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TRAPPING REGIONS AND AN ODE–TYPE PROOF OF THE EXISTENCE AND UNIQUENESS THEOREM FOR NAVIER–STOKES EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS ON THE PLANE

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Abstract. We present a new ODE–type method of passing to the limit with the dimension of Galerkin projection for dissipative PDEs. We apply this method to trapping regions derived by Mattingly and Sinai to give a new proof of the existence and uniqueness of solutions to Navier–Stokes equations with periodic boundary conditions on the plane.

1. Introduction

The goal of this paper is to present self-contained account of the ODE–type proofs from [**5, 9, 11**] of the existence and uniqueness of the Navier–Stokes systems with periodic boundary conditions on the plane. Mattingly and Sinai called their proof elementary (see title of [**9**]), but their proof was ODE–type (elementary in their sense) only up to the moment of getting the trapping regions for all Galerkin projections, but to pass to the limit with the dimensions of Galerkin projections they invoked the now standard results from [**1, 3, 13**] (which are not elementary – i.e. ODE–type). Here we fill in this gap by giving ODE–type arguments, which enable us to pass to the limit. Using ODE–type estimates based on the logarithmic norms we also obtained the uniqueness and an estimate for the Lipschitz constant of evolution induced by the Navier–Stokes equations. In fact we have proved that we have a continuous

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semidynamical system on the trapping region. The results we prove here are well known for Navier–Stokes system in 2D (see for example [**6, 5, 8, 4**]), but the method of getting estimates for Galerkin projections and the Lipschitz constant of the induced flow presented in section 5 is new.

Another goal of this paper is to prepare the ground for the rigorous study of the dynamics of the Navier–Stokes equations with periodic boundary conditions. The fact that we have here a semidynamical system on a compact set, and this system is approximated in a controlled way by finite-dimensional semidynamical systems is in our opinion of great importance, because it opens the possibility of applying finite-dimensional tools developed for the study of dynamics of ODEs.

The trapping regions described here for the Navier–Stokes equations are particular examples of the self-consistent a priori bounds introduced in [**14**] for the rigorous study of the dynamics of the dissipative PDEs, where Conley index type arguments where used to obtain the existence of multiple steady states for Kuramoto–Sivashinsky PDE (KS–equations). The tools developed in the present paper extend the ones given in [**14**]. For example they enable the Lipschitz constant of the flow induced by KS–equations to be computed effectively. This was already used to obtain proof of asymptotic stability of some steady states for the KS–equation in [**15**], the result which was previously known only on the numerical level.

A few words about a general construction of the paper: In sections 2 and 3 we recall the results from [**5, 9, 11**] about the existence of trapping regions for Navier–Stokes equations on the plane with periodic boundary conditions. Sections 4 and 5 contain ODE–type proofs of the convergence of the Galerkin scheme on trapping regions. The remaining sections contain the existence results for the Navier–Stokes equations on the plane and the Sannikov and Kaloshin [**11**] result in the dimension three.

2. Navier–Stokes equations

We will use the following notation. For $z \in \mathbb{C}$, by $\overline{z}$ we denote the conjugate of z. For any two vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ from $\mathbb{C}^n$ or $\mathbb{C}^\infty$ we set (if it makes sense)

$$\begin{aligned} (u|v) &= \sum_i u_i \overline{v_i}, \\ (u \cdot v) &= \sum_i u_i v_i. \end{aligned}$$

The general d–dimensional Navier–Stokes system (NSS) is written for d unknown functions $u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$ of d variables $x = (x_1, \ldots, x_d)$

and time t, and the pressure $p(t,x)$.

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{d} u_k\frac{\partial u_i}{\partial x_k} = \nu\triangle u_i-\frac{\partial p}{\partial x_i}+f^{(i)}, \tag{1}$$

$$\text{div } u = \sum_{i=1}^{d}\frac{\partial u_i}{\partial x_i}=0. \tag{2}$$

The functions $f^{(i)}$ are the components of the external forcing, $\nu>0$ is the viscosity.

We consider (1), (2) on the torus $\mathbb{T}^d=(\mathbb{R}/2\pi)^d$. This enables us to use Fourier series. We write

$$u(t,x)=\sum_{k\in\mathbb{Z}^d} u_k(t)e^{i(k,x)}, \qquad p(t,x)=\sum_{k\in\mathbb{Z}^d} p_k(t)e^{i(k,x)}. \tag{3}$$

Observe that $u_k(t)\in\mathbb{C}^d$, i.e. they are d–dimensional vectors and $p_k(t)\in\mathbb{C}$. We will always assume that $f_0=0$ and $u_0=0$.

Observe that (2) is reduced to the requirement $u_k\perp k$. Namely

$$\begin{aligned}\text{div } u &= \sum_{k\in\mathbb{Z}^d} i(u_k(t),k)e^{i(k,x)}=0,\\ (u_k,k) &= 0 \quad k\in\mathbb{Z}^d.\end{aligned}$$

To derive the evolution equation for $u_k(t)$ we will now compute the non-linear term in (1). We will use the following notation $u_k=(u_{k,1},\dots,u_{k,d})$

$$\begin{aligned}&\sum_l u_l\frac{\partial u}{\partial x_l} = \left(\sum_{k_1,l} u_{k_1,l}e^{i(k_1,x)}\right)\left(\sum_{k_2} ik_{2,l}u_{k_2}e^{i(k_2,x)}\right)\\ &= i\sum_{l,k_1,k_2} e^{i(k_1+k_2,x)}k_{2,l}\cdot u_{k_1,l}\cdot u_{k_2} = i\sum_{k_1,k_2} e^{i(k_1+k_2,x)}(u_{k_1}|k_2)u_{k_2}\\ &= i\sum_{k\in\mathbb{Z}^d}\left(\sum_{k_1}(u_{k_1}|k-k_1)u_{k-k_1}\right)e^{i(k,x)} = i\sum_{k\in\mathbb{Z}^d}\left(\sum_{k_1}(u_{k_1}|k)u_{k-k_1}\right)e^{i(k,x)}.\end{aligned}$$

We obtain the following infinite ladder of differential equations for u_k

$$\frac{du_k}{dt}=-i\sum_{k_1}(u_{k_1}|k)u_{k-k_1}-\nu k^2u_k-ip_kk+f_k. \tag{4}$$

Here f_k are components of the external forcing. Let $\sqcap_k$ denote the operator of orthogonal projection onto the $(d-1)$–dimensional plane orthogonal to k.

Observe that since $(u_k, k) = 0$, we have $\sqcap_k u_k = u_k$. We apply the projection $\sqcap_k$ to (4). The term $p_k k$ disappears and we obtain

$$\frac{du_k}{dt} = -i \sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1} - \nu k^2 u_k + \sqcap_k f_k. \tag{5}$$

The pressure is given by the following formula

$$-i \sum_{k_1} (u_{k_1}|k)(I - \sqcap_k) u_{k-k_1} - i p_k k + (I - \sqcap_k) f_k = 0. \tag{6}$$

Observe that solutions of (5) satisfy incompressibility condition $(u_k, k) = 0$. The subspace of real functions which can be defined by $\overline{u_{-k}} = u_k$ for all $k \in \mathbb{Z}^d$ is invariant under (5). In the sequel, we will investigate the equation (5) restricted to this subspace.

