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STRUCTURE OF ONE-DIMENSIONAL CHAIN-RECURRENT SETS OF FLOWS ON THE 2–SPHERE AND ON THE PLANE

by Piotr Oprocha

Abstract. The main subject of this paper is the topological structure of connected components of the set of all chain-recurrent points of flows in the 2–sphere and the plane. Such components for flows with finitely many stationary points on the 2–sphere are topologically finite graphs. We will extend this property onto a class of flows in the plane.

1. Introduction. Consider a flow in the sphere S^2. It is known that any limit set on the sphere is connected, compact, invariant and the flow restricted to it is chain-recurrent. If such a set consists of at least one nonstationary point, then it is one-dimensional. However, a chain-recurrent set on S^2 may not be locally an arc at its nonstationary points, while a limit set always is. From the topological point of view, limit sets and chain-recurrent sets may differ considerably.

It was proved in [**1**] that any one-dimensional chain-recurrent set of the flow in S^2 with finitely many stationary points is locally an arc at its nonstationary points. Moreover, it consists of finitely many orbits, and it is topologically a finite graph. In [**1**], there was also an example to the effect that assumption of finiteness of the set of stationary points is essential.

In the plane, it is possible to give an infinite set which does not focus to any point. So it is possible that some properties of chain-recurrent set obtained on S^2 may be (in some way) true for flows in the plane.

The main aim of this paper is to show that one-dimensional chain-recurrent components of the set $CR(\varphi)$ may be topologically viewed, in some cases, as infinite graphs [Theorem 13 in Section 4]. For the completeness of this paper,

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we also present the case of the 2–sphere using a different and, in our opinion simpler approach [Section 3].

2. Preliminaries. Let X be a metric space. By a *Jordan arc* (resp., a *Jordan curve*) we mean a homeomorphic image of the the closed interval $[a, b]$ (resp., the unit circle). A corresponding homeomorphism $\alpha : I \longrightarrow \Gamma \subset X$ will be called a *parameterization* of an arc Γ.

Let J be a Jordan arc with a parameterization α, and let $x, y \in J$, $x \neq y$. Let us denote $t_1 = \min(\alpha^{-1}(x), \alpha^{-1}(y))$ and $t_2 = \max(\alpha^{-1}(x), \alpha^{-1}(y))$. By the part of J between points x, y we mean the set $[x, y] = \alpha([t_1, t_2])$;

Let L_1, L_2, L_3 be Jordan arcs with end-points a and b. If $L_i \cap L_j = \{a, b\}$ for $i \neq j$, then the set $T = L_1 \cup L_2 \cup L_3$ is said to be a Θ–curve.

A set A is *locally an arc* if for any point $x \in A$ there exists a closed ball $B(x, r) \subset X$ such that $A \cap B$ is a Jordan arc.

Let $X = \mathbb{R}^2$ or $X = S^2$. A subset A of X is *one-dimensional* if $intA = \emptyset$ and A has no isolated points.

Let X be a metric space. We say that a continuous function $\varphi : \mathbb{R} \times X \longrightarrow X$ is a *flow* (*dynamical system*) if $\varphi(0, x) = x$ and $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for any s, t, x.

Through the rest of this paper, a pair (X, φ) will denote a dynamical system φ on some metric space (X, d), where d is the metric.

A point x is said to be

- *stationary* if $\varphi(t, x) = x$ for every t;
- *periodic* if there exists a $t > 0$ such that $\varphi(t, x) = x$ and x is not a stationary point.

By the *positive semiorbit (semitrajectory)* we mean the set $o^+(x) = \{\varphi(t, x) : t \geq 0\}$ and by *negative semiorbit* we mean the set $o^-(x) = \{\varphi(t, x) : t \leq 0\}$ The set $o(x) = o^+(x) \cup o^-(x)$ is an *orbit (trajectory)* of the point x.

For a given point x, we define the *positive limit set of* x as $L^+(x) = \{y \mid \exists t_n \to +\infty \,:\, \varphi(t_n, x) \to y\}$ and *negative limit set* as $L^-(x) = \{y \mid \exists t_n \to -\infty \,:\, \varphi(t_n, x) \to y\}$. The *limit set of* x is the set $L(x) = L^+(x) \cup L^-(x)$.

A set A is said to be *positively (negatively) invariant* if $o^+(x) \subset A$ ($o^-(x) \subset A$) for every $x \in A$. If A is both positively and negatively invariant then we call it *invariant.*

Let (X, φ) be a flow and let $x, y \in X$. Given $\varepsilon > 0$ and $T > 0$, an (ε, T)–chain from x to y is a pair of finite sets of points $\{x_0, \ldots, x_{p+1}\}$ and $\{t_0, \ldots, t_p\}$ such that $x = x_0$, $y = x_{p+1}$, $t_j > T$ and $d(\varphi(t_j, x_j), x_{j+1}) < \varepsilon$ for $j = 0, \ldots, p$.

