions for a given Euler–Lagrange type morphism to be locally variational, *i.e.* $\eta = E_{d_V\lambda}$ [**22**].

2.2. *Čech cohomology.* Suppose $\mathfrak{H}$ is a (paracompact Hausdorff) topological space. In the following we shall call *graded sheaf* over $\mathfrak{H}$ any countable family of sheaves $\mathcal{F}^* := \{\mathcal{F}^i\}_{i\in\mathbb{Z}}$ over $\mathfrak{H}$. A *resolution* of a given sheaf $\mathcal{S}$ is an exact sequence of sheaves of the form $0 \to \mathcal{S} \to \mathcal{F}^*$.

Set $H^q(\mathfrak{H}, \mathcal{S}) := \ker(\mathcal{C}^q(\mathcal{S})_{\mathfrak{H}} \to \mathcal{C}^{q+1}(\mathcal{S})_{\mathfrak{H}}) \;/\; \mathrm{im}(\mathcal{C}^{q-1}(\mathcal{S})_{\mathfrak{H}} \to \mathcal{C}^q(\mathcal{S})_{\mathfrak{H}})$, for each $q \in \mathbb{Z}$, with $\mathcal{C}^{-1}(\mathcal{S})_{\mathfrak{H}} := 0$. Here $\mathcal{C}^q(\mathcal{S})_{\mathfrak{H}}$ is the sheaf naturally induced by the sheaf of discontinuous sections of $\mathcal{S}$ [**9**]. The Abelian group $H^q(\mathfrak{H}, \mathcal{S})$ is called the cohomology group of $\mathfrak{H}$ of degree q with coefficients in the sheaf $\mathcal{S}$.

We say $H^*(\mathfrak{H}, \mathcal{S}) := \oplus_{i\in\mathbb{Z}} H^i(\mathfrak{H}, \mathcal{S})$ to be the *cohomology of* $\mathfrak{H}$ *with values in* $\mathcal{S}$. It is clear that $H^0(\mathfrak{H}, \mathcal{S}) = \mathcal{S}_{\mathfrak{H}}$. We say $\mathcal{S}$ to be *acyclic* if $H^q(\mathfrak{H}, \mathcal{S}) = 0$ for all $q \in \mathbb{Z}$, $q > 0$.

We remark that a resolution $0 \to \mathcal{S} \to \mathcal{F}^*$ naturally induces a cochain complex $0 \to \mathcal{S}_{\mathfrak{H}} \to \mathcal{F}^*_{\mathfrak{H}}$ via the global section functor. Hence, we can define the *derived groups* $H^q(\mathcal{F}^*_{\mathfrak{H}}) := \ker(\mathcal{F}^q_{\mathfrak{H}} \to \mathcal{F}^{q+1}_{\mathfrak{H}}) / \mathrm{im}(\mathcal{F}^{q-1}_{\mathfrak{H}} \to \mathcal{F}^q_{\mathfrak{H}})$, for all $q \in \mathbb{Z}$, with $\mathcal{F}^{-1}_{\mathfrak{H}} := \mathcal{S}_{\mathfrak{H}}$.

Let $0 \to \mathcal{S} \to \mathcal{F}^*$ be a resolution of $\mathcal{S}$. Then for each $q \in \mathbb{Z}$ there is a natural morphism $H^q(\mathcal{F}^*_{\mathfrak{H}}) \to H^q(\mathfrak{H}, \mathcal{S})$. If the sheaves of $\mathcal{F}^*$ are acyclic then the above morphism is an isomorphism (Abstract de Rham Theorem) [**9**].

We also recall that a *cochain complex* is a sequence of morphisms of Abelian groups of the form $0 \to \overset{0}{\Lambda} \to^{d_0} \overset{1}{\Lambda} \to^{d_1} \overset{2}{\Lambda} \to^{d_2} \dots$, such that $d_{k+1} \circ d_k = 0$. This last condition is equivalent to $\mathrm{im}\, d_k \subset \ker d_{k+1}$. A cochain complex is said to be an *exact sequence* if $\mathrm{im}\, d_k = \ker d_{k+1}$.

