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THE CLASSIFICATION OF TILING SPACE FLOWS

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Abstract. We consider the conjugacy of the natural flows on one-dimensional tiling spaces presented as inverse limits. We also draw connections between geometric models and the spectral information for such flows.

1. Introduction. Our goal here is to present some of the results on classifying the flows on one-dimensional substitution tiling spaces in [**8**] from the perspective of inverse limits, to emphasize the features of those results that follow from this perspective, to extend some of those results to more general tiling spaces, and to demonstrate how to provide a geometric model of the tiling space when it has pure point spectrum.

If $\mathcal{P} = \{P_1, ..., P_n\}$ is a collection of intervals (prototiles), then a tiling T of $\mathbf{R}$ based on $\mathcal{P}$ is a collection of intervals (tiles) $\{T_i\}_{i\in\mathbf{Z}}$ satisfying:

1. Each T_i a translate of some $P_j \in \mathcal{P}$
2. $\cup_{i\in\mathbf{Z}} T_i = \mathbf{R}$
3. $T_i \cap T_{i+1}$ is a singleton for each i.

There is a metric on $T(\mathcal{P})$, the tilings of $\mathbf{R}$ based on $\mathcal{P}$, by which two tilings T and T' are close if there is a small $\varepsilon > 0$ so that the tiles of T and T' in a large neighborhood of 0 agree up to translation by some number $< \varepsilon$ [**1**]. Given any $T = \{T_i\}_{i\in\mathbf{Z}} \in T(\mathcal{P})$ and $t \in \mathbf{R}$, $T - t = \{T_i - t\}_{i\in\mathbf{Z}} \in T(\mathcal{P})$, and so there is the natural continuous flow

$$\phi : \mathbf{R} \times T(\mathcal{P}) \to T(\mathcal{P}); (t, T) \overset{\phi}{\mapsto} T - t$$

that moves the origin of a tiling t units forward along the tiling after t units of time. Given a particular $T \in T(\mathcal{P})$, the *tiling space* $\mathcal{T}$ of T is the closure of the $\phi-$ orbit of T. The restriction $\phi|_{\mathcal{T}}$ is then the *natural flow* on $\mathcal{T}$.

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We begin by considering tiling spaces presented as an inverse limit space

$$K_0 \overset{f_0}{\leftarrow} K_1 \overset{f_1}{\leftarrow} K_2 \cdots \ \varprojlim \{K_i; f_i\} = \mathcal{T},$$

where each K_i is a PL wedge of n of circles and where each f_i is a PL local isometry, represented by the integral matrix (using row multiplication) M_i: $\mathbf{Z}^n \to \mathbf{Z}^n$ giving the homomorphism $(f_i)_* : H_1(K_{i+1}) \to H_1(K_i)$. We construct the inverse limit representation of $\mathcal{T}$ to reflect the natural flow structure. If $T \in T(\mathcal{P} = \{P_1, ..., P_n\})$, then K_0 is the wedge of n circles $K_0^1, ..., K_0^n$ with the circumference of K_0^i the length of P_i. If $\rho_0(x, y)$ denotes the minimum length of any arc joining the two points $x, y \in K_0$, then as a metric for K_0 we use

$$d_0(x, y) = \min\{L_1, \ldots, L_n, \rho_0(x, y), 1\},$$

where L_j is the length of K_0^j. Then for all $i > 0$ the circumferences of the circles $K_i^1, ..., K_i^n$ are determined by requiring the PL bonding maps f_j to be local isometries. This in turn determines metrics ρ_i and d_i by analogy. We then define a metric d for $\mathcal{T}$ by:

$$d\left(\langle x_i\rangle_{i=0}^{\infty}, \langle y_i\rangle_{i=0}^{\infty}\right) = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} d_i(x_i, y_i).$$

Then if $p_i : \mathcal{T} \to K_i$ denotes the projection, the natural flow ϕ on $\mathcal{T}$ projects to a branched flow on K_i , which is well-defined and locally isometric except at the branch point. We orient the circles in K_i to coincide with the direction of the flow. Moreover, if $y = \phi(t, x)$, then $d_i(x_i, y_i) \le t$ for all i and so $d(x, y) \le t$. We shall examine when two natural flows ϕ and ψ are conjugate for homeomorphic tiling spaces $\mathcal{T}$ and $\mathcal{S}$ with different choices of tile lengths.

2. Sufficient conditions for Conjugacy. In this section we provide manageable conditions for conjugacy. We first treat the "substitution" case with $M_k \equiv M$ and $L = (L_1, ..., L_n)$, $S = (S_1, ..., S_n)$ the circumferences of the circles wedged to form K_0 and J_0 used in the construction of $\mathcal{T}$ and $\mathcal{S}$ respectively. Here we are assuming that the bonding maps f_i for $\mathcal{T}$ and g_i for $\mathcal{S}$ determine the same association of circles, and so only differ in the lengths of the circles in the domain and range spaces.