If for any $\varepsilon > 0$ and $T > 0$ there exists an (ε, T)–chain from x to y, then we write xPy.

The set $\Omega^+(x) = \{y \mid xPy\}$ is called the *positive chain limit set of* x and the set $\Omega^-(x) = \{y \mid yPx\}$ *negative chain limit set of* x.

A point x is chain-recurrent if xPx. The set $CR(\varphi) = \{x \mid xPx\}$ is closed and invariant.

A closed set S containing x is called an *ε–section* through x if the set $U = \varphi((-\varepsilon, \varepsilon), S)$ is a neighborhood of x and $\varphi(t_1, S) \cap \varphi(t_2, S) = \emptyset$ for $-\varepsilon < t_1 < t_2 < \varepsilon$. We say that a set S is a *section* through x if there exists $\varepsilon > 0$ such that S is an ε–section through x. In the case of S^2 and $\mathbb{R}^2$, there always exists a section through any nonstationary point and this section is a Jordan arc (see [**4**, thm. 3.1]).

A closed set S is called a *section* if it is section through some $x \in S$.

If for any given section S containing y there exists a real number $t > 0$ such that $\varphi(t, y) \in S$, then $t_0 = \inf\{t > 0 \mid \varphi(t, y) \in S\} \neq \emptyset$ is said to be *the time of first return* of the point y to S. If such t does not exist, we say that y *does not return to S*.

Let S be a section and let $y \in S$ be a point such that the set $o(y) \cap S$ is nonempty but finite (i.e. $\varphi(t_i, x) \in S$ for times $t_1 < t_2 < \cdots < t_n$, $n \geq 1$, and $\varphi(t, x) \notin S$ for $t \neq t_i$). In this case, the time t_1 is called *time of the first intersection* of the orbit of y with S and the time t_n is called *time of the last intersection* of the orbit of y with S.

Observation 1. Let $X = \mathbb{R}^2$, $S \subset X$ be both a Jordan arc and a section and let $y \in S$ be a point returning to S. Let t_y denote the time of the first return of y and let $[y, \varphi(t, y)]$ be the part of S between y and $\varphi(t, y)$. In this case, the set $\Gamma = [y; \varphi(t, y)] \cup \varphi([0; t], y)$ is a Jordan curve dividing X into two connected open sets D and E with the common boundary Γ. The set D is positively invariant, set E is negatively invariant and one of the sets is compact (compare Fig. 1).

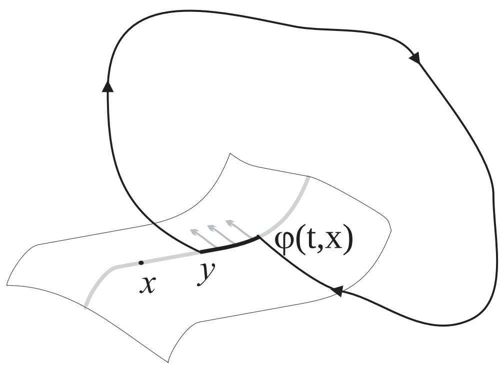

Figure 1. Setting of Observation 1.

Observation 2. Let $X = \mathbb{R}^2$, $S \subset X$ be both Jordan arc and a section, and let $y, z \in S$ be points such that $L^+(y) = \{p_0\}$,$L^+(z) = \{p_1\}$ and $p_0 \neq p_1$. If there exists an invariant Jordan arc α disjoint with S and connecting p_0 with p_1, while points y, z do not return to S, then the set $\Gamma = [y; z] \cup o^+(y) \cup o^+(z) \cup \alpha$ is a Jordan curve dividing X into two connected open sets D and E with boundary Γ. The set D is positively invariant, set E is negatively invariant and one of the sets is compact (compare Fig. 2 (a)).

Remark 3. We may make observations analogous to Observation 2, replacing $L^+(y)$ with $L^-(y)$ or $L^+(z)$ with $L^+(y)$ (in this case, we need to assume that $o^-(y) \cap S = \{y\}$ and $o^-(z) \cap S = \{z\}$). A similar situation arises if α is stationary point (and then $p_0 = p_1 = \alpha$). If for points y, z there is $L^+(y) = L^+(z) = \emptyset$, then point in infinity plays the role of α (see Fig. 2 (b) and (c)).