Suppose now that $\mathcal{S}$ is a sheaf of Abelian groups over $\mathfrak{H}$. Let $\mathfrak{U} := \{U_i\}_{i\in I}$, with $I \subset \mathbb{Z}$, be a countable open covering of $\mathfrak{H}$. We set $C^q(\mathfrak{U}, \mathcal{S})$ to be the set of

q–cochains with coefficients in $\mathcal{S}$. Let $\sigma = (U_{i_0}, \dots, U_{i_{q+1}}) \subset \mathfrak{U}$ be a q–simplex and $f \in C^q(\mathfrak{U}, \mathcal{S})$. The *coboundary operator* $\mathfrak{d} : C^q(\mathfrak{U}, \mathcal{S}) \to C^{q+1}(\mathfrak{U}, \mathcal{S})$ is the map defined by

$$\mathfrak{d} f(\sigma) := \sum_{i=0}^{q+1} (-1)^i r^{|\sigma_i|}_{|\sigma|} f(\sigma_i) \, ,$$

where $\sigma_j := (U_{i_0}, \dots, U_{i_{j-1}}, U_{i_{j+1}}, \dots U_{i_{q+1}})$, for $0 \le j \le q+1$, r is the restriction mapping of $\mathcal{S}$ and $|\sigma_j|$ denotes the lenght of σ_j (see [**8**]).

For all $q \in \mathbb{Z}$ the set $C^q(\mathfrak{U}, \mathcal{S})$ can be endowed with an Abelian group structure in a natural way. It is rather easy to verify that $\mathfrak{d}$ is a group morphism, such that $\mathfrak{d}^2 = 0$. Hence we have the cochain complex $C^0(\mathfrak{U}, \mathcal{S}) \to C^1(\mathfrak{U}, \mathcal{S}) \to C^2(\mathfrak{U}, \mathcal{S}) \to \dots$

Definition 2.1. We say the derived groups $H^*(\mathfrak{U}, \mathcal{S})$ of the above cochain complex to be the *Čech cohomology* of the covering $\mathfrak{U}$ with coefficients in $\mathcal{S}$.

The above cohomology is a combinatorial object and it depends on the choice of a covering $\mathfrak{U}$. Let $\mathfrak{U} := \{U_i\}_{i \in I}$, $\mathfrak{V} := \{V_j\}_{j \in J}$, with $I, J \subset \mathbb{Z}$, be two countable coverings of $\mathfrak{H}$. Then we say that $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ if there exists a map $f : J \to I$ such that $V_j \subset U_{f(j)}$. Then there is a group morphism $H^*(\mathfrak{U}, \mathcal{S}) \to H^*(\mathfrak{V}, \mathcal{S})$, so that we can define the *Čech cohomology* of $\mathfrak{H}$ with coefficients in $\mathcal{S}$ to be the direct limit $H^*(\mathfrak{H}, \mathcal{S}) := \lim\limits_{\mathfrak{U}} H^*(\mathfrak{U}, \mathcal{S})$.

2.3. *Cohomology of the variational sequence.* We recall now the theory of variational sequences on finite order jet spaces, as it was developed by Krupka [**16**]. By an abuse of notation, we denote by $d \ker h$ the sheaf generated by the presheaf $d \ker h$. Set $\overset{*}{\Theta}_r := \ker h + d \ker h$.

Definition 2.2. The quotient sequence

$$0 \longrightarrow I\!R_{\boldsymbol{Y}} \longrightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} \overset{n}{\Lambda}_r / \overset{n}{\Theta}_r \xrightarrow{\mathcal{E}_n} \overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r \xrightarrow{\mathcal{E}_{n+1}} \overset{n+2}{\Lambda}_r / \overset{n+2}{\Theta}_r \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

is called the r–th order *variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$. It turns out that it is an exact resolution of the constant sheaf $I\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$ [**16**].

Let us now consider the cochain complex

$$\begin{aligned} 0 \longrightarrow I\!R_{\boldsymbol{Y}} \longrightarrow \dots \xrightarrow{\mathcal{E}_{n-1}} (\overset{n}{\Lambda}_r / \overset{n}{\Theta}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_n} \\ \xrightarrow{\mathcal{E}_n} (\overset{n+1}{\Lambda}_r / \overset{n+1}{\Theta}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_{n+1}} (\overset{n+2}{\Lambda}_r / \overset{n+2}{\Theta}_r)_{\boldsymbol{Y}} \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0 \end{aligned} \tag{6}$$