Theorem 1. *The natural flows on $\mathcal{T}$ and $\mathcal{S}$ are conjugate if there exists an integer k so that*

$$\lim_{i\to\infty}\left(LM^{i+k} - SM^i\right) = (0, \ldots, 0).$$

Proof. We first show that any two such flows meeting the condition for $k = 0$ are conjugate. To start, we construct a homeomorphism $h_0 : \mathcal{T} \to \mathcal{S}$ induced by the PL homeomorphism $\lambda_0^0 : K_0 \to J_0$ which maps K_0^j linearly and orientation preserving onto J_0^j, thereby determining a sequence of homeomorphisms $\lambda_0^j : K_j \to J_j$ making the following diagram and its vertical inverse commute

$$\begin{array}{ccccccc} K_0 & \overset{f_0}{\leftarrow} & K_1 & \overset{f_1}{\leftarrow} & K_2 & \cdots & \mathcal{T} \\ \downarrow \lambda_0^0 & & \downarrow \lambda_0^1 & & \downarrow \lambda_0^2 & & \downarrow h_0 \\ J_0 & \overset{g_0}{\leftarrow} & J_1 & \overset{g_1}{\leftarrow} & J_2 & \cdots & \mathcal{S} \end{array}.$$

For each $i = 1, 2, ...$ there is an analogous PL homeomorphism $\lambda_i^i : K_i \to J_i$ which maps each K_i^j linearly and orientation preserving onto J_i^j, but this homeomorphism does not lead to a complete diagram as before since there are no well-defined commuting vertical maps for $k < i$. However, as the homeomorphism type of an inverse limit is unchanged by dropping off any finite number of initial factors in the defining sequence, the commutative diagram

$$\begin{array}{ccccccc} K_i & \overset{f_i}{\leftarrow} & K_{i+1} & \overset{f_{i+1}}{\leftarrow} & K_{i+2} & \cdots & \mathcal{T} \\ \downarrow \lambda_i^i & & \downarrow \lambda_i^{i+1} & & \downarrow \lambda_i^{i+2} & & \downarrow h_i \\ J_i & \overset{g_i}{\leftarrow} & J_{i+1} & \overset{g_{i+1}}{\leftarrow} & J_{i+2} & \cdots & \mathcal{S} \end{array}$$

induces a homeomorphism $h_i : \mathcal{T} \to \mathcal{S}$. The homeomorphism h_i identifies the supertiles of order i. Moreover, each h_i induces the same correspondence of path components, moving points to varying places in the same flow orbit.

We now proceed to show that $\{h_i\}$ forms a Cauchy sequence of homeomorphisms in the space of homeomorphisms $\mathcal{T} \to \mathcal{S}$ in the sup metric D. The vectors LM^i and SM^i give the circumferences of the circles in K_i and J_i. Since we have constructed the metrics to locally correspond to length in K_i and J_i, we then see that the map λ_i^i distorts length and hence distance by at most μ_i, the maximum difference in the entries of LM^i and SM^i. Moreover, comparing the construction of h_i with that of h_{i+k}, we see that

$$D\left(h_i, h_{i+k}\right) \leq \sum_{\ell=0}^{i} \frac{\mu_i + \mu_{i+k}}{2^{\ell+1}} + \sum_{\ell=i+1}^{i} \frac{1}{2^{\ell+1}} < \mu_i + \mu_{i+k} + \frac{1}{2^i}.$$

Hence, $\{h_i\}$ is a Cauchy sequence of homeomorphisms. As $\mathcal{T}$ and $\mathcal{S}$ are compact metric spaces, the space of homeomorphisms $\mathcal{T} \to \mathcal{S}$ is complete relative to D, and so $\{h_i\}$ converges to a homeomorphism $h : \mathcal{T} \to \mathcal{S}$, which then conjugates the natural flows on $\mathcal{T}$ and $\mathcal{S}$ since the $\{h_i\}$ preserve time up to $\{\mu_i\} \to 0$ over supertiles of order i.

Now assume that the condition is met for some $k > 0$. The k^{th} iterate of the shift map of $\mathcal{T}$ conjugates the natural flow on $\mathcal{T}$ with the natural flow on

$$K_k \overset{f_k}{\leftarrow} K_{k+1} \overset{f_{k+1}}{\leftarrow} K_{k+2} \cdots \varprojlim \{K_i; f_i\}_{i \geq k} = \mathcal{T}',$$

where the circumferences of the circles in K_k are given by LM^k. Then the natural flows on $\mathcal{T}'$ and $\mathcal{S}$ are conjugate by the $k = 0$ case. The case $k < 0$ can be handled similarly to construct a conjugacy $\mathcal{S} \to \mathcal{T}$. □

In [**8**] a *substitution tiling space flow* is the special flow (suspension) under a function f of a substitution subshift on a finite alphabet, $\{a_1, ..., a_n\}^{\mathbf{Z}}$, where the function f depends only on the letter corresponding to $0 \in \mathbf{Z}$. The above result does not apply directly to all the substitution tiling spaces considered in [**8**], but as the following shows, the above result can be applied to this more general setting.