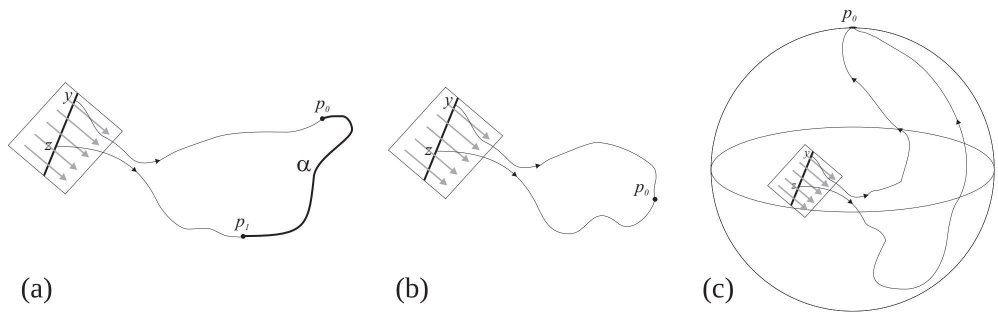

Figure 2. (a) $L^+(y) = \{p_0\}$,$L^+(z) = \{p_1\}$ and $p_0 \neq p_1$, (b) $L^+(y) = L^+(z) = \{p_0\}$, (c) $L^+(y) = L^+(z) = \emptyset$.

Observe that if $X = S^2$, then both sets D and E are compact. Next, we will state, without proofs, Observations 4, 5 and 6, which correspond to the analogous theorems presented in [**1**] in the case of $X = S^2$.

Observation 4. If $x \in \mathbb{R}^2$ is a nonperiodic but chain-recurrent point, then $L^+(x)$ and $L^-(x)$ are empty or consist of stationary points only.

Observation 5. Let $A \subset \mathbb{R}^2$ be a one-dimensional closed, connected, invariant and nonempty chain-recurrent set. If A contains a periodic orbit C, then $A = C$.

Observation 6. Let φ be a flow on $\mathbb{R}^2$ and let A be a one-dimensional compact, connected and nonempty chain-recurrent set. If A contains no stationary point, then A is a periodic orbit.

The following is a kind of a folklore theorem.

OBSERVATION 7. Let $S \subset X$ be both a Jordan arc and a ε–section, and let U denote the set $\varphi([-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}], S)$. For every $y \in U$, let us set $P(y) = \varphi(t_y, y) \in S$, where $t_y \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. Then P is continuous function.

3. One-dimensional chain-recurrent sets in the 2–sphere. The following theorem summarizes the main results of paper [**1**] concerning flows in S^2. Yet, our proof is quite different and shorter as it mostly uses properties of Jordan curves.

Let φ be a dynamical system in S^2 with finitely many stationary points.

THEOREM 8. *If Y is a one-dimensional connected component of $CR(\varphi)$, then Y consists of finitely many orbits.*

PROOF. When Y contains a periodic point, the theorem follows from Observation 5, so we may assume that there are no periodic points in Y.

Suppose that there exists a sequence $\{x_n\} \subset Y$ consisting of points with disjoint orbits. There are finitely many stationary points, so in the case of S^2, by virtue of Observation 4, there is

$$L^+(x_n) = \{p_0\}\ ,\ L^-(x_n) = \{p_1\} \quad \forall\, n \in \mathbb{N}$$

where p_0 and p_1 are stationary points (not necessarily different). We may also assume, that $x_n \to x \in Y$ and $x \notin o(x_n)$ for every n.

First suppose that $p_0 \neq p_1$. We will make recursive construction (for shortness, we will present the first three steps only):

1. Define two Jordan arcs $\Gamma_{1,1} = \overline{o(y_0)}$ and $\Gamma_{1,2} = \overline{o(y_1)}$, where y_0 and y_1 are any two elements of $\{x_n\}$. Observe that these arcs together form a Jordan curve Γ_1, which by Schönflies theorem (see [**10**, p. 71]) is the common boundary of two open discs D and E such that $S^2 = D \cup E \cup \Gamma_1$. It is easy to see that these discs are invariant sets. By A_1 we denote this of the discs which does not contain x.
2. Take a point $y_2 \in \{x_n\}$ lying outside $\overline{A_1}$ (e.g., $y_2 \notin \overline{A_1}$). Set $\Gamma_{1,3} = \overline{o(y_2)}$ and observe that the Jordan arcs $\Gamma_{1,1}$, $\Gamma_{1,2}$ and $\Gamma_{1,3}$ form a Θ–curve. By the Θ–curve Theorem (see [**2**], C.22 and [**9**]), we get an open and connected set A_2 disjoint from A_1. Moreover, $\partial\overline{A_1 \cup A_2} = \Gamma_{1,1} \cup \Gamma_{1,3}$, or $\partial\overline{A_1 \cup A_2} = \Gamma_{1,2} \cup \Gamma_{1,3}$. If first condition is true, we set $\Gamma_{2,1} = \Gamma_{1,1}$ and $\Gamma_{2,2} = \Gamma_{1,3}$. Otherwise, $\Gamma_{2,1} = \Gamma_{1,2}$ and $\Gamma_{2,2} = \Gamma_{1,3}$.
3. Take a point $y_3 \in \{x_n\}$ lying outside $\overline{A_2}$ and set $\Gamma_{2,3} = \overline{o(y_3)}$. By applying the Θ–curve Theorem to $\Gamma_{2,1}$, $\Gamma_{2,2}$ and $\Gamma_{2,3}$, we get open and connected set A_3 disjoint from $A_1 \cup A_2$. Moreover, $\partial\overline{(A_1 \cup A_2) \cup A_3} = \Gamma_{2,1} \cup \Gamma_{2,3}$, or $\partial\overline{(A_1 \cup A_2) \cup A_3} = \Gamma_{2,2} \cup \Gamma_{2,3}$. In the first case we set $\Gamma_{3,1} = \Gamma_{2,1}$ i $\Gamma_{3,2} = \Gamma_{2,3}$. Otherwise, $\Gamma_{3,1} = \Gamma_{2,2}$ and $\Gamma_{3,2} = \Gamma_{2,3}$.