Definition 1. Energy of $\{u_k, \quad k \in \mathbb{Z}^d\}$ is

$$E(\{u_k, \quad k \in \mathbb{Z}^d\}) = \sum_{k \in \mathbb{Z}^d} |u_k|^2.$$

Definition 2. Enstrophy of $\{u_k, \quad k \in \mathbb{Z}^d\}$ is

$$V(\{u_k, \quad k \in \mathbb{Z}^d\}) = \sum_{k \in \mathbb{Z}^d} |k|^2 |u_k|^2.$$

3. Construction of trapping regions from [**5, 9**]

The idea in [**5, 9**] is to construct a trapping region for each Galerkin projection and this trapping region give uniform bounds enabling passing to the limit. The *trapping region* for an ODE (here the Galerkin projection of Navier–Stokes equations) is a set such that the vector field on its boundary is pointing inside, hence no trajectory can leave it in forward time. In the sequel we consider only the Galerkin projection onto the set of modes O, such that if $k \in O$ then $-k \in O$. We will call such projections *symmetric*. This restriction comes from the observation made in Section 2 that for Galerkin projection on such O, the space of real function is invariant under (5).

Lemma 1. $d = 2$. *For any solution of (5) (such that all necessary Fourier series converge) or the symmetric Galerkin projection of (5) we have*

$$\frac{dV\{u_k(t)\}}{dt} \leq -2\nu V(\{u_k(t)\}) + 2V(F)\sqrt{V(\{u_k(t)\})}, \tag{7}$$

where $V(F) = \sqrt{\sum |k|^2 f_k^2}$.

The proof can be found in many text-books, see also [**12**].

Inequality (7) shows that

$$\frac{dV\{u_k(t)\}}{dt} < 0, \qquad \text{when} \qquad V > V^* = \left(\frac{V(F)}{\nu}\right)^2. \tag{8}$$

LEMMA 2. *Assume that* $\{u_k, k \in \mathbb{Z}^d\}$ *is such that for some* $D < \infty$, $\gamma > 1 + \frac{d}{2}$

$$|u_k| \leq \frac{D}{|k|^\gamma}, \quad \textit{and} \quad V(\{u_k\}) \leq V_0. \tag{9}$$

Then for $d \geq 3$

$$\left|\sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1}\right| \leq \frac{C\sqrt{V_0}D}{|k|^{\gamma - \frac{d}{2}}}, \tag{10}$$

where the constant C *depends only on* γ *and dimension* d,

for $d = 2$ *for any* $\epsilon > 0$

$$\left|\sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1}\right| \leq \frac{C(\epsilon,\gamma)\sqrt{V_0}D}{|k|^{\gamma - \frac{d}{2} - \epsilon}}, \tag{11}$$

PROOF. In order to estimate the sum $|\sum_{k_1} (u_{k_1}|k) \sqcap_k u_{k-k_1}|$ we will use the following inequality

$$|(u_{k_1}|k) \sqcap_k u_{k-k_1}| = |(u_{k_1}|k - k_1) \sqcap_k u_{k-k_1}| \leq |u_{k_1}|\ |k - k_1|\ |u_{k-k_1}| \tag{12}$$

We consider three cases.

Case I. $|k_1| \leq \frac{1}{2}|k|$.

Here $|k - k_1| \geq \frac{1}{2}|k|$ and therefore $|u_{k-k_1}|\ |k - k_1| \leq \frac{D}{|k-k_1|^{\gamma-1}} \leq \frac{2^{\gamma-1}D}{|k|^{\gamma-1}}$. Now observe that

$$\sum_{|k_1| \leq \frac{1}{2}|k|} |u_{k_1}| = \sum_{|k_1| \leq \frac{1}{2}|k|} |k_1|\ |u_{k_1}| \frac{1}{|k_1|} \leq \sqrt{\sum |k_1|^2 |u_{k_1}|^2} \cdot \sqrt{\sum_{|k_1| < \frac{1}{2}|k|} \frac{1}{|k_1|^2}} \tag{13}$$

The sum $\sum_{|k_1| < \frac{1}{2}|k|} \frac{1}{|k_1|^2}$ can be estimated from above by a constant times an integral of $\frac{1}{r^2}$ over the ball of radius $\frac{1}{2}|k|$ with the ball around the origin removed. Hence for $d = 2$ we have

$$\sum_{|k_1| \leq \frac{1}{2}|k|} \frac{1}{|k_1|^2} \leq C \int_1^{|k|/2} \frac{rdr}{r^2} \leq C \ln |k|. \tag{14}$$

For $d \geq 3$ there is

$$\sum_{|k_1| \leq \frac{1}{2}|k|} \frac{1}{|k_1|^2} \leq C \int_1^{|k|/2} \frac{r^{d-1} dr}{r^2} \leq C|k|^{d-2}. \tag{15}$$

From all the above computations it follows that for $d \geq 3$ holds

$$\left| \sum_{|k_1| \leq \frac{|k|}{2}} (u_{k_1}|k) \sqcap_k u_{k-k_1} \right| \leq \frac{2^{\gamma-1} D}{|k|^{\gamma-1}} \sqrt{V_0} \sqrt{C} |k|^{\frac{d}{2}-1} = \frac{2^{\gamma-1} D \sqrt{V_0} \sqrt{C}}{|k|^{\gamma-\frac{d}{2}}}. \tag{16}$$

For $d = 2$ there is

$$\left| \sum_{|k_1| \leq \frac{|k|}{2}} (u_{k_1}|k) \sqcap_k u_{k-k_1} \right| \leq \frac{2^{\gamma-1} D}{|k|^{\gamma-1}} \sqrt{V_0} \sqrt{C} \sqrt{\ln |k|} < \frac{C \sqrt{V_0} D}{|k|^{\gamma-1-\epsilon}}. \tag{17}$$

Case II. $\frac{1}{2}|k| < |k_1| \leq 2|k|$.

$$|u_{k_1}| < \frac{D}{|k_1^\gamma|} < \frac{D}{\left(\frac{|k|}{2}\right)^\gamma} = \frac{2^\gamma D}{|k|^\gamma}. \tag{18}$$

Hence

$$\sum_{\frac{1}{2}|k| < |k_1| \leq 2|k|} |u_{k_1}| \cdot |u_{k-k_1}| \cdot |k - k_1| \leq \frac{2^\gamma D}{|k|^\gamma} \sum_{\frac{1}{2}|k| < |k_1| \leq 2|k|} |u_{k-k_1}| \cdot |k - k_1|. \tag{19}$$

We interpret $\sum_{\frac{1}{2}|k|<|k_1|\leq 2|k|} |u_{k-k_1}| \cdot |k-k_1|$ as a scalar product of $|u_{k-k_1}| \cdot |k-k_1|$ and 1, hence, by the Schwarz inequality,

$$\sum_{\frac{1}{2}|k| < |k_1| \leq 2|k|} |u_{k-k_1}| \cdot |k - k_1| \leq \sqrt{\sum_{|k_1| \leq 3|k|} |u_{k_1}|^2 |k_1|^2} \cdot \sqrt{C(3|k|)^d}, \tag{20}$$

where C is such that $C(3|k|)^d$ is greater than or equal to the number of such vectors in $\mathbb{Z}^d$ which are contained in the ball of radius $3|k|$ around the origin.