and denote by $H^k_{\mathrm{VS}}(\boldsymbol{Y})$ its k–th cohomology group. The variational sequence is a soft resolution of the constant sheaf $I\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$, hence the cohomology of the sheaf $I\!R$ is naturally isomorphic to the cohomology of the cochain complex above. Also, the de Rham sequence gives rise to a cochain complex of global sections, the cohomology of which is naturally isomorphic to the cohomology of the sheaf $I\!R_{\boldsymbol{Y}}$ on $\boldsymbol{Y}$, as an application of the Abstract de Rham Theorem. Then, by a composition of isomorphisms, for all $k \geq 0$ we get a natural isomorphism $H^k_{\mathrm{VS}}(\boldsymbol{Y}) \simeq H^k_{\mathrm{dR}}\boldsymbol{Y}$ [**16**].

The quotient sheaves in the variational sequence can be conveniently represented [**22**]. The sheaf morphism h yields the natural isomorphisms

$$I_k : \overset{k}{\Lambda}_r/\overset{k}{\Theta}_r \to \overset{k}{\mathcal{H}}{}^h_{r+1} := \overset{k}{\mathcal{V}}_r : [\alpha] \mapsto h(\alpha)\,, \qquad k \leq n\,,$$
$$I_k : (\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r) \to (\overset{k-n}{\mathcal{C}}{}_r \wedge \overset{n}{\mathcal{H}}{}^h_{r+1})/h(\overline{d\ker h}) := \overset{k}{\mathcal{V}}_r : [\alpha] \mapsto [h(\alpha)]\,, \qquad k > n\,.$$

Let $s \leq r$. Then we have the injective sheaf morphism (see [**16**]) $\chi^r_s : (\overset{k}{\Lambda}_s/\overset{k}{\Theta}_s) \to (\overset{k}{\Lambda}_r/\overset{k}{\Theta}_r) : [\alpha] \mapsto [\pi^{r*}_s \alpha]$, where $[\alpha]$ denotes the equivalence class of a form α on $J_s\boldsymbol{Y}$.

3. Čech cochains valued in the sheaves of the variational sequence. We are interested in the case in which the topology of $\boldsymbol{Y}$ is non–trivial; in particular we shall be concerned with an application of Čech cohomology to the cases $H^{n+1}_{\mathrm{dR}}\boldsymbol{Y} \neq 0$ and $H^n_{\mathrm{dR}}\boldsymbol{Y} \neq 0$.

The following results hold true (see *e.g.* [**3**], and [**4**] Chap. II).

THEOREM 3.1. *Let us consider the variational sequence* (6) *and let* $\boldsymbol{K}_r := Ker\,\mathcal{E}_n$ *and* $H^1(\boldsymbol{Y}, \boldsymbol{K}_r)$ *be the first Čech cohomology group of* $\boldsymbol{Y}$ *with values in* $\boldsymbol{K}_r$. *Then the long exact sequence obtained from the short exact sequence*

$$0 \longrightarrow \boldsymbol{K}_r \longrightarrow \overset{n}{\mathcal{V}}_r \xrightarrow{\mathcal{E}_n} \mathcal{E}_n(\overset{n}{\mathcal{V}}_r) \longrightarrow 0$$

gives rise to the exact sequence

$$0 \longrightarrow \Gamma(\boldsymbol{Y}, \boldsymbol{K}_r) \longrightarrow \Gamma(\boldsymbol{Y}, \overset{n}{\mathcal{V}}_r) \longrightarrow \Gamma(\boldsymbol{Y}, \mathcal{E}_n(\overset{n}{\mathcal{V}}_r)) \xrightarrow{\delta} H^1(\boldsymbol{Y}, \boldsymbol{K}_r) \longrightarrow 0\,.$$

THEOREM 3.2. *Let us consider the variational sequence* (6) *and let* $\boldsymbol{T}_r := Ker\,d_H$ *and* $H^1(\boldsymbol{Y}, \boldsymbol{T}_r)$ *be the first Čech cohomology group of* $\boldsymbol{Y}$ *with values in* $\boldsymbol{T}_r$. *Then the long exact sequence obtained from the short exact sequence*

$$0 \to \boldsymbol{T}_r \to \overset{n-1}{\mathcal{V}}{}_r \xrightarrow{d_H} d_H(\overset{n-1}{\mathcal{V}}{}_r) \to 0$$

gives rise to the exact sequence

$$0 \to \Gamma(\boldsymbol{Y}, \boldsymbol{T}_r) \to \Gamma(\boldsymbol{Y}, \overset{n-1}{\mathcal{V}}{}_r) \to \Gamma(\boldsymbol{Y}, d_H(\overset{n-1}{\mathcal{V}}{}_r)) \xrightarrow{\delta'} H^1(\boldsymbol{Y}, \boldsymbol{T}_r) \to 0\,.$$

Here and above we have used the standard notation denoting by $\Gamma(\boldsymbol{Y},\cdot)$ the corresponding modules of global sections.