By the recurrent use of this construction, we will get a family $\{A_n\}$ of open, invariant and pairwise disjoint sets. In the case $p_0 = p_1$ a construction of such a family is similar. There are finitely many stationary points, so there exists N such that the set A_N does not contain a stationary point (when such a set is constructed, we may stop the procedure). Observe that p_0 and p_1 are the only stationary points in the set $D = \overline{A_N}$. There are no stationary points inside D, so there are no periodic points either. If $z \in D$, then, its positive and negative limit sets by the Poincaré–Bendixson theorem, must contain at least one of the points p_0 and p_1, which implies that z is a chain-recurrent point. The set D consists of chain-recurrent points only and is connected, so $D \subset Y$, which means that Y is not one-dimensional. $\square$

Remark 9. Every one-dimensional connected component of $CR(\varphi)$, may be seen, from topological point of view, as finite graph whose vertices are stationary points, and edges are orbits of nonstationary points.

Example 10. Observe that when (S^2, φ) has infinitely many stationary points, then Theorem 8 is not true. As an example, we may consider the dynamical system from Fig. 3, where points $x, y, z_1, z_2, \ldots$ and $u_1, u_2, \ldots$ are stationary.

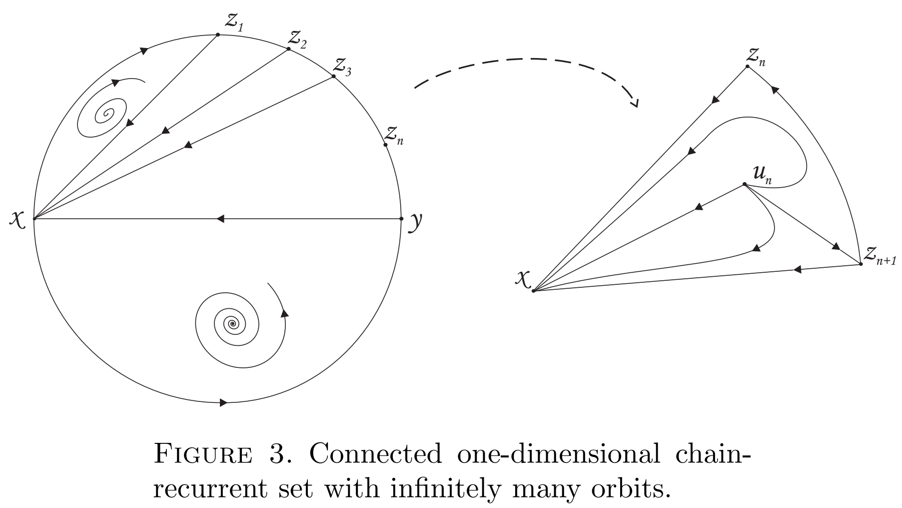

Figure 3. Connected one-dimensional chain-recurrent set with infinitely many orbits.

For that system, there is $CR(\varphi) = \bigcup_{n\in\mathbb{N}}[z_n, x] \cup S^1 \cup [x, y]$. Observe that this set is connected, compact, one-dimensional, but it consists of infinitely many orbits. The dynamical system in Fig. 3 was described in [**1**].