Finally we obtain

$$\sum_{\frac{1}{2}|k| < |k_1| \leq 2|k|} |u_{k_1}| \cdot |u_{k-k_1}| \cdot |k - k_1| \leq \frac{2^\gamma D \tilde{C} \sqrt{V_0}}{|k|^{\gamma - \frac{d}{2}}}. \tag{21}$$

Case III. $|k_1| > 2|k|$. Here $|k - k_1| > |k|$.

$$\sum |u_{k_1}||k-k_1||u_{k-k_1}| \le \frac{1}{|k|}\sum |u_{k_1}||k_1||k-k_1||u_{k-k_1}|$$
$$\le \frac{1}{|k|}\sqrt{\sum |u_{k_1}|^2|k_1|^2}\sqrt{\sum |u_{k-k_1}|^2|k-k_1|^2}$$
$$\le \frac{\sqrt{V_0}}{|k|}\sqrt{\sum_{|k_1|>2|k|}\frac{D^2}{|k_1|^{2\gamma-2}}} = \frac{\sqrt{V_0}D}{|k|}\sqrt{\sum_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}}}.$$

To estimate $\sum_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}}$ observe that there is (we denote all constant factors depending on γ by C)

$$\begin{aligned}\sum_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}} &\le C\int_{|k_1|>2|k|}\frac{1}{|k_1|^{2\gamma-2}}d^dk_1 = C\int_{2|k|}^{\infty}\frac{1}{r^{2\gamma-2}}r^{d-1}dr\\ &= C\int_{2|k|}^{\infty} r^{-(2\gamma-2-d+1)} = C|k|^{-(2\gamma-2-d)}.\end{aligned}$$

Observe that we used here the assumption $\gamma > 1 + \frac{d}{2}$, which guarantees that $2\gamma - 2 - d + 1 > 1$, thus the integral converges.

Hence for the case III we obtain

$$\left|\sum_{|k_1|>2|k|}(u_{k_1}|k)\sqcap_k u_{k-k_1}\right| \le \frac{\sqrt{V_0}DC}{|k|^{\gamma-\frac{d}{2}}}. \tag{22}$$

Adding cases I,II,III we obtain for $d \ge 3$

$$\left|\sum_{k_1}(u_{k_1}|k)\sqcap_k u_{k-k_1}\right| \le \frac{C\sqrt{V_0}D}{|k|^{\gamma-\frac{d}{2}}}. \tag{23}$$

For $d = 2$ we obtain

$$\left|\sum_{k_1}(u_{k_1}|k)\sqcap_k u_{k-k_1}\right| \le \frac{C\sqrt{V_0}D}{|k|^{\gamma-\frac{d}{2}-\epsilon}}. \tag{24}$$

□

Lemma 3. *Assume that $\gamma > d$. Then*

$$\sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}}\frac{1}{|k_1|^{\gamma}|k-k_1|^{\gamma}} \le \frac{C_Q(d,\gamma)}{|k|^{\gamma}}. \tag{25}$$

Proof. We consider three cases.
Case I. $|k_1| < \frac{|k|}{2}$, hence $|k-k_1| \geq \frac{|k|}{2}$.
There is

$$\sum_{|k_1|<\frac{|k|}{2}} \leq \sum_{|k_1|<\frac{|k|}{2}} \frac{1}{|k_1|^\gamma}\frac{2^\gamma}{|k|^\gamma} < \frac{2^\gamma}{|k|^\gamma} C \int_1^\infty \frac{r^{d-1}}{r^\gamma} dr.$$

The improper integral $\int_1^\infty \frac{r^{d-1}}{r^\gamma} dr$ converges, because $\gamma > d$. Hence

$$\sum_{|k_1|<\frac{|k|}{2}} < \frac{C_I(d,\gamma)}{|k|^\gamma}.$$

Case II. $\frac{|k|}{2} < |k_1| \leq 2|k|$.

$$\begin{aligned}\sum_{\frac{|k|}{2}<|k_1|\leq 2|k|} &\leq \frac{2^\gamma}{|k|^\gamma} \sum_{\frac{|k|}{2}<|k_1|\leq 2|k|} \frac{1}{|k-k_1|^\gamma} \\ &< \frac{2^\gamma}{|k|^\gamma} \sum_{|k_1|\leq 3|k|} \frac{1}{|k_1|^\gamma} < \frac{2^\gamma}{|k|^\gamma} C \int_1^\infty \frac{r^{d-1}}{r^\gamma} dr.\end{aligned}$$

Hence

$$\sum_{\frac{|k|}{2}<|k_1|\leq 2|k|} < \frac{C_{II}(d,\gamma)}{|k|^\gamma}.$$

Case III. $2|k| < |k_1|$, hence $|k-k_1| > |k|$.

$$\sum_{2|k|<|k_1|} < \frac{1}{|k|^\gamma} \sum \frac{1}{|k_1|^\gamma} < \frac{C_{III}(d,\gamma)}{|k|^\gamma}.$$

□

3.1. The construction of the trapping region I. We take $V_0 > V^*$, $\gamma \geq 2.5$ and K such that $f_k = 0$ for $|k| > K$. We set

$$N(V_0, K, \gamma, D) = \left\{ \{u_k\} \mid V(\{u_k\}) \leq V_0, \qquad |u_k| \leq \frac{D}{|k|^\gamma}, \quad |k| > K \right\} \tag{26}$$

We prove the following theorem.

Theorem 4. *Let $d = 2$ and $C = C(\epsilon = \frac{1}{2}, \gamma)$ be a constant from Lemma 2. If $K > \frac{C^2 V_0}{\nu^2}$ and $D > \sqrt{V_0} K^{\gamma-1}$, then $N = N(V_0, K, \gamma, D)$ is a trapping region for each Galerkin projection.*

Proof. Observe that for $D \geq \sqrt{V_0}K^{\gamma-1}$ for all $\{u_k\} \in N$ there holds

$$|u_k| \leq \frac{D}{|k|^\gamma}. \tag{27}$$

To prove this observe that (27) holds for $|k| > K$ by the definition of N. For $|k| \leq K$ we proceed as follows: since $V(\{u_k\}) \leq V_0$ then $|k|^2|u_k|^2 \leq V_0$. So we have

$$|u_k| \leq \frac{\sqrt{V_0}}{|k|} \leq \frac{D}{|k|^\gamma}, \quad |k| \leq K \tag{28}$$

for D such that $\sqrt{V_0}|k|^{\gamma-1} \leq D$ for all $|k| \leq K$.

We will now show that on the boundary of N (we are considering the Galerkin projection) the vector field is pointing inside. For points $V(\{u_k\}) = V_0$ it follows from (8). For points such that $u_k = \frac{D}{|k|^\gamma}$ for some $|k| > K$ from Lemma 2 (with $\epsilon = 1/2$) we have

$$\frac{d|u_k|}{dt} \leq \frac{C\sqrt{V_0}D}{|k|^{\gamma-\frac{3}{2}}} - \nu|k|^2\frac{D}{|k|^\gamma} < 0, \tag{29}$$

which is satisfied when

$$C\sqrt{V_0} < \nu|k|^{1/2}. \tag{30}$$

Observe that (30) holds for $|k| \geq K$ if $K > \frac{C^2V_0}{\nu^2}$. □

Remark 1. Observe that it was of crucial importance in the proof that the constant D entered linearly in the estimate in Lemma 2 and, due to this fact, it did not appear in (30). For example assume that the estimate of the nonlinear part will be of the form $\frac{D^2C}{|k|^{\gamma-\frac{3}{2}}}$; then instead of (30) there would be

$$CD < \nu|k|^{1/2}$$

which will require that $K > \frac{C^2D^2}{\nu^2}$ which might be incompatible with $D > \sqrt{V_0}K^{\gamma-1}$.

This shows how important it was to use the enstrophy in these estimates.