Corollary 1. *If $\mathcal{S}$ and $\mathcal{T}$ are one-dimensional substitution tiling spaces generated by the same substitution but with tile lengths given by L and S respectively, then the natural flows on $\mathcal{T}$ and $\mathcal{S}$ are conjugate if there exists an integer k so that*

$$\lim_{i\to\infty}\left(LM^{i+k} - SM^i\right) = (0, \dots, 0),$$

where the matrix M represents the substitution.

Proof. Let K_i be the supertiles in $\mathcal{T}$ of order i wedged at a single point, with lengths given by LM^i. Let $f_i : K_{i+1} \to K_i$ be the map determined by the substitution, essentially what is referred to in [**3**] as the *map of the rose* (only here we vary lengths in the K_i). The inverse limit

$$K_0 \overset{f_0}{\leftarrow} K_1 \overset{f_1}{\leftarrow} K_2 \quad \cdots \quad \mathcal{T}'$$

is not homeomorphic to $\mathcal{T}$ unless the original substitution is proper (forces the border), see [**1**], [**3**]. Consider, however, the natural mapping $p : \mathcal{T} \to \mathcal{T}'$, $p(T) = \langle p_i(T)\rangle_{i=0}^{\infty}$, where $p_i(T)$ assigns to the tiling T the position of the origin in T within its i^{th} order supertile, which is well defined by the results of [**12**], [**13**]. Then two tilings T, T' have the same p value only if the origins of T and T' are in the same position relative to all order supertiles, which can only happen if the flow orbits of T and T' are asymptotic in either the forward or backward time direction, as mentioned in [**3**]. It then follows that the mapping p respects the time structure of the flow and identifies at most finitely many asymptotic flow orbits. Similarly we construct $\mathcal{S}'$ and $q : \mathcal{S} \to \mathcal{S}'$. Under the stated condition, we can then construct a length preserving homeomorphism $h' : \mathcal{T}' \to \mathcal{S}'$ as before. What is more, h' associates the path components in $\mathcal{T}'$ and $\mathcal{S}'$ which are images of more than one orbit from the original spaces. Thus, the map h' lifts to a homeomorphism $h : \mathcal{T} \to \mathcal{S}$ which maps the orbits identified in $\mathcal{T}'$ and $\mathcal{S}'$ as determined by h'. Since p and q preserve length along orbits, it then follows that h is a conjugacy. □

Substitutions having a Pisot matrix representation have been well studied, see, e.g., [**2**]. Any matrix of Pisot type is diagonalizable and has a single eigenvalue of modulus greater than one. Thus, if M is an $n \times n$ matrix of Pisot type with dominant eigenvector $\mathbf{v}_1$, then the natural flows corresponding to $(L_1, L_2, ..., L_n) = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$ and $(S_1, S_2, ..., S_n) = b_1\mathbf{v}_1 + + \cdots + b_n\mathbf{v}_n$ are conjugate if $a_1 = b_1$, where $\{\mathbf{v}_i\}$ is a basis of left eigenvectors. In fact, all that is necessary to conclude conjugacy up to a linear rescaling is that the only eigenvalue of M that has modulus 1 or greater is the Perron eigenvalue. Hence, up to a time-scale factor any two such flows are conjugate, generalizing the result of [**14**] for the Fibonacci substitution.

If the substitution is an invertible substitution on two letters, then the inverse limit space is homeomorphic to a suspension of a Sturmian subshift for some quadratic irrational α [**2**]; in other contexts such a space is referred to as a Denjoy continua [**4**]. Any such Sturmian subshift has discrete spectrum, and since the matrix of such a substitution (being unimodular) will have one eigenvalue larger and one eigenvalue smaller than 1 in absolute value, the above results imply that the natural flow on any such tiling space has pure discrete spectrum.

We now treat the general (not necessarily substitution) case $(M_1, M_2, \dots)$ with corresponding bonding maps $(f_1, f_2, \dots)$.

