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NOTES ON EXTENDED RECURRENT AND EXTENDED QUASI-RECURRENT MANIFOLDS

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Dedicated to Professor Dr. Witold Roter on his seventieth birthday.

Abstract. A local structure theorem for conformally flat manifolds of dimension $n > 4$ with condition of recurrent type imposed on Riemann curvature tensor is proved. It appears that the condition describes almost exactly the subprojective manifolds.

1. Introduction. Recently M. Prvanović ([**6**]) introduced a type of manifold (M, g) whose $(0,4)$ curvature tensor R satisfies

$$\begin{aligned} &\nabla_Z R(X,Y,U,V) \\ &= A(Z)\left[R(X,Y,U,V) + (\beta - \psi)\, G(X,Y,U,V)\right] \\ &+\tfrac{\beta}{2}\left[A(X)G(Z,Y,U,V) + A(Y)G(X,Z,U,V)\right. \\ &\left.+A(U)G(X,Y,Z,V) + A(V)G(X,Y,U,Z)\right], \end{aligned} \tag{1}$$

where β, ψ are functions on M, A is a closed form satisfying $\beta A(\frac{\partial}{\partial x^r}) = \frac{\partial \psi}{\partial x^r}$ and $G(X,Y,U,V) = g(Y,U)g(X,V) - g(Y,V)g(X,U)$. She proved that in a neighbourhood of a generic point the associated 1–form A is concircular, i. e.

$$(\nabla A)(X,Y) = Fg(X,Y) + HA(X)A(Y)$$

holds for some functions F, H, and found the local form of the metric ([**6**]). The condition (1) can be considered as a generalisation of the well known notion of a recurrent manifold ($\nabla_Z R = a(Z)R$, ([**8**])) as well as of a generalised recurrent

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manifold introduced by Dubey ([**2**])

$$\nabla_Z R(X,Y,U,V) = K(Z)R(X,Y,U,V) + L(Z)G(X,Y,U,V). \tag{2}$$

On the other hand, M.C. Chaki ([**1**]) introduced and studied a type of manifolds satisfying

$$\begin{aligned}&\nabla_Z R(X,Y,U,V)\\ &= 2a(Z)R(X,Y,U,V) + a(X)R(Z,Y,U,V)\\ &+a(Y)R(X,Z,U,V) + a(U)R(X,Y,Z,V) + a(V)R(X,Y,U,Z)\end{aligned}$$

known as pseudo-symmetric (in the sense of Chaki) or quasi-recurrent.

In the paper, we consider manifolds satisfying a condition including the above ones. For such manifolds being simultaneously conformally flat and of dimension $n > 4$ we prove the local structure theorem. It appears that the condition describes almost exactly the subprojective manifolds.

All manifolds under consideration are connected, smooth, Hausdorff and their metrics need not to be definite.

2. Preliminaries. Using the symmetries of the curvature tensor R as well as the Bianchi's identities the condition (1) yields

$$\begin{aligned}&\nabla_Z R(X,Y,U,V)\\ &= 2a(Z)R(X,Y,U,V)\\ &+2b(Z)G(X,Y,U,V) + b(X)G(Z,Y,U,V)\\ &+b(Y)G(X,Z,U,V) + b(U)G(X,Y,Z,V) + b(V)G(X,Y,U,Z)\end{aligned} \tag{3}$$

for some covectors a and b (c.f. [**4**]). By the same method, one can prove even a more general statement.

Lemma 1. *If*

$$\begin{aligned}\nabla_{X_1} R(X_2,X_3,X_4,X_5) = &\sum_{\sigma\in S_5} \overset{\sigma}{a}\,(X_{\sigma(1)})R(X_{\sigma(2)},X_{\sigma(3)},X_{\sigma(4)},X_{\sigma(5)})\\ &+\sum_{\sigma\in S_5} \overset{\sigma}{b}\,(X_{\sigma(1)})G(X_{\sigma(2)},X_{\sigma(3)},X_{\sigma(4)},X_{\sigma(5)})\end{aligned}$$

for some covectors $\overset{\sigma}{a}$ *and* $\overset{\sigma}{b}$, σ *being a permutation, then there exist covectors* a *and* b *such that*

$$\begin{aligned}&\nabla_Z R(X,Y,U,V)\\ &= 2a(Z)R(X,Y,U,V) + a(X)R(Z,Y,U,V)\\ &+a(Y)R(X,Z,U,V) + a(U)R(X,Y,Z,V) + a(V)R(X,Y,U,Z)\\ &+2b(Z)G(X,Y,U,V) + b(X)G(Z,Y,U,V)\\ &+b(Y)G(X,Z,U,V) + b(U)G(X,Y,Z,V) + b(V)G(X,Y,U,Z)\end{aligned} \tag{4}$$

holds.