3.2. The construction of the trapping region II – exponential decay.

Theorem 5. *Assume that $\gamma \geq 2.5$, $d = 2$. Then the set*

$$N_e = N(V_0, K, \gamma, D) \cap \left\{\{u_k\} \mid |u_k| \leq \frac{D_2}{|k|^\gamma}e^{-a|k|} \text{ for } |k| > K_e\right\}, \tag{31}$$

where $N(V_0, K, \gamma, D)$ is a trapping region from Theorem 4, $D_2 > D$, $K_e > \frac{C_Q(d,\gamma)D_2}{\nu}$ (C_Q was obtained in Lemma 3) and $0 < a < \frac{1}{K_e}\ln\frac{D_2}{D}$ is a trapping region for each symmetric Galerkin projection.

PROOF. The set N_e constructed so that for all $|k| \le K_e$ the trapping (the vector field is pointing toward the interior of N_e on the boundary) is obtained from $N(V_0, K, \gamma, D)$ and for $|k| > K_e$ it results from the new exponential estimates.

Observe that a is such that $\frac{D_2}{|k|^\gamma} e^{-a|k|} > \frac{D}{|k|^\gamma}$ for all $|k| \le K_e$. This solves the trapping for $|k| \le K_e$.

Hence to prove the trapping it is enough to consider the boundary points such that $|u_k| = \frac{D_2}{|k|^\gamma} e^{-a|k|}$ for some $k > K_e$. For such a point and $|k|$ there is

$$\frac{d|u_k|}{dt} \le \left| \sum (u_{k_1}|k) \sqcap_k u_{k-k_1} \right| - \nu |k|^2 |u_k|$$
$$\le \sum |u_{k-1}||k||u_{|k-k_1|}| - \nu|k|^2|u_k| \le D_2^2 |k| \sum \frac{e^{-a|k_1|} e^{-a|k-k_1|}}{|k_1|^\gamma |k-k_1|^\gamma} - \nu |k|^2 |u_k|.$$

Observe that $e^{-a|k_1|}e^{-a|k-k_1|} \le e^{-a|k|}$. From this and Lemma 3 we obtain

$$\frac{d|u_k|}{dt} < \frac{D_2^2 C_Q(\gamma, d)}{|k|^{\gamma-1}} e^{-a|k|} - \nu |k|^2 |u_k|.$$

Hence $\frac{d|u_k|}{dt} < 0$, when

$$|u_k| = \frac{D_2}{|k|^\gamma} e^{-a|k|} > \frac{C_Q D_2^2}{\nu |k|^{\gamma+1}} e^{-a|k|},$$

which is equivalent to

$$|k| > K_e = \frac{C_Q D_2}{\nu}.$$

□

3.3. Trapping region III – exponential decay in time.

THEOREM 6. *Let $t_0 > 0$. Assume that $\gamma \ge 2.5$, $d = 2$. Then the set*

$$N_e = N(V_0, K, \gamma, D) \cap \left\{ \{u_k\} \mid |u_k| \le \frac{D_3}{|k|^\gamma} e^{-a_3 |k| t} \text{ for } |k| > K_e \right\}, \tag{32}$$

where $N(V_0, K, \gamma, D)$, is a trapping region from Theorem 4, $D_3 > D$, $K_e > \frac{D_3 C_Q(d,\gamma)}{\nu}$ (C_Q was obtained in Lemma 3) and $0 < a_3 < \frac{1}{K_e t_0} \ln \frac{D_3}{D}$ is a trapping region for each symmetric Galerkin projection for $0 \le t \le t_0$.

PROOF. The set N_e is constructed so that for all $|k| \le K_e$ the trapping property is obtained from $N(V_0, K, \gamma, D)$ and for $|k| > K_e$ it results from the new exponential estimates.

To be sure that the boundary of N_e for $|k| < K_e$ is obtained from $N(V_0, K, \gamma, D)$, we require that

$$\frac{D}{|k|^\gamma} < \frac{D_3}{|k|^\gamma} e^{-a_3|k|t}, \quad \text{for } 0 \le t \le t_0 \text{ and } |k| \le K_e. \tag{33}$$

Easy computations show that (33) holds iff $a_3 < \frac{1}{K_e t_0} \ln \frac{D_3}{D}$.

To obtain the trapping property for $|k| > K_e$ we need to show that $\frac{d|u_k|}{dt} < 0$ if $|u_k| = \frac{D_3}{|k|^\gamma} e^{-a_3 t}$, for some $0 \leq t \leq t_0$ and $|k| > K_e$).

$$
\begin{aligned}
\frac{d|u_k|}{dt} &\leq \sum |u_{k_1}||k||u_{k-k_1}| - \nu |k|^2 |u_k| \\
&\leq |k| D_3^2 \sum \frac{e^{-a_3|k_1|t} e^{-a_3|k-k_1|t}}{|k_1|^\gamma |k-k_1|^\gamma} - \nu |k|^2 |u_k| \\
&\leq |k| e^{-a_3|k|t} D_3^2 \sum \frac{1}{|k_1|^\gamma |k-k_1|^\gamma} - \nu |k|^2 |u_k| \\
&\leq \frac{e^{-a_3|k|t} D_3^2 C_Q(d,\gamma)}{|k|^{\gamma-1}} - \nu |k|^2 |u_k|
\end{aligned}
$$

Hence $\frac{d|u_k|}{dt} < 0$ if

$$
\frac{D_3^2 C_Q(d,\gamma)}{\nu |k|^{\gamma+1}} e^{-a_3|k|t} < |u_k| = \frac{D_3}{|k|^\gamma} e^{-a_3|k|t}, \tag{34}
$$

which is equivalent to

$$
\frac{D_3 C_Q}{\nu} < |k|. \tag{35}
$$

Hence for $K_e \geq \frac{D_3 C_Q}{\nu}$ we obtain the trapping. □

4. Passing to the limit for Galerkin projections via the Ascoli–Arzela Lemma

The goal of this section is to present a relatively simple argument for the passing to the limit with Galerkin projections. The argument given in this section does not give any control of how the Galerkin projections converge and we cannot obtain the uniqueness using it. In section 5 we will introduce some new assumptions (which are easily satisfied for NS in 2D) which will give us much better control of the limit process.

All what follows in this section was essentially proved in [**14**]. We will also use some conventions used there.

Let H be a Hilbert space. Let $e_1, e_2, \ldots$ be an orthonormal basis in H.

Let $A_n : H \to H$ denote the projection onto 1–dimensional subspace $\langle e_n \rangle$, i.e., $x = \sum A_n(x) e_n$ for all $x \in H$. By V_n we will denote the space spanned by $\{e_1, \ldots, e_n\}$. Let P_n denote the projection onto V_n and $Q_n = I - P_n$.

Definition 3. Let $W \subset H$, $F : \mathrm{dom}(F) \to H$ and W be closed. We say that W and F satisfy conditions C1,C2,C3 if

C1 There exists $M \geq 0$ such that $P_n(W) \subset W$ for $n \geq M$
C2 Let $\hat{u_k} = \max_{x \in W} |A_k x|$. Then $\hat{u} = \sum \hat{u_k} e_k \in H$. In particular, $|\hat{u}| < \infty$.
C3 The function $x \mapsto F(x)$ is continuous on W and $f = \sum_k f_k e_k$, given by $f_k = \max_{x \in W} |A_k F(x)|$ is in H. In particular, $|f| < \infty$.

Observe that condition C2 implies that the set W is compact. Conditions C2 and C3 guarantee good behavior of F with respect to passing to the limit. For example, $F \circ P_n$ converges uniformly to F on W. We here have a continuous function on the compact set, which is a perfect setting for a study of the dynamics of $x' = F(x)$ (see [**14**] for more details).