4. One-dimensional chain-recurrent sets on the plane. Throughout this section, we will consider a dynamical system φ in the plane with the following conditions:

(W1) *The stationary points are isolated from one another.*

(W2) *If a point x is stationary, then the set $A = \{y \mid \exists\ z\ :\ \{x, y\} \subset L(z)\}$ contains a finite number of stationary points and periodic orbits.*

(W3) *If p_0 and p_1 are stationary points in the same connected component of $CR(\varphi)$, then there exists a Jordan arc $\alpha \subset CR(\varphi)$ with end-points p_0 and p_1.*

(W4) *If Y is a one-dimensional connected component of $CR(\varphi)$ containing point y with $L(y) = \emptyset$, then $Y = o(y)$*

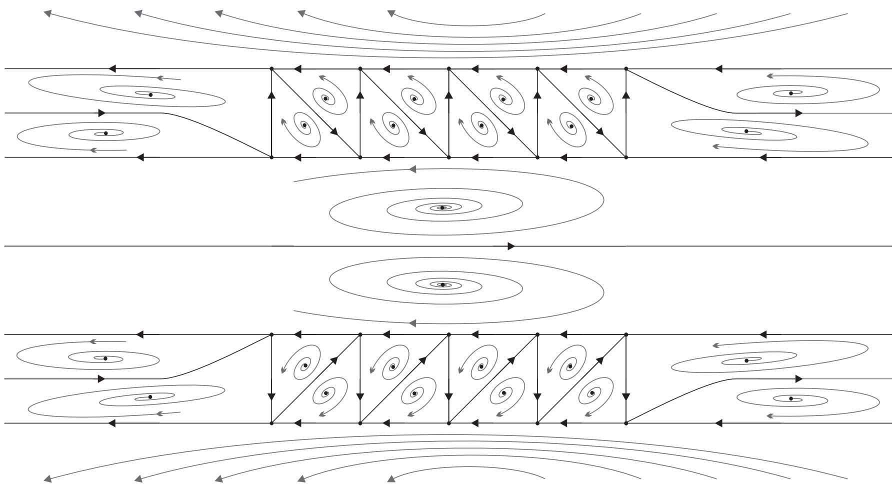

FIGURE 4. Dynamical system in the plane fulfilling conditions **(W1)**–**(W4)**.

LEMMA 11. *Let S be a section through x and let $\{x_n\}$ be a sequence of points converging to x. Then there exist sequences of times $\{t_n\}$ and points $\{y_n\}$ such that $\varphi(x_n, t_n) = y_n \in S$ and $y_n \to x$.*

PROOF. It is a consequence of Observation 7. □

THEOREM 12. *If Y is a one-dimensional connected component of the set $CR(\varphi)$, then it is locally an arc in its nonstationary points.*

PROOF. We may assume that every $y \in Y$ has the nonempty limit set; otherwise, by condition **(W4)**, there is nothing to prove. By Observation 5, we may also assume that there are no periodic points in Y.

Let $x \in Y$ be any nonstationary point. Then for some $\varepsilon > 0$ there exists such an ε–section S through x which is a Jordan arc.

Suppose that Y is not locally an arc in x. There exists $\{x_n \mid n \in \mathbb{N}\} \subset S \cap Y$ converging monotonically to x on S. The point x is not periodic, so by Observation 4 the sets $L^+(x)$ and $L^-(x)$ contain stationary points only or are empty. The set $L(x)$ is nonempty, and so are the sets $L(x_n)$. Suppose that $L^+(x) \neq \emptyset$ (when $L^+(x) = \emptyset$, then $L^-(x) \neq \emptyset$ and the proof is similar). The set $L^+(x)$ consist of stationary points which are by **(W1)** isolated, so $L^+(x) = \{p_0\}$ for some stationary point p_0.

Observe that the set $o(x_n) \cap S$ is finite. Otherwise, $L^+(x) \cap S \neq \emptyset$ but there are no stationary points in S. Thus we may assume that $o(x_m) \cap o(x_n) = \emptyset$ for $m \neq n$.

Fix any $N \in \mathbb{N}$ and $y \in L(x_N)$. The points y and p_0 lie in the same connected component of $CR(\varphi)$, so by **(W3)** there exists a Jordan arc $\alpha \subset Y$ with end-points p_0 and y. The set Y is one-dimensional, so α is invariant.