Furthermore we have ([**3**], Lemma 4.1, Theorem 4.2, [**16**], [**22**]) for $s\leq r$:

$$H^{n+1}_{dR}(\boldsymbol{Y})\simeq H^{n+1}_{VS}(\boldsymbol{Y})\simeq H^1(\boldsymbol{Y},\boldsymbol{K}_r)\simeq H^1(\boldsymbol{Y},\chi^{r*}_s\boldsymbol{K}_s)\,,$$

and

$$H^{n}_{dR}(\boldsymbol{Y})\simeq H^{n}_{VS}(\boldsymbol{Y})\simeq H^1(\boldsymbol{Y},\boldsymbol{T}_r)\simeq H^1(\boldsymbol{Y},\chi^{r*}_s\boldsymbol{T}_s)\,.$$

Remark 3.3. As a straightforward application of the Abstract de Rham Theorem, we have the following.

Let $\eta\in(\overset{n+1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ be a global section such that $\mathcal{E}_{n+1}(\eta)=0$. Suppose, moreover, that $H^{n+1}_{\mathrm{dR}}\boldsymbol{Y}\ni\delta\eta=0$. Then, there exists a global section $\lambda\in(\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $\mathcal{E}_n(\lambda)=\eta$ (see *e.g.* [**2**]).

Analogously, let $\lambda\in(\overset{n}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ be a global section such that $\mathcal{E}_n(\lambda)=0$, *i.e.* λ is variationally trivial. Suppose, moreover, that $H^{n}_{\mathrm{dR}}\boldsymbol{Y}\ni\delta'\lambda=0$. Then, there exists a global section $\beta\in(\overset{n-1}{\mathcal{V}}_r)_{\boldsymbol{Y}}$ such that $\mathcal{E}_{n-1}(\beta)=\lambda$, where $\mathcal{E}_{n-1}=d_H$ (see *e.g.* [**16, 22**]). $\square$

If the topology of $\boldsymbol{Y}$ is trivial, so that, in particular, $H^{n+1}_{\mathrm{dR}}\boldsymbol{Y}=0$ and $H^{n}_{\mathrm{dR}}\boldsymbol{Y}=0$ hold true, then each global Euler–Lagrange morphism η is globally variational and each global variationally trivial Lagrangian λ is the horizontal differential of a form β.

If the topology of $\boldsymbol{Y}$ is non–trivial, *i.e.* $H^{n+1}_{VS}(\boldsymbol{Y})\simeq H^1(\boldsymbol{Y},\boldsymbol{K}_r)\neq 0$, then the inverse problem for a given global Euler–Lagrange morphism η can be solved only locally, so that in general we can write $\eta=\mathcal{E}_n(\lambda)$ only locally (provided, of course, that the corresponding cohomology class of η is non–trivial). More precisely this means that around each point a Lagrangian $\lambda_{\boldsymbol{U}}$ is defined only on an open subset $\boldsymbol{U}\subset\boldsymbol{Y}$, so that $\eta|_{\boldsymbol{U}}=\mathcal{E}_n(\lambda_{\boldsymbol{U}})$. We are then naturally faced with the following situation which is in fact often encountered in physical applications: there exists a countable open covering $\{\boldsymbol{U}_i\}_{i\in\mathbb{Z}}$ in $\boldsymbol{Y}$ together with a family of local Lagrangians λ_i over each subset $\boldsymbol{U}_i\subset\boldsymbol{Y}$ (which, a priori, do not glue together into a global Lagrangian λ). Let then $\mathfrak{U}:=\{\boldsymbol{U}_i\}_{i\in I}$, with $I\subset\mathbb{Z}$, be any countable open covering of $\boldsymbol{Y}$ and $\lambda=\{\lambda_i\}_{i\in I}$ a 0–cochain of Lagrangians in Čech cohomology with values in the sheaf $\overset{n}{\mathcal{V}}_r$, *i.e.* $\lambda\in C^0(\mathfrak{U},\overset{n}{\mathcal{V}}_r)$. By an abuse of notation we shall denote by η_λ the 0–cochain formed by the restrictions $\eta_i=\mathcal{E}_n(\lambda_i)$.