LEMMA 7. *Assume that $W \subset H$ and F satisfy C1,C2,C3. Let $x : [0, T] \to W$ be such that for each n*

$$\frac{dA_n x}{dt} = A_n(F(x)). \tag{36}$$

Then

$$x' = F(x). \tag{37}$$

PROOF. Let us set $x_k = A_k x$. Let us fix $\epsilon > 0$ and $t \in [0, T]$. For any n there is

$$\begin{aligned} \left| \frac{x(t+h) - x(t)}{h} - F(x) \right| &\leq \left| \frac{P_n x(t+h) - P_n x(t)}{h} - P_n F(x) \right| \\ &+ \left| \frac{1}{h} \sum_{k=n+1}^{\infty} (x_k(t+h) - x_k(t)) e_k \right| + |Q_n F(x)| \end{aligned} \tag{38}$$

We will estimate the three terms on the right hand side separately. From **C3** for a given $\epsilon > 0$ it follows that there exists n_0 such that $n > n_0$ implies

$$|Q_n(F(x))| < \epsilon/3.$$

From now on we fix $n > n_0$. Condition **C3** and the mean value theorem imply

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} \frac{1}{h} (x_k(t+h) - x_k(t)) e_k \right| &= \left| \sum_{k=n+1}^{\infty} \frac{dx_k}{dt} (t + \theta_k h) e_k \right| \\ &\leq \left| \sum_{k=n+1}^{\infty} f_k e_k \right| < \epsilon/3. \end{aligned}$$

Finally, for h sufficiently small,

$$\left| \frac{1}{h} (P_n x(t+h) - P_n x(t)) - P_n F(x) \right| < \epsilon/3$$

and hence the desired limit is obtained. □

Lemma 8. *Assume that $W \subset H$ and the function F satisfy C1,C2,C3. Let $x_0 \in W$. Assume that for each n a function $x_n : [0,T] \to P_n(W)$ is a solution of the problem (Galerkin projection of $x' = F(x)$)*

$$x_n' = P_n(F(x)), \quad x_n(0) = P_n(x_0). \tag{39}$$

Assume also that x_n converges uniformly to $x^* : [0,T] \to W$.

Then x^* solves the following initial value problem

$$x' = F(x), \quad x(0) = x_0. \tag{40}$$

Proof. We first show that for all n and $t \in [0,T]$ holds

$$P_n x^*(t) = P_n x_0 + \int_0^t P_n F(x^*(s)) ds. \tag{41}$$

Let us fix n. Observe that for each $m \geq n$ the following equality holds

$$P_n x_m(t) = P_n x_0 + \int_0^t P_n F(x_m(s)) ds. \tag{42}$$

Since the series x_m converges uniformly to x^*, then also $P_n x_m$ converges uniformly to $P_n x^*$. Observe that also the functions $P_n F(x_m)$ converge uniformly to $P_n F(x^*)$ as the composition of the uniformly continuous function $P_n F$ (because F is a continuous function on the compact set W) with a uniformly convergent sequence, hence also the integral in (42) converges (uniformly in $t \in [0,T]$) to $\int_0^t P_n F(x^*(s))$. This proves (41). Differentiation of (41) gives

$$\frac{dP_n x^*}{dt} = P_n F(x^*). \tag{43}$$

The assertion follows from Lemma 7. □

Theorem 9. *Assume that $W \subset H$ and the function F satisfy C1,C2,C3. Let $x_0 \in W$. Assume that for each n a function $x_n : [0,T] \to P_n(W)$ is a solution of the problem (Galerkin projection of $x' = F(x)$)*

$$x_n' = P_n(F(x)), \quad x_n(0) = P_n(x_0). \tag{44}$$

Then there exists $x^* : [0,T] \to W$, such that x^* solves the following initial value problem

$$x' = F(x), \quad x(0) = x_0. \tag{45}$$

Proof. The idea goes as follows. First we try to pick up a convergent subsequence from $\{x_n\}$ using the Ascoli–Arzela compactness Lemma. Then we show that the limit function x^* solves (45).

Observe first that, due to the compactness of W and since $x_n(t) \in W$ for $t \in [0,T]$, the sequence $\{x_n\}$ is contained in a compact set. Observe that the derivatives $x_n'(t)$ are uniformly bounded by $|F(W)|$, hence the sequence of functions x_n is equicontinuous. From the Ascoli–Arzela Lemma it follows that there exists a subsequence converging uniformly to $x^* : [0,T] \to W$. Without loss of generality we can assume that the whole sequence x_n converges uniformly to x^*. It is obvious that $x^*(0) = x_0$. The assertion of the theorem follows from Lemma 8. □

5. Passing to the limit, an analytic argument

The goal of this section is to present another argument for the existence of the limit of Galerkin projections. Compared with Section 4, we assume more about the function F and we add a new condition D on the trapping regions; these new conditions are satisfied for the Navier–Stokes system and the trapping regions constructed in section 3. We obtain better results on the convergence plus the uniqueness and the Lipschitz constant for the induced flow.

We will here use the notations introduced in Section 4. We investigate the Galerkin projections of the following problem

$$x' = F(x) = L(x) + N(x), \tag{46}$$

where L is a linear operator and N is a nonlinear part of F. We assume that the basis $e_1, e_2, \dots$ of H is built of eigenvectors of L. We assume that the corresponding eigenvalues λ_k (i.e. $Le_k = \lambda_k e_k$) can be ordered in such a way that

$$\lambda_1 \geq \lambda_2 \geq \dots, \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = -\infty.$$

Hence we can have only a finite number of positive eigenvalues.

5.1. Estimates based on logarithmic norms.

The goal of this subsection is to recall some results about one-sided Lipschitz constants of the flows induced by ODEs.

Definition 4. [**7**, Def. I.10.4] Let Q be a square matrix; we call

$$\mu(Q) = \lim_{h>0, h\to 0} \frac{\|I + hQ\| - 1}{h}$$

the *logarithmic norm* of Q.

Theorem 10. [**7**, Th. I.10.5] *The logarithmic norm is obtained by the following formulas*

- *for Euclidean norm*

$$\mu(Q) = \textit{the largest eigenvalue of} \quad 1/2(Q + Q^T).$$

- *for max norm* $\|x\|_\infty = \max_k |x_k|$

$$\mu(Q) = \max_k \left(q_{kk} + \sum_{i \neq k} |q_{ki}| \right)$$

- *for norm* $\|x\|_1 = \sum_k |x_k|$

$$\mu(Q) = \max_i \left(q_{ii} + \sum_{k \neq i} |q_{ki}| \right)$$

Consider now the differential equation

$$x' = f(x), \quad f \in C^1. \tag{47}$$

Let $\varphi(t, x_0)$ denote the solution of equation (47) with the initial condition $x(0) = x_0$. By $\|x\|$ we denote a fixed arbitrary norm in $\mathbb{R}^n$.

The following theorem was proved in [**7**, Th. I.10.6] (for a non-autonoumous ODE, here we restrict ourselves to the autonomous case only and we use a different notation).

THEOREM 11. *Let* $y : [0, T] \to \mathbb{R}^n$ *be a piecewise* C^1 *function and* $\varphi(\cdot, x_0)$ *be defined for* $t \in [0, T]$. *Suppose that the following estimates hold:*

$$\mu\left(\frac{\partial f}{\partial x}(\eta)\right) \leq l(t), \quad \text{for } \eta \in [y(t), \varphi(t, x_0)],$$
$$\left\|\frac{dy}{dt}(t) - f(y(t))\right\| \leq \delta(t).$$

Then for $0 \leq t \leq T$ *there is*

$$\|\varphi(t, x_0) - y(t)\| \leq e^{L(t)} \left(\|y(0) - x_0\| + \int_0^t e^{-L(s)} \delta(s) ds \right),$$

where $L(t) = \int_0^t l(s) ds$.