Moreover, applying the second Bianchi identity to (3) we get

$$a(X)R(Y,Z)+a(Y)R(Z,X)+a(Z)R(X,Y)=0.$$

Making use of the last equality we are in a position to transform (1) into (4). A similar result holds for (2).

Any manifold satisfying (3) is conformally recurrent ($\nabla_Z C = a(Z)C$). The converse statement is not true in general: a conformally recurrent manifold with recurrent Ricci tensor need not to satisfy (3).

For a given $(0,2)$ tensor A and a generalised curvature tensor R define the tensor $P\,(A,R)$ with components

$$\begin{aligned}&P\,(A,R)_{hijklm} = \\ &2\,(A_{lm}R_{hijk}+A_{hi}R_{jklm}+A_{jk}R_{lmhi})+ \\ &A_{hm}R_{lijk}-A_{hl}R_{mijk}+A_{im}R_{hljk}-A_{il}R_{hmjk}+A_{jm}R_{hilk}-A_{jl}R_{himk}+ \\ &A_{km}R_{hijl}-A_{kl}R_{hijm}+A_{ji}R_{hklm}-A_{jh}R_{iklm}+A_{ki}R_{jhlm}-A_{kh}R_{jilm}.\end{aligned}$$

Lemma 2. ([**3**, p. 194–195]) *If $A_{lm}=-A_{ml}$ and $P\,(A,R)_{hijklm}=0$, then either $A_{lm}=0$ for all l, m or $R_{hijk}=0$ for all h, i, j, k.*

Lemma 3. ([**8**]). *The curvature tensor of an arbitrary manifold (M,g) satisfies the equation*

$$R_{hijk,[lm]}+R_{jklm,[hi]}+R_{lmhi,[jk]}=0.$$

Lemma 4. ([**7**]) *If c_j , p_j and B_{hijk} are numbers satisfying*

$$c_l B_{hijk}+p_h B_{lijk}+p_i B_{hljk}+p_j B_{hilk}+p_k B_{hijl}=0,$$

$$B_{hijk}=-B_{ihjk}=B_{jkhi}, \qquad B_{hijk}+B_{hjki}+B_{hkij}=0,$$

then either $c_j+2b_j=0$ for all j or $B_{hijk}=0$ for all h,i,j,k.

Differentiating covariantly (4) and making use of Lemma 3, we obtain

$$P(A,R)+P(B,G)=0,$$

where $A=da$ and $B=db$. In virtue of Lemma 2 it is clear that if (M,g) is not of constant curvature, then the forms a and b either both are gradients or both are not. On the other hand, if (4) holds on a manifold of constant curvature, then a and b are proportional by virtue of Lemma 4.

A space of affine connection is said to be subprojective ([**5**, p. 164]) if both:

- under a mapping onto pseudo-euclidean space the image of each geodesic is contained in two-dimensional plane;
- all such planes have either a common point or are parallel to each other.

Lemma 5. ([**5**, p. 184]) *A Riemannian manifold $(M,\ g)$, $dimM \geq 3$, is subprojective if and only if it is conformally flat and its $(0,2)$ Ricci tensor S satisfies*

$$S(X,Y)-\frac{TrS}{2(n-1)}g(X,Y)=P(v)g(X,Y)+Q(v)X(v)Y(v) \tag{5}$$

for some non-constant function v.