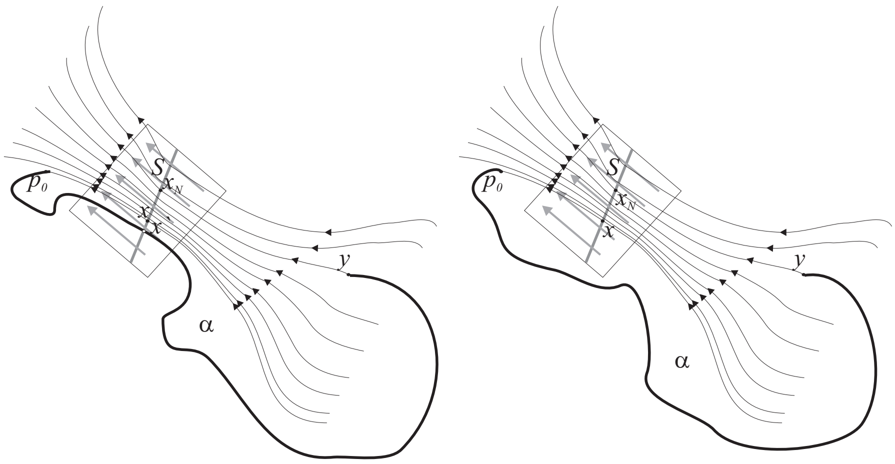

FIGURE 5. Case 1. and Case 2.

There are two cases possible (see Fig. 5).

1. $\alpha \cap S \neq \emptyset$.
   Let x' be the point from $\alpha \cap S$ first to p_0 and let α' be a subarc of α connecting p_0 with x'. Observe that $\Gamma = [x, x'] \cup o^+(x) \cup \alpha'$ is a Jordan curve, so by Schönflies theorem, Γ is the common boundary of two open discs D and E, where D is positively invariant and E is negatively invariant.

2. $\alpha \cap S = \emptyset$.
   If $y = L^+(x)$, then we take $\Gamma = [x, x_N] \cup o^+(x) \cup \alpha \cup o^+(x_N)$. Otherwise, $\Gamma = [x, x_N] \cup o^+(x) \cup \alpha \cup o^-(x_N)$. Observe that Γ is a Jordan curve and by Observation 2 it is the common boundary of the sets D and E, as in (1).

Suppose that D is bounded (otherwise E is bounded and the proof is similar). We may suppose that $x_n \in \overline{D}$ for all n. The set $\overline{D}$ is compact so, as by **(W1)** stationary points are isolated, it contains finitely many stationary points. The set $\overline{D}$ is positively invariant and bounded, hence the sets $L^+(x_n) \neq \emptyset$ and by Observation 4 consist of stationary points. We may assume that $L^+(x_n) = \{z_1\}$ for all n, where $z_1 \in \overline{D}$ is some stationary point. As we said before, the orbit $o(x_n)$ intersects with S a finite number of times. Let s_n and t_n be the times of the first and last intersection of $o(x_n)$ with S. We may assume that sequences $\{s_n\}$ and $\{t_n\}$ are monotonic. Let

$$\Gamma_n = [\varphi(t_n, x_n), \varphi(t_{n+1}, x_{n+1})] \cup o^+(t_n x_n) \cup o^+(t_{n+1} x_{n+1}) \cup \{z_1\} \ ;$$

and observe that Γ_n is a Jordan curve contained in $\overline{D}$. Let $D_n \subset D$ be a connected open set with the boundary Γ_n given by 2. The set $D_n \subset D$, so $\overline{D_n}$ is compact and thus contains finitely many stationary points.

First we claim that $D_m \cap o(x_n) = \emptyset$. Suppose that $D_m \cap o(x_n) \neq \emptyset$. By $S \cap D_n = \emptyset$, there is $x_n \notin D_n$, so there exists $s > 0$ such that $\varphi(s, x_n) \in \partial D$. If $\varphi(s, x_n) \in (\varphi(t_m, x_m), \varphi(t_{m+1}, x_{m+1}))$ then $s = t_n$ and the sequence $\{t_n\}$ is not monotonic. If $x_n \in o^+(x_m)$ then $o(x_n) = o(x_m)$, but $o(x_m) \cap D_m = \emptyset$, a contradiction. When $x_n \in o^+(x_{m+1})$ the proof is analogous, which completes the proof of the claim.

Observe that $D_n \cap D_m = \emptyset$. If we set $B = D_m \cap D_n$, then B is open and closed. If it is also nonempty, then, as a subset of a connected set, it must be equal to it and then $m = n$.

There are finitely many stationary points in D so we may suppose that there are no stationary points in D_n for all n. This implies that there are no periodic orbits in $\overline{D_n}$. The point z_1 is the only stationary point in $\overline{D_n}$, so $L^+(p) = \{z_1\}$ for all $p \in \overline{Dn}$. Taking a subsequence we may encounter the following two situations.

1. $L^-(x_n) = \emptyset$ for all n.
   In this case, the set

$$\Gamma_n = [\varphi(s_n, x_n), \varphi(s_{n+1}, x_{n+1})] \cup o^-(\varphi(s_n, x_n)) \cup o^-(\varphi(s_{n+1}, x_{n+1}))$$

   is a Jordan curve, so there exists a negatively invariant open set E_n such that $\partial E_n = \Gamma_n$. As in the case of D_n one may show that $E_n \cap E_m = \emptyset$ for $m \neq n$.