From the above theorem one easily derives the following.

LEMMA 12. *Let* $y : [0, T] \to \mathbb{R}^n$ *be a piecewise* C^1 *function and* $\varphi(\cdot, x_0)$ *be defined for* $t \in [0, T]$. *Suppose that* Z *is a convex set such that the following estimates hold:*

$$y([0, T]), \varphi([0, T], x_0) \in Z,$$
$$\mu\left(\frac{\partial f}{\partial x}(\eta)\right) \leq l, \quad \text{for } \eta \in Z,$$
$$\left\|\frac{dy}{dt}(t) - f(y(t))\right\| \leq \delta.$$

Then for $0 \leq t \leq T$ *there is*

$$\|\varphi(t,x_0) - y(t)\| \leq e^{lt}\|y(0) - x_0\| + \delta\frac{e^{lt}-1}{l}, \quad \text{if } l \neq 0.$$

For $l = 0$*, there is*

$$\|\varphi(t,x_0) - y(t)\| \leq \|y(0) - x_0\| + \delta t.$$

5.2. Application to Galerkin projections – uniqueness and another proof of convergence.

Definition 5. We say that $W \subset H$ and $F = N + L$ satisfy condition D if the following condition holds

D There exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \ldots$

$$1/2\sum_{i=1}^{\infty}\left|\frac{\partial N_k}{\partial x_i}\right|(W) + 1/2\sum_{i=1}^{\infty}\left|\frac{\partial N_i}{\partial x_k}\right|(W) + \lambda_k \leq l. \tag{48}$$

The main idea behind condition **D** is to ensure that the logarithmic norms for all Galerkin projections are uniformly bounded.

Theorem 13. *Assume that* $W \subset H$ *and* F *satisfy conditions* C1,C2,C3,D *and* W *is convex. Assume that* $P_n(W)$ *is a trapping region for the* n*–dimensional Galerkin projection of (46) for all* $n > M_1$*. Then*

1. **Uniform convergence and existence** *For a fixed* $x_0 \in W$*, let* $x_n : [0,\infty] \to P_n(W)$ *be a solution of* $x' = P_n(F(x))$*,* $x(0) = P_n x_0$*. Then* x_n *converges uniformly on compact intervals to a function* $x^* : [0,\infty] \to W$*, which is a solution of (46) and* $x^*(0) = x_0$*. The convergence of* x_n *on compact time intervals is uniform with respect to* $x_0 \in W$*.*
2. **Uniqueness within** W*. There exists only one solution of the initial value problem (46),* $x(0) = x_0$ *for any* $x_0 \in W$ *such that* $x(t) \in W$ *for* $t > 0$*.*
3. **Lipschitz constant***. Let* $x : [0,\infty] \to W$ *and* $y : [0,\infty] \to W$ *be solutions of (46), then*

$$|y(t) - x(t)| \leq e^{lt}|x(0) - y(0)|.$$

4. **Semidynamical system.** *The map* $\varphi : \mathbb{R}_+ \times W \to W$*, where* $\varphi(\cdot, x_0)$ *is the unique solution of equation (46) such that* $\varphi(0,x_0) = x_0$ *defines a semidynamical system on* W*, namely*
   - φ *is continuous*
   - $\varphi(0,x) = x$
   - $\varphi(t,\varphi(s,x)) = \varphi(t+s,x)$

Proof. By $|x|_n$ we will denote $|P_n(x)|$, i.e. Euclidean norm in $\mathbb{R}^n$.
Let

$$\delta_n = \max_{x\in W} |P_n(F(x)) - P_n(F(P_n x))|.$$

Obviously $\delta_n \to 0$ for $n \to \infty$, because $F \circ P_n$ converges uniformly to F on W.

Let us consider the logarithmic norm of the vector field for the n–dimensional Galerkin projection. We will estimate it using the Euclidean norm on $P_n H = \mathbb{R}^n$(which coincides with the norm inherited from H). Since

$$\left[\frac{\partial P_n(L+N)}{\partial(x_1 \dots x_n)}\right]_{ij} = \frac{\partial N_i}{\partial x_j} + \delta_{ij}\lambda_j, \tag{49}$$

we need to estimate the largest eigenvalue of the following matrix $Q_n(x)$ for $x \in P_n(W)$,

$$Q_{n,ij}(x) = \frac{1}{2}\frac{\partial N_i}{\partial x_j}(x) + \frac{1}{2}\frac{\partial N_j}{\partial x_i}(x) + \delta_{ij}\lambda_j, \quad \text{for } i,j = 1,\dots,n \tag{50}$$

where δ_{ij} is the Kronecker symbol, i.e., $\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$ otherwise.

To estimate the largest eigenvalue of Q_n, we will use the Gershgorin theorem (see [**10**, Property 5.2]), which states that all eigenvalues of a square $n \times n$-matrix A, $\sigma(A)$, satisfy

$$\sigma(A) \subset \cup_{j=1}^{n} \{z \in \mathbb{C} : \quad |z - A_{jj}| < \Sigma_{i,i\neq j} |A_{ij}|\}. \tag{51}$$

From the above equation and condition **D** it follows immediately that eigenvalues of Q_n are less than or equal to l_n, where

$$l_n = \max_{k=1,\dots,n} \max_{x\in P_n W} \sum_{i=1}^{n} \left(1/2\left|\frac{\partial N_k}{\partial x_i}(x)\right| + 1/2\left|\frac{\partial N_i}{\partial x_k}(x)\right|\right) + \lambda_k. \tag{52}$$

From assumption **D**, it follows that l_n are uniformly bounded, namely

$$l_n \le l, \quad \text{for all } n. \tag{53}$$

Let us take $m \ge n$. Let $x_n : [0,T] \to P_n W$ and $x_m : [0,T] \to P_m W$ be the solutions of n- and m–dimensional projections of (46). From Lemma 12 it follows immediately that (we treat here $P_n x_m$ as a perturbed 'solution' y)

$$|x_n(t) - P_n(x_m(t))|_n \le e^{lt}|x_n(0) - P_n x_m(0)| + \delta_n \frac{e^{lt}-1}{l}. \tag{54}$$

To prove the uniform convergence of $\{x_n\}$ starting from the same initial condition, observe that

$$\begin{aligned} |x_n(t) - x_m(t)| &\le |x_n(t) - P_n(x_m(t))|_n + |(I - P_n)x_m(t)| \\ &\le \delta_n \frac{e^{lt}-1}{l} + |(I-P_n)x_m(t)| \le \delta_n \frac{e^{lT}-1}{l} + |(I-P_n)W|. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in $\mathcal{C}([0,T],H)$, hence it converges uniformly to $x^*:[0,T]\to W$. From Lemma 8 it follows that $\frac{dx^*}{dt}=F(x)$.

Uniqueness. Let $x:[0,T]\to W$ be a solution of (46) with the initial condition $x(0)=x_0$. We will show that x_n converge to x. We apply Lemma 12 to n–dimensional projection and the function $P_n x(t)$. We obtain

$$|x_n(t)-P_n(x(t))|_n \le \delta_n \frac{e^{lt}-1}{l}. \tag{55}$$

Since the tail $(I-P_n)x(t)$ is uniformly converging to zero as $n\to\infty$, we see that $x_n\to x$ uniformly.