Remark 3.4. Let $\mathfrak{d}\lambda=\{\lambda_{ij}\}=(\lambda_i-\lambda_j)|_{U_i\cap U_j}$. We stress that $\mathfrak{d}\lambda=0$ if and only if λ is globally defined on $\boldsymbol{Y}$. Analogously, if $\eta\in C^0(\mathfrak{U},\overset{n+1}{\mathcal{V}}_r)$, then $\mathfrak{d}\eta=0$ if and only if η is global.

Remark 3.5. Let $\lambda \in C^0(\mathfrak{U}, \overset{n}{\mathcal{V}}_r)$ and let $\eta_\lambda := \mathcal{E}_n(\lambda) \in C^0(\mathfrak{U}, \overset{n+1}{\mathcal{V}}_r)$ be as above. Then $\mathfrak{d}\lambda = 0$ implies $\mathfrak{d}\eta_\lambda = 0$, but the converse is *not* true, in general. This is due to the $I\!\!R$–linearity of all the operations involved in the variational sequence. Therefore $\mathfrak{d}\eta_\lambda = \eta_{\mathfrak{d}\lambda} = 0$ implies only $\mathfrak{d}\lambda \in C^1(\mathfrak{U}, \boldsymbol{K}_r)$.

We shall in fact be concerned with the case

$$\mathfrak{d}\eta_\lambda = 0\,, \qquad \mathfrak{d}\lambda \neq 0\,. \tag{7}$$

Definition 3.6. We shall call a Čech cochain λ of Lagrangians satisfying condition (7) a *non–global* Lagrangian.

Definition 3.7. A non–global Lagrangian is said to be *topologically non–trivial* if the cohomology class of η_λ in the first Čech cohomology group is non–trivial, *i.e.* $\delta\eta_\lambda \neq 0$.

It is clear that a non–global Lagrangian is defined modulo a refinement of $\mathfrak{U}$. In particular, $\mathfrak{U}$ can be chosen to be a good covering of $\boldsymbol{Y}$ (on a differentiable manifold there exists always a good covering, see *e.g.* [**8**]), on which all (local) de Rham cohomologies are trivial. Then Remark 3.3 can be reformulated as follows.

Proposition 3.8. *(A) Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a global variationally trivial Lagrangian. Then for any good cover $\mathfrak{U}$ there exists a 0–cochain $\beta \in C^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_r)$ such that $\lambda = d_H\beta$. Thus $\mathfrak{d}\beta \in C^1(\mathfrak{U}, \boldsymbol{T}_r)$ defines a unique cohomology class $[\mathfrak{d}\beta]_{\check{C}} := \delta'\lambda \in H^1(\boldsymbol{Y}, \boldsymbol{T}_r) \simeq H^n_{dR}\boldsymbol{Y}$. If, moreover, this cohomology class is trivial, then there exists a 0–cochain $\gamma \in C^0(\mathfrak{U}, \overset{n-2}{\mathcal{V}}_r)$ such that $d_H\mathfrak{d}\gamma = \mathfrak{d}\beta$ with $\beta' = \beta - d_H\gamma$ a global morphism and $\lambda = d_H\beta'$. (B) Let $\lambda \in C^0(\mathfrak{U}, \overset{n}{\mathcal{V}}_r)$ be a non–global Lagrangian. Then $\mathfrak{d}\lambda$ defines a unique cohomology class $[\mathfrak{d}\lambda]_{\check{C}} := \delta\eta_\lambda \in H^1(\boldsymbol{Y}, \boldsymbol{K}_r) \simeq H^{n+1}_{dR}\boldsymbol{Y}$. If, moreover, this cohomology class is trivial then there exists a 0–cochain $\nu \in C^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_r)$ such that*

$$\mathfrak{d}\lambda = d_H\mathfrak{d}\nu \quad \textit{and} \quad \mathfrak{d}(\lambda - d_H\nu) = 0\,.$$

Thus $\lambda' = \lambda - d_H\nu$ is a global Lagrangian and $\mathcal{E}_n(\lambda') = \mathcal{E}_n(\lambda)$.