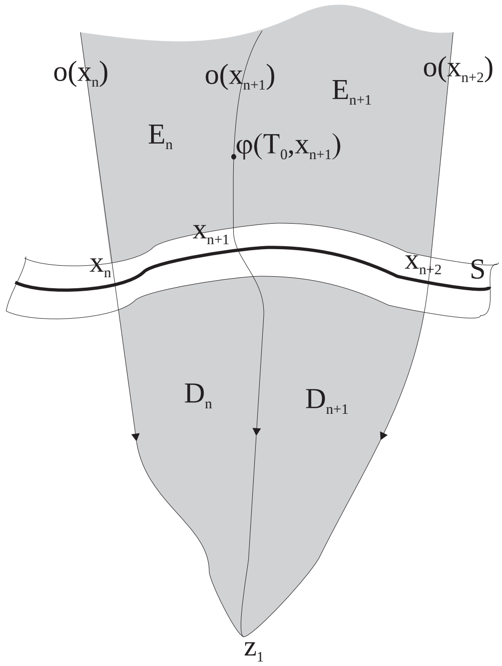

FIGURE 6. Situation when $L^-(x_n) = \emptyset$ for all n.

By **(W2)** we may assume, that if p lies in the arc $(\varphi(s_n, x_n), \varphi(s_{n+1}, x_{n+1}))$, then $L^+(p) = \emptyset$.

Take $U = \varphi((-\varepsilon, \varepsilon), S)$ and observe that the set $(\overline{E_n} \cup \overline{E_{n+1}} \cup \overline{D_n} \cup \overline{D_{n+1}}) \setminus U$ contains two connected components, one of which is compact. The distance between those components is $2\delta > 0$. Let us take $T_0 < -2\varepsilon$. The points x_{n+1} and $\varphi(T_0, x_{n+1})$ lie in the same connected component of the set $CR(\varphi)$, so for any $\lambda \in (0, \delta)$ and $T > \varepsilon$, there exists a (λ, T)–chain from x_{n+1} to $\varphi(T_0, x_{n+1})$. Observe that every such chain must have such point outside $(\overline{E_n} \cup \overline{E_{n+1}} \cup \overline{D_n} \cup \overline{D_{n+1}})$ and every following point of the chain does not lie in $\overline{D_n} \cup \overline{D_{n+1}}$. Let A_t be a $(\frac{1}{t}, t)$–chain from x_{n+1} to $\varphi(T_0, x_{n+1})$ where $t > \varepsilon$ and $\frac{1}{t} < \delta$. We may assume that for every t there exist points $a_{1,t}, a_{2,t} \in A_t$ such that $d(a_{1,t}, o^-(x_n)) < \frac{1}{t}$ and $d(a_{2,t}, o^-(x_{n+1})) < \frac{1}{t}$ $(d(a_{1,t}, o^-(x_{n+1})) < \frac{1}{t}$ and $d(a_{2,t}, o^-(x_{n+2})) < \frac{1}{t})$, and points of A_t lying between $a_{1,t}$ and $a_{2,t}$ are in $E_n \setminus U$ (in $Int E_{n+1} \setminus U$). Let p be any point from $(\varphi(s_n, x_n), \varphi(s_{n+1}, x_{n+1}))$ (from $(\varphi(s_{n+1}, x_{n+1}), \varphi(s_{n+2}, x_{n+2}))$ in the second case). The set $L^-(p)$ is empty, so $o^-(p)$ dissects E_n (E_{n+1})

into two open connected components. It implies that for every t there exist $a_t \in A_t$ and $p_t \in o^-(p)$ such that $d(\varphi(a_t,t),p_t) < \frac{1}{t}$. We can construct a $(\frac{2}{t},t)$–chain from x_n to p (for t large enough), so x_nPp.

On the other hand $L^+(p)=\{z_1\}$, which implies pPz_1. The points z_1 and x_n lie in the same connected component of $CR(\varphi)$, so z_1Px_n and then pPx_n, which means that p is a chain-recurrent point.

The points p and z_1 lie in the same connected component of $CR(\varphi)$; since p was an arbitrary point in the arc $(\varphi(s_n,x_n),\varphi(s_{n+1},x_{n+1}))$, then $(\varphi(s_n,x_n),\varphi(s_{n+1},x_{n+1}))\subset Y$ (similarly in the second case), so the set Y is not one-dimensional, which contradicts the assumptions concerning Y.