Lipschitz constant on W. From Lemma 12 applied to n–dimensional Galerkin projection for different initial conditions (we denote the functions by x_n and y_n and the initial conditions x_0 and y_0), we obtain

$$|x_n(t)-y_n(t)| \le e^{lt}|P_n x_0 - P_n y_0|. \tag{56}$$

Let $x_n\to x$ and $y_n\to y$. Then passing to the limit in (56) gives

$$|x(t)-y(t)| \le e^{lt}|x_0-y_0|. \tag{57}$$

Assertion 4 follows easily from the previous ones. □

6. Existence theorems for Navier–Stokes system in 2D

6.1. Some easy lemmas about Fourier series. The following three lemmas are easy exercises in elementary Fourier series theory [**2**].

Lemma 14. *Let $u\in C^n(\mathbb{T}^d,\mathbb{C})$ and let u_k for $k\in\mathbb{Z}^d$ be the Fourier coefficient of u. Then there exists M, such that*

$$|u_k| \le \frac{M}{|k|^n}.$$

Lemma 15. *Assume that $|u_k|\le\frac{M}{|k|^\gamma}$ for $k\in\mathbb{Z}^d$. If $n\in\mathbb{N}$ is such that $\gamma-n>d$, then the function $u(x)=\sum_{k\in\mathbb{Z}^d}u_k e^{ikx}$ belongs to $C^n(\mathbb{T}^d,\mathbb{C})$. The series*

$$\frac{\partial^s u}{\partial x_{i_1}\dots x_{i_s}} = \sum_{k\in\mathbb{Z}^d} u_k \frac{\partial^s}{\partial x_{i_1}\dots x_{i_s}} e^{ikx}$$

converges uniformly for $0\le s\le n$.

Lemma 16. *Assume that for some $\gamma>0$, $a>0$ and $D>0$ there is $|u_k|\le\frac{De^{-a|k|}}{|k|^\gamma}$ for $k\in\mathbb{Z}^d\setminus\{0\}$.*

Then the function $u(x)=\sum_{k\in\mathbb{Z}^d}u_k e^{ikx}$ is analytic.

Let $H = \left\{\{u_k\} \mid \sum_{k\in\mathbb{Z}^d} |u_k|^2 < \infty\right\}$. Obviously H is a Hilbert space. Let F be the right-hand side of (5)

$$F(u)_k = -i\sum_{k_1}(u_{k_1}|k) \sqcap_k u_{k-k_1} - \nu k^2 u_k + \sqcap_k f_k. \tag{58}$$

For a general $u \in H$, we cannot claim that $F(u) \in H$. But when $|u_k|$ decreases fast enough, the following holds

LEMMA 17. *Let* $W(D,\gamma) = \left\{u \in H \mid |u_k| \le \frac{D}{|k|^\gamma}\right\}$. *Then*

1. *if* $\gamma > \frac{d}{2}$, *then* $W(D,\gamma)$ *satisfies condition C2.*
2. *if* $\gamma - 2 > \frac{d}{2}$ *and* $\gamma > d$, *then the function* $F : W(D,\gamma) \to H$ *is continuous and condition C3 is satisfied on* $W(D,\gamma)$.
3. *if* $\gamma > d+1$, *then condition D is satisfied on* $W(D,\gamma)$.

PROOF. To prove Assertion 1, it is enough to show that $W(d,\gamma)$ is bounded, closed (obvious) and is component-wise bounded by some $v = \{v_k\}$, such that $v \in H$. We set $v_k = \frac{D}{|k|^\gamma}$. Observe that $v \in H$, because

$$\sum_{k\in\mathbb{Z}^d} |v_k|^2 \le CD^2 \sum_{n=1}^{\infty} \frac{n^{d-1}}{n^{2\gamma}} \tag{59}$$

and the series converges when $2\gamma - (d-1) > 1$. This concludes the proof of Assertion 1.

To prove Assertion 2, we may assume that $f = 0$ (it is just a constant vector in H). From Lemma 3 if follows immediately that for $u \in W$ there is

$$|F(u)_k| \le \frac{C}{|k|^{\gamma-1}} + \frac{\nu D}{|k|^{\gamma-2}} \le \frac{B}{|k|^{\gamma-2}}.$$

Hence $F(u) \in W(B,\gamma-2) \subset H$, when $\gamma - 2 > \frac{d}{2}$. Hence $F(W(D,\gamma)) \subset W(B,\gamma-2)$. Since the convergence in $W(B,\gamma-2)$ is equivalent to component-wise convergence, the same holds for the continuity. It is obvious that $F(u)_k$ is continuous on $W(d,\gamma)$, because the series defining it is uniformly convergent, hence F is continuous on $W(d,\gamma)$.

We now prove Assertion 3. Observe that

$$\frac{\partial N_k}{\partial u_{k_1}} = (\cdot|k) \sqcap_k u_{k-k_1} + (u_{k-k_1}|k) \sqcap_k . \tag{60}$$

We will here treat u_k as one dimensional object, but the argument is generally correct, i.e., treating u_k as a vector would introduce only an additional constant and not affect the proof. We estimate

$$\left|\frac{\partial N_k}{\partial u_{k_1}}\right|(W) \le \frac{2D|k|}{|k-k_1|^\gamma}. \tag{61}$$

Hence the sum, $S(k)$, appearing in condition D can be estimated as follows

$$\begin{aligned} S(k) &= 1/2 \sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \left|\frac{\partial N_k}{\partial u_{k_1}}\right|(W) + 1/2 \sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \left|\frac{\partial N_{k_1}}{\partial u_k}\right|(W) \\ &\leq D|k| \sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \frac{1}{|k-k_1|^\gamma} + D \sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \frac{|k_1|}{|k-k_1|^\gamma}. \end{aligned}$$

Now observe that

$$\sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \frac{1}{|k-k_1|^\gamma} < \sum_{k_1\in\mathbb{Z}^d, k_1\neq 0} \frac{1}{|k|^\gamma} = C(d,\gamma) < \infty, \qquad \text{for } \gamma > d. \tag{62}$$

To estimate the sum $\sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \frac{|k_1|}{|k-k_1|^\gamma}$, we show that there exists a constant A such that

$$\frac{|k_1|}{|k-k_1|} < A|k|, \quad \text{for } k, k_1 \in \mathbb{Z}^d \setminus \{0\},\ k \neq k_1. \tag{63}$$

Observe that, for $|k_1| \leq 2|k|$, $k_1 \neq 0$, $k_1 \neq k$, we can estimate the denominator by 1, hence we obtain

$$\frac{|k_1|}{|k-k_1|} \leq 2|k|. \tag{64}$$

For $|k_1| > 2|k|$, there is

$$\frac{|k_1|}{|k-k_1|} = \frac{1}{\left|\frac{k_1}{|k_1|} - \frac{k}{|k_1|}\right|} \leq \frac{1}{1-\frac{|k|}{|k_1|}} \leq 2. \tag{65}$$

So we may take $A = 2$.

Now we estimate as follows

$$\sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \frac{|k_1|}{|k-k_1|^\gamma} \leq A|k| \sum_{k_1\in\mathbb{Z}^d\setminus\{0,k\}} \frac{1}{|k-k_1|^{\gamma-1}} < AC(d,\gamma-1)|k|, \tag{66}$$

provided $\gamma - 1 > d$.

So there is $S(k) < (DC(d,\gamma) + ADC(d,\gamma-1))\,|k|$ and since $\lambda_k = -\nu|k|^2$, we see that there exists l satisfying condition D. □

6.2. Existence theorems. We set the dimension $d = 2$. We again assume that the force f is such that $f_k = 0$ for $|k| > K$ (in [**9**] a more general force is treated).

Observe that from Lemma 17 it follows that we need $\gamma > 3$ for conditions C1, C2, C3, D on the trapping regions constructed in Section 3 to be satisfied.