Proof. It follows from the application of the Poincaré Lemma, the standard Čech cohomology arguments [**9**] and the Abstract de Rham Theorem (see also [**3**]; in [**17**] was shown that, even more, $\beta \in \mathcal{C}^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_{r-1})$). □

Example 3.9. (Einstein theory) From the above Proposition it follows that a topologically trivial non–global Lagrangian is always equivalent to a global one. This is *e.g.* the case of the Hilbert–Einstein Lagrangian, which is a second order global Lagrangian in the bundle $\boldsymbol{Y} = \mathrm{Lor}(\boldsymbol{X})$ of Lorentzian metrics over $\boldsymbol{X}$. The Hilbert–Einstein Lagrangian is equivalent to a sheaf of non–global first order Einstein's Lagrangians. In this case, the cohomology class corresponding to the Euler–Lagrange morphism is trivial and this fact does not depend on the topology of the space-time manifold (see also [**12**]).

EXAMPLE 3.10. (Chern–Simons theory) Let $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{X}, \boldsymbol{G})$ be a principal bundle over an odd dimensional manifold $\boldsymbol{X}$ with a structure group $\boldsymbol{G}$ (*e.g.* any simple Lie group). To any connection one–form ω we can associate the Chern-Simons form [**10**] from which by pull-back along any (local) section one gets a (local) Lagrangian on $\boldsymbol{X}$. Since the Chern-Simons form is not tensorial the local Lagrangians are not gauge-invariant. In spite of this fact, the corresponding Euler-Lagrange equations (*i.e.* the vanishing curvature equations for ω) are invariant and global. Moreover, in this case, an invariant Lagrangian does not exist at all. The existence of global Lagrangians relies on the choice of a global section on $\boldsymbol{P}$ (see *e.g.* [**6, 7**] and references quoted therein).

3.1. *Symmetries and conservation laws.* Making use of the sheaf isomorphisms (7) and of the decomposition formulae (4) and (5), in [**13**] it was proved that the Lie derivative operator with respect to the r-th order prolongation $j_r\Xi$ of a projectable vector field (Ξ, ξ) can be conveniently represented on the quotient sheaves of the variational sequence in terms of an operator, the *variational Lie derivative* $\mathcal{L}_{j_r\Xi}$, as follows:

if $p = n$ and $\lambda \in \overset{n}{\mathcal{V}}_r$, then

$$\mathcal{L}_{j_r\Xi}\lambda = \Xi_V \rfloor \mathcal{E}_n(\lambda) + d_H(j_r\Xi_V \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)\,; \tag{8}$$

if $p = n+1$ and $\eta \in \overset{n+1}{\mathcal{V}}_r$, then

$$\mathcal{L}_{j_r\Xi}\eta = \mathcal{E}_n(\Xi_V \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi_V)\,. \tag{9}$$

DEFINITION 3.11. Let (Ξ, ξ) be a projectable vector field on $\boldsymbol{Y}$. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian and $\eta \in \overset{n+1}{\mathcal{V}}_r$ an Euler–Lagrange morphism. Then Ξ is called a *symmetry* of λ (respectively, a *generalized or Bessel–Hagen symmetry*, of η) if $\mathcal{L}_{j_{r+1}\Xi}\,\lambda = 0$ (respectively, if $\mathcal{L}_{j_{2r+1}\Xi}\,\eta = 0$).

Let now $\eta \in \overset{n+1}{\mathcal{V}}_r$ be an Euler–Lagrange morphism and let $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$ be a section. We recall that σ is said to be *critical* if $\eta \circ j_{2r+1}\sigma = 0$, *i.e.* if it is a solution of the Euler–Lagrange equations $(j_{2r+1}\sigma)^*\mathcal{E}_n(\lambda) = 0$.

Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian and (Ξ, ξ) a symmetry of λ. Then, by Equation (8), *i.e.* the first Noether's theorem, we have

$$0 = \Xi_V \rfloor \mathcal{E}_n(\lambda) + d_H(j_r\Xi_V \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)\ .$$

Suppose that the section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$ fulfils $(j_{2r+1}\sigma)^*(\Xi_V \rfloor \mathcal{E}_n(\lambda)) = 0$, then we have the conservation law $d((j_{2r}\sigma)^*(j_r\Xi_V \rfloor p_{d_V\lambda} + \xi \rfloor \lambda)) = 0$.