LEMMA 6. ([**5**, p. 176]) *If* (M,g) *is a subprojective Riemannian manifold, then in a neighbourhood of each point there exists a coordinate system* $x^1,\dots,x^n$ *such that the metric takes the form either*

$$ds^2=(dx^1)^2+p^2(x^1)ds_1^2, \tag{6}$$

where $ds_1^2=f_{ab}dx^adx^b$ *is a metric of an* $(n-1)$*–dimensional space of constant curvature, or*

$$ds^2=2dx^1dx^2+p^2(x^1)ds_2^2, \tag{7}$$

where ds_2^2 *is a metric of* $(n-2)$*–dimensional pseudoeuclidean space.*

3. Conformally flat manifolds. Let M, $dimM\geq 3$, be a conformally flat manifold. Then on M the following well-known relations hold:

$$\begin{aligned} R_{hijk}=&\tfrac{1}{n-2}\left(g_{ij}R_{hk}-g_{ik}R_{hj}+g_{hk}R_{ij}-g_{hj}R_{ik}\right)+\\ &\tfrac{r}{(n-1)(n-2)}(g_{ij}g_{hk}-g_{ik}g_{hj}),\end{aligned} \tag{8}$$

$$R_{ij,k}-R_{ik,j}-\frac{1}{2(n-1)}\left(g_{ij}r_{,k}-g_{ik}r_{,j}\right)=0, \tag{9}$$

where $R_{ij}=S\left(\partial_i,\partial_j\right)$, $r=TrS$.

Following the considerations made in the proof of [**3**, Theorem 2] we obtain

THEOREM 7. *Let* M, $dimM\geq 3$, *be a conformally flat manifold whose curvature tensor satisfies (4) and the fundamental forms* a, b *are locally gradients. If* $a_l(x)\neq 0$, $x\in M$, *then there exists a neighbourhood of* x *such that*

$$R_{ij}=Fg_{ij}+Ha_ia_j, \tag{10}$$

F, H *being functions and*

$$\begin{aligned} R_{ij,l}=&2F(a_ig_{lj}+a_jg_{il})+4Ha_ia_ja_l\\ &+\tfrac{r}{n-1}(2g_{ij}a_l-g_{lj}a_i-g_{il}a_j)+2ng_{ij}b_l+(n-2)(g_{lj}b_i+g_{il}b_j).\end{aligned} \tag{11}$$

REMARK 1. Manifolds satisfying (10) are called quasi-Einstein.

We also have

THEOREM 8. *Let* M *be a manifold with vanishing Weyl conformal curvature tensor and suppose that the Ricci tensor and its covariant derivative satisfy (10) and (11). Then:*

(a)
$$R_{hijk} = \frac{1}{n-2}\left[\left(2F - \frac{r}{n-1}\right)(g_{ij}g_{hk} - g_{ik}g_{hj}) + H\left(g_{ij}a_h a_k - g_{ik}a_h a_j + g_{hk}a_i a_j - g_{hj}a_i a_k\right)\right],$$

i.e. M is of almost constant curvature;

(b) relation (4) holds on M.

PROPOSITION 9. *Let M be a conformally flat manifold, $dimM > 4$, whose curvature satisfy (4) and the fundamental form a is locally a gradient. If $a(x) \neq 0$, $R(x) \neq 0$, $x \in M$, then there exist a neighbourhood U and a function a defined on U satisfying $a_l = a_{,l}$, such that $F = F(a)$, $H = H(a)$, $B = a_r a^r = B(a)$, where F, H are defined by (10).*

PROOF. Differentiating covariantly (10) and substituting into (9) we get

(12)
$$0 = F_k g_{ij} + H_k a_i a_j + H a_{i,k} a_j - F_j g_{ik} - H_j a_i a_k - H a_{i,j} a_k - \tfrac{1}{2(n-1)}\left[g_{ij}\left(nF_k + H_k B + 2H a_{r,k}a^r\right) - g_{ik}\left(nF_j + H_j B + 2H a_{r,j}a^r\right)\right],$$

where $F_k = F_{,k}$, $H_k = H_{,k}$. Contracting (12) with g^{ij} we obtain

$$\frac{n-2}{2}F_k = \frac{-1}{2}H_k B + H_r a^r a_k + H a^r_{,r} a_k = 0, \tag{13}$$

whence, multiplying by a_i and alternating in (i,k), we find

$$(n-2)\left(F_k a_i - F_i a_k\right) = -B\left(H_k a_i - H_i a_k\right). \tag{14}$$

Moreover, transvecting (12) with a^i and applying (13) we get

$$B\left(H_k a_j - H_j a_k\right) = -H\left(a_j a_{r,k}a^r - a_k a_{r,j}a^r\right). \tag{15}$$