Theorem 18. *If for some D and $\gamma > 3$*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} \tag{67}$$

then the solution of (5) is defined for all $t > 0$ and there exists a constant D', such that

$$|u_k(t)| \leq \frac{D'}{|k|^\gamma}, \quad t > 0. \tag{68}$$

The following theorem tells that if we start with analytic initial conditions, the solution will remain analytic (in space variables).

Theorem 19. *If for some D, $\gamma > 3$ and $a > 0$*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} e^{-a|k|}, \tag{69}$$

then the solution of (5) is defined for all $t > 0$ and there exist constants D' and $a' > 0$ such that

$$|u_k(t)| \leq \frac{D'}{|k|^\gamma} e^{-a'|k|}, \quad t > 0. \tag{70}$$

The next theorem states that the solution starting from regular initial conditions becomes analytic immediately.

Theorem 20. *Assume that for some D, $\gamma > 3$ and $a > 0$ the initial conditions satisfy*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma}. \tag{71}$$

Then the solution of (5) is defined for all $t > 0$ and for any $t_0 > 0$ one can find constants D' and $a' > 0$ such that

$$|u_k(t_0)| \leq \frac{D'}{|k|^\gamma} e^{-a'|k|}. \tag{72}$$

Proof of Theorem 18. Observe first that the enstrophy of $\{u_k(0)\}$ is finite. Let us take $V_0 > \max(V(\{u_k\}), V^*)$. From Theorem 4 it follows that there exist K and D' such that $\{u_k(0)\}$ belongs to the trapping set $N = N(V_0, K, \gamma, D')$. Observe that $N \subset W(D', \gamma)$, hence we can pass to the limit with solutions obtained from Galerkin projections (see Theorem 13). □

Proof of Theorem 19. The proof is essentially the same as for Theorem 18, with the only difference being that we now use Theorem 5 instead of Theorem 4. □

Proof of Theorem 20. The global existence was proved in Theorem 18. To prove the estimate for $|u_k(t_0)|$, we use Theorem 6 to obtain

$$|u_k(t_0)| \leq \frac{D'}{|k|^\gamma} e^{-a|k|t_0}, \tag{73}$$

which finishes the proof. □

Theorem 21. $d = 2$. *If* $u_0 \in C^5$*, then the classical solution of the NS equations such that* $u(0,x) = u_0(x)$ *exists for all* $t > 0$ *and is analytic in space variables for* $t > 0$.

Proof. From Lemma 14 it follows that the Fourier coefficients of u_0, $\{u_{0,k}\}$, satisfy assumptions of Theorem 18 with $\gamma = 5$. Hence there exists a solution, $\{u_k(t)\}$ of (5) in H such that $u_k(0) = u_{0,k}$.

Let us set $u(t,x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} u_k(t) e^{ikx}$. It is easy to see that $u(t,x)$ is a classical solution of the Navier–Stokes system, because the Fourier series for all terms in the NS equations converges fast enough (compare proof of Lemma 7).

From Theorem 20 and Lemma 16 it follows that the function $u(t_0, \cdot)$ is analytic in space variables for any $t_0 > 0$. □

The following theorem is an easy consequence of Theorem 13.

Theorem 22. *Assume* $d = 2$ *and* $\gamma > 3$*. If* W *is any of the trapping regions defined in Theorems 4 and 5, then the Navier–Stokes system induces a semidynamical system on* W.

7. Trapping regions in 3D for small initial data

In this section we recall the method by Sannikov and Kaloshin [**11**] for constructing a trapping region for small initial data in dimension 3.

Let us state a result which is not contained in [**11**] but can be easily obtained using the technique presented there.

We set the dimension $d = 3$. We assume the force f is zero.

Theorem 23. *For any* $\gamma > 3.5$*, there exists* $D_0 = D_0(\gamma, \nu)$ *such that for all* $D < D_0$*, if*

$$|u_k(0)| \leq \frac{D}{|k|^\gamma} \tag{74}$$

then the solution of (5) is defined for all $t > 0$ *and*

$$|u_k(t)| \leq \frac{D}{|k|^\gamma}, \quad t > 0. \tag{75}$$

Proof. Let

$$W = \left\{ \{u_k\} \mid |u_k| \leq \frac{D}{|k|^\gamma} \right\}. \tag{76}$$

From Lemma 3 it follows that for $\{u_k\} \in W$ there is

$$\frac{d|u_k|}{dt} \leq \left| \sum (u_{k_1}|k) \sqcap_k u_{k-k_1} \right| - \nu|k|^2|u_k| \leq \frac{D^2 C_Q(3,\gamma)}{|k|^{\gamma-1}} - \nu|k|^2|u_k|. \tag{77}$$

Hence W is a trapping region if for every k there is

$$\frac{D^2 C_Q(3,\gamma)}{|k|^{\gamma-1}} - \frac{\nu D}{|k|^{\gamma-2}} < 0. \tag{78}$$

We obtain

$$\frac{D C_Q(3,\gamma)}{\nu} < |k|, \quad k \in \mathbb{Z}^3 \setminus \{0\}. \tag{79}$$

Hence if

$$D < D_0 = \frac{\nu}{C_Q(3,\gamma)}, \tag{80}$$

then W is a trapping region for all projections of the Navier Stokes equations. From Lemma 17 it follows that conditions C1,C2,C3 are satisfied (it is easy to see that condition D holds if $\gamma > 4$.) Hence we can pass to the limit with the dimension of Galerkin projection to obtain a desired solution. □

One can easily state a similar theorem for analytic initial condition.

Let us comment on the Sannikov and Kaloshin result presented in [**11**]. They constructed the trapping region of the form $|u_k| \leq \frac{D}{|k|^2} e^{-v|k|t}$, $t \geq 0$, where $v > 0$. The methods developed in this paper require more compactness at $t = 0$ to be directly applicable to this trapping region.

8. Conclusions and outlook

As already discussed in the introduction, the tools developed here and in [**14**] enable the topological finite-dimensional tools developed to study the dynamics of ODEs to be applied to dissipative PDEs.

To be able to apply other dynamical-system tools, such as the hyperbolicity concept, one needs C^1-information about the induced flow. We believe one can get such information for the Navier–Stokes equations with periodic boundary conditions on the plane using the framework presented here. For example, the Lipschitz constant, which we have obtained in this paper represents this kind of data. But we may definitively expect much more. The natural question to ask here is the following.

Suppose that all assumptions of Theorem 13 are satisfied. Let φ^n be a semidynamical system induced by the n-th Galerkin projection. Let us consider the variational matrix for φ^n given by

$$V^n_{ij}(t,x) = \frac{\partial \varphi^n_i}{\partial x_j}(t,x). \tag{81}$$

QUESTION. Do $V^n_{ij}(t,P_n x)$ converge? And if they converge, then what use we can make of this fact in the context of the method of self-consistent a priori bounds developed in [**14**]? We hope to answer this question in a subsequent paper.

To see why we expect convergence here, let us remark, that V^n satisfies the following differential equation

$$\frac{dV^n_{ij}}{dt} = \lambda_i V^n_{ij} + \sum_k \frac{\partial N_i}{\partial x_k} V^n_{kj}. \tag{82}$$

Hence we can see that there is here the same strong damping as for the original equation (46). Observe that the bound for Lipschitz constant for (46) and its Galerkin projections is also a uniform bound for the norms of matrices V^n on any finite time interval. Once we have a strong damping and a priori bounds for V^n, we expect that we can use logarithmic norms to control the convergence of V^n's [**16**].

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