2. $L^-(x_n)\neq\emptyset$ for all n. By Observation 4 and **(W2)**, we may assume that $L^-(x_n)=\{z_2\}$ for all n, where z_2 is a stationary point. Observe that the set

$$\Gamma_n=[\varphi(s_n,x_n),\varphi(s_{n+1},x_{n+1})]\cup o^-(\varphi(s_n,x_n))\cup o^-(\varphi(s_{n+1},x_{n+1}))\cup\{z_2\}$$

is a Jordan curve, so there exists a negatively invariant open set E_n with boundary Γ_n. We may show as before that $E_n\cap E_m=\emptyset$, when $n\neq m$. By **(W2)** we may assume that, if $p\in(s_nx_n,s_{n+1}x_{n+1})$, then $L^+(p)=\{z_2\}$ for every n. We may also assume that every $\overline{E_n}$ is compact (at most one of the sets is unbounded). Taking a subsequence, we may encounter following two situations.

(a) $o(x_n)\cap S=\{x_n\}$ for all n. In this case, for every n there is $t_n=s_n=0$ and

$$\forall p\in[x_n,x_{n+1}]\qquad L^+(p)=\{z_1\}\ ,\ L^-(p)=\{z_2\},$$

which implies

$$[x_n,x_{n+1}]\subset\Omega^+(z_1)\cap\Omega^-(z_2)\subset Y.$$

The set Y is invariant and contains $\varphi((-\varepsilon,\varepsilon),(x_n,x_{n+1}))$, thus it is not one-dimensional.

(b) $o(x_n)\cap S\supsetneq\{x_n\}$ for all n.
Let r be the Poincaré map on S. Observe that for every n there must be the same number of intersection times of the orbit of x_n with S. Denote that number by k. By the definition of t_n and s_n, there is $r^{k-1}(\varphi(s_n,x_n))=\varphi(t_n,x_n)$ for all n. The map r is continuous, so there exists a neighborhood I of $\varphi(s_n,x_n)$ on S mapped by r^k into a neighborhood J of point $\varphi(t_n,x_n)\subset S$. Then $L^+(p)=\{z_1\}$ and $L^-(p)=\{z_1\}$ for all $p\in I$, so Y is not one-dimensional.

This completes the proof of Theorem 12. $\square$

Theorem 13. *Let Y be a one-dimensional connected component of $CR(\varphi)$ and let $z \in Y$ be a stationary point. The set $A(z) = \{x \in Y \backslash \{z\} \ : \ z \in L(x)\}$ is nonempty and consists of finitely many orbits.*

Proof. As before, we will assume that Y does not contain a point with the empty limit set or periodic orbit. If $x \in Y\backslash\{z\}$, then the set $L(x)$ consists of stationary points. If $z \in L(x)$, then $A(z) \neq \emptyset$; otherwise by **(W3)**, there exists a Jordan arc with end-points z and $p \in L(x)$, which, by one-dimensionality of Y, is invariant set. It implies that $A(z)$ is nonempty.

To prove the remaining claim of Theorem 13, suppose that A contains infinitely many orbits. Then there exists a sequence $\{x_n \mid n \in \mathbb{N}\} \subset Y$ such that $z \in L(x_n)$. Assume that $z \in L^+(x_n)$ (in the other case the proof is analogous).

By **(W1)** stationary points are isolated, so $L^+(x_n) = \{z\}$. Let B be closed ball such that z is the only stationary point of the flow lying in B. Observe that $o(x_n)$ intersects ∂B a finite number of times; otherwise $\partial B \cap L^+(x_n) \neq \emptyset$ which is a contradiction.

Let t_n denote the time of the last intersection of $o(x_n)$ with ∂B, and let $y_n = \varphi(t_n, x_n)$. We may assume that there exists $y \in \partial B$ such that $y_n \to y$. Point y is chain-recurrent and $o^+(y) \subset B$. If there existed $t > 0$ such that $\varphi(t, y) \notin B$, then by continuity of φ, $y_N \notin B$ either for some N large enough, what contradict with $o^+ y_n \subset B$.

There is no periodic point in B (otherwise $z \notin L^+(y_n)$), thus by Poincaré–Bendixson theorem, the set $L^+(y)$ must contain stationary points, which implies that $z \in B$ as it is the only stationary point in B. The point $y \in Y$, so Y is not locally an arc in its nonstationary points, which contradicts the claim of Theorem 12. This ends the proof. □

Remark 14. Every connected component of $CR(\varphi)$ is topologically an infinite graph. For every vertex, there is a finite number of edges terminating at this vertex. Some of the edges may go to the infinity.

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Jagiellonian University
Institute of Mathematics
Reymonta 4
30-059 Kraków, Poland
and
Akademia Górniczo-Hutnicza
Faculty of Applied Mathematics
Al. Mickiewicza 30
30-059 Kraków, Poland
e-mail: oprocha@agh.edu.pl