On the other hand, substituting (10) into the left hand side of (11), we have

$$\begin{aligned} &F_l g_{hk} + H_l a_h a_k + H a_{h,l} a_k + H a_h a_{k,l} \\ &= 2F\left(a_h g_{lk} + a_k g_{hl}\right) + 4H a_h a_k a_l + \tfrac{r}{n-1}\left(2g_{hk}a_l - g_{lk}a_h - g_{hl}a_k\right) \\ &\quad + 2n g_{hk} b_l + (n-2)\left(g_{lk}b_h + g_{hl}b_k\right), \end{aligned} \tag{16}$$

whence, by contraction with g^{hk},

$$nF_l + H_l B + 2H a_{r,l}a^r = \left[(2n+4)F + 6HB\right]a_l + 2(n+2)(n-1)b_l.$$

Multiplying by a_i and alternating in (i,l), in virtue of (15) and (14), we obtain

$$a_i F_l - a_l F_i = (n+2)\left(a_i b_l - a_l b_i\right). \tag{17}$$

Moreover, multiplying (16) by a_m alternating in (l,m) and applying (17), we obtain

$$(n-2)\left(b_l a_m - b_m a_l\right) g_{hk} = a_h U_{klm} + a_k U_{hlm} - b_h V_{klm} - b_k V_{hlm}$$

for some tensors U_{klm} and V_{klm}. Suppose that $b_l a_m - b_m a_l \neq 0$ at a point $x \in M$. Then we can choose at x two vectors v^l, w^l such that $(b_l a_m - b_m a_l) v^l w^m(x) \neq 0$. Transvecting the last equation with $v^l w^m$ we get

$$g_{hk} = a_h t_k + a_k t_h - b_h u_k - b_k u_h,$$

whence $rank\,[g_{hk}] \leq 4$ results, a contradiction. Therefore $b_l a_m - b_m a_l = 0$ and, in virtue of (17),

$$a_i F_l - a_l F_i = 0$$

holds. Hence $F = F(a)$, $b = b(a)$ follow, where a is a function such that $a_{,l} = a_l$.

Now we shall prove $H = H(a)$. Multiplying (16) by a_m and alternating in (h, m), by the use of $F_l = F' a_l$ and $b_l = b' a_l$, we get

$$\begin{aligned} &\left(F' - \tfrac{2r}{n-1} - 2nb'\right) a_l (a_m g_{hk} - a_h g_{mk}) + H a_k (a_{h,l} a_m - a_{m,l} a_h) \\ &\quad = \left(2F - \tfrac{r}{n-1} + (n-2)b'\right) a_k (a_m g_{hl} - a_h g_{ml}). \end{aligned} \tag{18}$$

Symmetrizing in (k, l), we find

$$\begin{aligned} &H(a_{h,k} a_m - a_{m,k} a_h) \\ &\quad = \left(2F - F' + \tfrac{r}{n-1} + (3n-2)b'\right) (a_m g_{hk} - a_h g_{mk}). \end{aligned} \tag{19}$$

Substituting (19) into (18) we easily find

$$\left(F' - \frac{2r}{n-1} - 2nb'\right) a_l = 0 \tag{20}$$

and, consequently,

$$H(a_{h,k} a_m - a_{m,k} a_h) = \left(2F - \frac{r}{n-1} + (n-2)b'\right) (a_m g_{hk} - a_h g_{mk}). \tag{21}$$

Now, multiplying (16) by a_m and alternating in (l, m) we obtain

$$\begin{aligned} &a_h a_k (H_l a_m - H_m a_l) + H a_h (a_m a_{k,l} - a_l a_{k,m}) + H a_k (a_m a_{h,l} - a_l a_{h,m}) \\ &= \left(2F - \tfrac{r}{n-1} + (n-2)b'\right) (a_h a_m g_{lk} + a_k a_m g_{hl} - a_h a_l g_{mk} - a_k a_l g_{hm}), \end{aligned}$$

which, by substituting (21), yields $a_k (H_l a_m - H_m a_l) = 0$. This proves $H = H(a)$.