The above implies that $\delta'(\mathcal{L}_{j_r\Xi}\lambda) \equiv \delta'(\Xi_V \rfloor \eta_\lambda) \equiv 0$.

Definition 3.12. Let $\lambda \in \overset{n}{\mathcal{V}}_r$ be a Lagrangian and Ξ a global symmetry of λ. Then a sheaf morphism of the type $\epsilon(\lambda, \Xi) = (j_r\Xi_V \rfloor p_{d_V\lambda} + \xi \rfloor \lambda) \in \overset{n-1}{\mathcal{V}}_r$ is said to be a *canonical* or *Noether current*.

Remark 3.13. Notice that if λ is globally defined, then for any global symmetry Ξ of λ the morphism $\epsilon(\lambda, \Xi)$ can be globally defined too (see *e.g.* [**15**]). If $H^{n+1}_{\mathrm{dR}}\boldsymbol{Y} \neq 0$, *i.e.* the topology of $\boldsymbol{Y}$ is not trivial, then given a globally defined Euler–Lagrange morphism with non–trivial cohomology class, we cannot find a corresponding globally defined Lagrangian via the inverse problem, so that in this case the corresponding Noether conserved current ϵ is not global.

Remark 3.14. Let $\eta \in \overset{n+1}{\mathcal{V}}_r$ and let Ξ be a generalized symmetry of η. Then, by Equation (9), we have $0 = \mathcal{E}_n(\Xi_V \rfloor \eta) + \tilde{H}_{d\eta}(j_{2r+1}\Xi_V)$. Suppose that η is locally variational, *i.e.* $\tilde{H}_{d\eta} = 0$; then we have $\mathcal{E}_n(\Xi_V \rfloor \eta) = 0$. This implies that $\Xi_V \rfloor \eta$ is variationally trivial. Therefore $\Xi_V \rfloor \eta$ is locally of the type $\Xi_V \rfloor \eta = d_H\beta$, where $\beta \in \mathcal{C}^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_{r+1})$ (in [**14, 17**] was shown that, even more, $\beta \in \mathcal{C}^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_{r-1})$).

Suppose that the section $\sigma : \boldsymbol{X} \to \boldsymbol{Y}$ fulfils $(j_{2r+1}\sigma)^*(\Xi_V \rfloor \eta) = 0$. Then we have $d((j_{2r}\sigma)^*\beta) = 0$ so that, as in the case of Lagrangians, if σ is critical, then β is conserved along σ.

Definition 3.15. Let $\eta \in \overset{n+1}{\mathcal{V}}_r$ be an Euler–Lagrange morphism and Ξ a symmetry of η. Then a sheaf morphism of the type β fulfilling the conditions of the above Remark is called a *generalized conserved current*.

Notice that, for locally variational Euler–Lagrange morphisms, *i.e.* $\eta = \eta_\lambda \equiv \mathcal{E}_n(\lambda)$ or, equivalently $\tilde{H}_{d\eta}(j_{2r+1}\Xi_V) = 0$. This implies $\mathcal{L}_{j_{2r}\Xi}\eta_\lambda = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda) = \mathcal{E}_n(\mathcal{L}_{j_r\Xi}\lambda)$.

Remark 3.16. Since $(j_r\sigma)^*d_H\epsilon = d((j_r\sigma)^*\epsilon) = 0$ any solution σ defines a corresponding cohomology class $\sigma(\epsilon) \equiv [(j_r\sigma)^*\epsilon]_{\check{C}} \in H^{n-1}_{dR}\boldsymbol{X}$. If all these cohomology classes are trivial then the corresponding current is called trivial (otherwise it is called topological). It is obvious that currents admitting (global) superpotentials [**11**] are trivial in the above sense. Non–trivial currents are more interesting and lead to topological charges (see e.g. [**19**]). Notice that if $H^{n-1}_{dR}\boldsymbol{X} = 0$ then topological charges do not appear.

Due to $\mathcal{E}_n \mathcal{L}_{j_r\Xi} = \mathcal{L}_{j_{2r+1}\Xi}\mathcal{E}_n$, a symmetry of a Lagrangian λ is also a symmetry of its Euler–Lagrange morphism $\mathcal{E}_n(\lambda)$ but the converse is not true. If (Ξ, ξ) is a generalized symmetry of λ the corresponding current is not longer a canonical Noether conserved current for λ, in general.