THEOREM 2. *The natural flows on*

$$K_0 \overset{f_1}{\leftarrow} K_1 \overset{f_2}{\leftarrow} K_2 \quad \cdots \quad \mathcal{T} \;\sim\; L = (L_1, \dots, L_n)$$

and

$$J_0 \overset{g_1}{\leftarrow} J_1 \overset{g_2}{\leftarrow} J_2 \quad \cdots \quad \mathcal{S} \;\sim\; S = (S_1, \dots, S_n)$$

are conjugate if

$$\lim_{i \to \infty} (LM_1 \cdots M_i - SM_1 \cdots M_i) = (0, \dots, 0).$$

PROOF. Just as in the $k = 0$ case of Theorem 1, construct a Cauchy sequence of homeomorphisms $\{h_i\}$ converging to a conjugacy. □

The Denjoy continua topologically classified in [**4**] and [**10**] are examples of tiling spaces to which the above would apply.

3. Spectral Information and Geometric Models. A detailed treatment of the spectral analysis of the natural flows on one-dimensional tiling spaces is presented in [**8**]. The goal of this section is to indicate how these results may be understood from our current perspective and how these results can be used to construct geometric models of the tiling spaces in the sense of [**6**]. In general, determining the discrete spectrum of a flow allows one to determine a maximal semi-conjugate flow with pure discrete spectrum. In the case of substitution tiling space flows, the substitution homeomorphism can be modelled by the shift map on an inverse limit representation, [**1**].

We explore this connection by examining the natural flows on tiling spaces arising from substitutions of constant length on 2 letters. The spectral analysis on the associated subshifts was carried out in [**9**]. As we shall see, when the flow has pure point spectrum, not only is the flow measure theoretically conjugate to a natural flow on an n-adic solenoid, but the shift map is also measure theoretically conjugate to the shift map on the same n-adic solenoid.

Let σ be a substitution of constant length n on $\{a, b\}$: $|\sigma(a)| = |\sigma(b)| = n$:

$$\sigma \sim \begin{pmatrix} n_a & n_b \\ n - n_a & n - n_b \end{pmatrix},$$

where n_a is the number of $a's$ in $\sigma(a)$ and similarly for n_b. Then there is the substitution subshift $(\mathcal{S}, s)$ of $\left(\{a, b\}^{\mathbf{Z}}, s\right)$ associated to σ, see, e.g., [**2**]. Let $\mathcal{T} \sim (L_1, L_2)$ be the tiling space obtained from the special flow under function $f : \mathcal{S} \to (0, \infty)$ with

$$f\left(\langle x_i \rangle_{i \in \mathbf{Z}}\right) = \begin{cases} L_1, & \text{if } x_0 = a \\ L_2, & \text{if } x_0 = b \end{cases}.$$

Then as in Corollary 1 there is the natural length-preserving map $p : \mathcal{T} \to \mathcal{T}'$ onto the associated space

$$K_0 \overset{f_0}{\leftarrow} K_1 \overset{f_1}{\leftarrow} K_2 \quad \cdots \quad \mathcal{T}' \sim L$$

with the bonding maps f_i the maps of the rose and with the factor spaces K_i the wedge of 2 circles of lengths LM^i.

With S_i a circle of length $L_1 \cdot n^i$, let Σ be the n -adic solenoid

$$S_0 \overset{g_1}{\leftarrow} S_1 \overset{g_2}{\leftarrow} S_2 \quad \cdots \quad \Sigma$$

where g_i is a length-preserving $n-$fold covering. If $L_1 = L_2$, then there is the natural map $q : \mathcal{T}' \to \Sigma$ which is induced by the mappings $q_i : K_i \to S_i$ which "fold" the two circles in K_i onto the circle S_i. The commutativity of the related diagrams shows that the shift map on Σ is semi-conjugate to the substitution homeomorphism of the original tiling space $\mathcal{T}$. By the results of [**9**], $q \circ p$ is a measure-theoretic isomorphism which is one-to-one off of a set of measure 0 in $\mathcal{T}$.

Thus, Σ provides a model of both the flow on $\mathcal{T}$ and of the substitution homeomorphism on $\mathcal{T}$. Due to cohomological considerations, Σ cannot be embedded in a surface, but the shift on Σ can be realized up to conjugacy as the expanding attractor of a hyperbolic map on a solid (three-dimensional) torus domain. The flow on Σ can also be realized up to conjugacy as a minimal set of a flow on a solid torus domain, but this flow on Σ is uniformly Lyapunov stable; whereas, the tiling space flow on $\mathcal{T}$ is not Lyapunov stable. Thus, while Σ does provide a measure theoretic model of $\mathcal{T}$, it does not share all the significant dynamic properties.

It follows from the results of [**8**] (see also [**5**]) that when $L_1/L_2 \notin \mathbf{Q}$ the resulting tiling space flow on $\mathcal{T}$ is weakly mixing, and so there is no such projection onto a solenoid, or even any periodic flow. Hence, the choice of L makes a critical difference in the dynamics for this type of substitution.

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