Finally, since $r = nF + HB$, using (20), we obtain $B = B(a)$. This completes the proof. □

THEOREM 10. *(a) Let M, $dim M > 4$, be a conformally flat manifold whose curvature satisfies (4) but is not recurrent and the fundamental form a is locally a gradient. If $a(x) \neq 0$, $R(x) \neq 0$, $x \in M$, then there exists a coordinate* *neighbourhood* $(U,\ (x^j))$ *such that the metric of* M *takes the form (6) where* $p = p(x^1)$ *is a function in* x^1 *variable only such that*

$$E + p'(x)^2 \neq 0 \quad or \quad p''(x) \neq 0 \tag{22}$$

and

$$pp'' \neq p'^2 + E \tag{23}$$

(b) Let U *be an open subset of* R^n, $n > 4$, *endowed with a metric* g *of the form (6) such that (23) is satisfied. Then* (U, g) *is a non-recurrent conformally flat manifold satisfying (4).*

Proof. (a) In virtue of Theorem 7, Proposition 9 and Lemmas 5 and 6 we state that in some neighbourhood of x the manifold must be subprojective and the metric is of the form either (7) or (6). In the first case a straightforward computations show us that the curvature tensor is recurrent.

On the other hand, for the metric (6) the only components of the Christoffel symbols and the curvature tensor which may not vanish are

$$\Gamma^c_{ab} = \overline{\Gamma}^c_{ab}, \quad \Gamma^1_{ab} = -pp' f_{ab}, \quad \Gamma^c_{1b} = \tfrac{p'}{p} f_{ab},$$
$$R_{abcd} = p^2 \left(E + p'^2\right) f_{abcd}, \quad R_{1bc1} = pp'' f_{bc},$$

where $a, b, c, ... = 2, ..., n$, $E = [(n-1)(n-2)]^{-1}\, \overline{r}$, the dash denotes objects in the metric $ds_1^2 = f_{ab} dx^a dx^b$ and $f_{abcd} = f_{bc} f_{ad} - f_{bd} f_{ac}$.

Computing the components of the covariant derivative of R and making use of (4) we obtain pairs of equations:

$$\begin{aligned} R_{abcd,1} &= 2pp'\left(pp'' - p'^2 - E\right) f_{abcd}, \\ R_{abcd,1} &= 2a_1 R_{abcd} + 2b_1 G_{abcd}; \end{aligned} \tag{24}$$

$$\begin{aligned} R_{abcd,e} &= 0, \\ R_{abcd,e} &= 2a_e R_{abcd} + a_a R_{ebcd} + a_b R_{aecd} + a_c R_{abed} + a_d R_{abce} \\ &\quad + 2b_e G_{abcd} + b_a G_{ebcd} + b_b G_{aecd} + b_c G_{abed} + b_d G_{abce}; \end{aligned} \tag{25}$$

$$\begin{aligned} R_{1bcd,1} &= 0, \\ R_{1bcd,1} &= a_c R_{1b1d} + a_d R_{1bc1} + b_c G_{1b1d} + b_d G_{1bc1}; \end{aligned} \tag{26}$$

$$\begin{aligned} R_{1bcd,e} &= pp'\left(pp'' - p'^2 - E\right) f_{ebcd}, \\ R_{1bcd,e} &= a_1 R_{ebcd} + b_1 G_{ebcd}; \end{aligned} \tag{27}$$

$$\begin{aligned} R_{1bc1,1} &= \left(pp''' - p'p''\right) f_{bc}, \\ R_{1bc1,1} &= 4a_1 R_{1bc1} + 4b_1 G_{1bc1}; \end{aligned} \tag{28}$$

$$\begin{aligned} R_{1bc1,e} &= 0, \\ R_{1bc1,e} &= a_b R_{1ec1} + a_c R_{1be1} + b_b G_{1ec1} + b_c G_{1be1}; \end{aligned} \tag{29}$$

Pair (24) is equivalent to (27) and (26) to (29). Then (25) and (26) yield $b_e = \frac{p''}{p} a_e$ and $a_e \left(pp'' - p'^2 - E\right) = 0$. If $pp'' \neq p'^2 + E$, then $a_e = b_e = 0$ and the system (24)–(28) has a unique solution with respect to a_1, b_1. Otherwise we get $p'p'' \left(1 - p'^2\right) = 0$.

(b) Straightforward calculation. $\square$

Corollary 11. *It can be easily seen that a manifold M endowed with metric (6) is locally symmetric and non-flat if and only if $pp'' = p'^2 + E$ on M. Thus, if M is a subprojective and non-recurren manifold then the condition (4) holds on M and conversely.*

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