Instead we can state the following (see [**20, 21**] for the local version).

PROPOSITION 3.17. *Let* (Ξ, ξ) *be a generalized symmetry for a (global) Lagrangian* $\lambda \in \overset{n}{\mathcal{V}}_r$. *Thus the canonical Noether current is not conserved in general. If the cohomology class* $\delta'(\Xi_V \rfloor \eta_\lambda) \in H^n_{\mathrm{dR}}\boldsymbol{Y}$ *is trivial then there exists a global conserved current associated with* (Ξ, ξ).

PROOF. When $\mathcal{L}_{j_r\Xi}\lambda = 0$ then we are in the standard Noether case. If $\mathcal{L}_{j_r\Xi}\lambda \neq 0$ then $\mathcal{L}_{j_r\Xi}\lambda$ is variationally trivial with the trivial cohomology class $\delta'(\mathcal{L}_{j_r\Xi}\lambda)$ (Remark 3.13). Hence, there exits a global morphism (Proposition 3.8 (A)) $\beta \equiv \beta(\lambda, \Xi)$ such that $\mathcal{L}_{j_r\Xi}\lambda = d_H\beta(\lambda, \Xi)$ and

$$\Xi_V \rfloor \eta_\lambda = d_H(\epsilon(\lambda, \Xi) - \beta(\lambda, \Xi))\,. \tag{10}$$

Thus $\tilde{\epsilon}(\lambda, \Xi) := \epsilon(\lambda, \Xi) - \beta(\lambda, \Xi)$ is global and conserved. □

As a result it is then possible to get a realization of the corresponding conservation law associated with this generalized symmetry, in terms of a (non–canonical) conserved current which is global.

DEFINITION 3.18. We call the above non–canonical conserved current an *improved Noether current*.

REMARK 3.19. We stress that if $H^n_{\mathrm{dR}}\boldsymbol{Y} = 0$, the improved Noether current $\tilde{\epsilon}(\lambda, \Xi)$ is always conserved and globally defined. If $H^n_{\mathrm{dR}}\boldsymbol{Y} \neq 0$ this is not true, in general. More precisely, for $\delta'(\Xi_V \rfloor \eta_\lambda) \neq 0$ we have $\beta(\lambda, \Xi) \in C^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_r)$ with non–trivial cohomolgy class $[\mathfrak{d}\beta(\lambda, \Xi)]_{\check{C}} = \delta'(\Xi_V \rfloor \eta_\lambda)$ and therefore, $\tilde{\epsilon}(\lambda, \Xi)$ is conserved but not global. This is *e.g.* the case where topological charges can appear.

We are in a similar situation for a non–global Lagrangian $\lambda \in C^0(\mathfrak{U}, \overset{n}{\mathcal{V}}_r)$. In this case both components of $\tilde{\epsilon}(\lambda, \Xi)$, the canonical and the improved one, are non–global too. However, Equation (10) still holds true for $\tilde{\epsilon}(\lambda, \Xi) \in C^0(\mathfrak{U}, \overset{n-1}{\mathcal{V}}_r)$ with $[\mathfrak{d}\tilde{\epsilon}(\lambda, \Xi)]_{\check{C}} = \delta'(\Xi_V \rfloor \eta_\lambda)$ and it provides us the following.

PROPOSITION 3.20. *Let* (Ξ, ξ) *be a global generalized symmetry for a non–global Lagrangian* $\lambda \in C^0(\mathfrak{U}, \overset{n}{\mathcal{V}}_r)$. *Then the improved Noether current is conserved and, in general, non–global. It is possible to improve it further to a global conserved current provided that* $\delta'(\Xi_V \rfloor \eta_\lambda) = 0$.

Proof. Since $[\mathfrak{d}\tilde{\epsilon}(\lambda, \Xi)]_{\check{C}} = 0$ then, thanks to Proposition 3.8 (A), it can be globalized. □

Remark 3.21. If $\delta(\eta_\lambda) \neq 0$, then $\delta'(\Xi_V \rfloor \eta_\lambda) = 0$ is not true, in general. Therefore, in order to get a global conserved quantity for topologically non–trivial Lagrangians some of our assumptions need to be relaxed. For example, one could consider 0–cochains of symmetries instead of global projectable vector fields. This may cover some physically interesting cases, like translations or angular momentum, *e.g.* . This will be the subject of our future investigations.

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