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THE SEMIDYNAMICAL SYSTEM NEAR A CLOSED NEGATIVELY STRONGLY INVARIANT SET

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Abstract. In this paper we define some kinds of dissipativity of the semidynamical systems. We describe the behaviour of such semidynamical systems in the vicinity of a closed, negatively strongly invariant set in a metric space.

1. Introduction. In a dynamical system motion is defined for positive and negative values of time. In a semidynamical system motion is defined only for positive values of time. However, we can ask about "the past" of a given point x. We may consider "the past" of a point x and investigate the behaviour of the semidynamical system there, as well as negative limit sets $L^-_\sigma(x)$. It is possible that there exist more (even infinitely many) such sets; it depends on a negative semisolutions through x.

In the first part of this paper we define some kinds of dissipativity and investigate connections among them. The situation in semidynamical systems is more complicated than that in dynamical systems, since we must consider not only one trajectory through x, but all negative semitrajectries σ through x. Dissipativity is useful to study persistence, which plays an important role in mathematical ecology.

In the second part we describe the behaviour of a semidynamical system near a closed, negatively strongly invariant set. H. I. Freedman, S. Ruan and M. Tang ([**8**]) investigated the behaviour of a continuous flow in the vicinity of a closed, positively invariant subset in a metric space. Their results generalize the theorems obtained by Ura and Kimura (1960) and Bhatia (1969). In this paper we obtain a similar theorem for the semidynamical system.

Although the problem becomes more complicated because there may be many semisolutions σ through x, our results are similar to those for a dynamical system.

We prove that in each sphere of radius ε and with a centre belonging to a closed, negatively strongly invariant set E we can find such point y, for which there exists a limit set contained in the closed ball $B[E,\varepsilon]$. We only have to assume that there exists a point $x \notin E$ such that the first negative prolongation of x has a non-empty intersection with set E and the semidynamical system is locally negatively strongly dissipative at x. It means that there exist a compact neighbourhood U of x and a compact set V such that all negative semitrajectories through points from U will be eventually contained in V.

A similar theorem, where the set E is replaced by its boundary, is also presented.

Subsequently, several conclusions drawn from the presented theorems are discussed. In closing, two theorems and illustrating examples are given, which give a classification of possible behaviour of the semidynamical system near a closed, negatively strongly invariant set E, and the boundary of such set E, under some assumptions defining the properties of semidynamical system.

2. Definitions and notations. In this section we give some basic notations and definitions on semidynamical systems which we require for this paper.

A *semidynamical system* on a metric space X with metric d is a triplet $(X, \mathbf{R}^+, \pi)$ where $\pi : X \times \mathbf{R}^+ \to X$ is a continuous mapping such that:
(i) $\pi(x, 0) = x$ for all $x \in X$
(ii) $\pi(\pi(x,t), s) = \pi(x, t+s)$ for all $x \in X$ and all $s, t \in \mathbf{R}^+$.
The *positive trajectory* of $x \in X$ is defined as $\{\pi(x,t) : t \in \mathbf{R}^+\}$ and denoted by $\pi^+(x)$.

Replacing $\mathbf{R}^+$ by $\mathbf{R}$ we get a definition of dynamical system. Obviously every dynamical system is a semidynamical system.

A point $x \in X$ is called a *start point* if $x \neq \pi(y,t)$ for any $y \in X$ and any $t > 0$.

A function $\sigma : I \to X$, where I is a non-empty interval in $\mathbf{R}$, is called a *solution* if $\pi(\sigma(t), s) = \sigma(t+s)$ whenever $t \in I$, $t+s \in I$ and $s \in \mathbf{R}^+$. If $0 \in I$ and $\sigma(0) = x$ then a solution is called a *solution through* x. The solution σ through x is called a *left solution through* x if the maximum of the domain of σ is equal to 0. A solution is called a *left maximal solution* if it is a left solution and it is maximal (with respect to inclusion) relative to the property of being a left solution. If a solution σ is maximal (relative to the property of being a solution, with respect to inclusion), then its image is called a *trajectory through* x. Note that in such case $[0, \infty)$ is contained in the domain of a solution.

When the semidynamical system has no start points, then we define a *negative escape time* $N(x)$ of x as

$$N(x) = \inf\{s \in (0, +\infty] : (-s, 0] \text{ is the domain of a left maximal solution through } x\}.$$

Let X be locally compact and the semidynamical system $(X, \mathbf{R}^+, \pi)$ has no start points, then the semidynamical system is isomorphic to a semidynamical system $(X, \mathbf{R}^+, \pi')$ which has infinite negative escape time for each $x \in X$ (see [**6**], compare also [**5**]).

In this paper by a *solution* (through x) we mean a solution with a domain equal to $\mathbf{R}$. By a *positive (negative) semisolution* through x we mean a suitable solution defined on $[0, \infty)$ $((-\infty, 0])$; their images are called *positive* (*negative*) *semitrajectories*. Note that for any x there is precisely one positive semisolution through x, however there may exist even infinitively many negative semisolutions through x.

Let $M \subset X$ be a non-empty set, σ be a negative semitrajectory and there exists $t_0 \leq 0$ such that $\sigma(t_0) \in M$. Then we say that the negative semitrajectory σ *exits the set* M if there exists $T \leq 0$ such that $\sigma(t) \notin M$ for any $t < T$.

Let $A \subset \mathbf{R}^+$ and $M \subset X$. Let us put $F(M, A) = \{y \in X : \pi(y, t) \in M \text{ for some } t \in A\}$. If $M = \{x\}$ and $A = \{t\}$, we write $F(x, t)$ instead of $F(\{x\}, \{t\})$. If the semidynamical system $(X, \mathbf{R}^+, \pi)$ is defined on a locally compact metric space without start points and have infinite negative escape time for each $x \in X$ then the function $F : X \times \mathbf{R}^+ \to \mathcal{P}(X)$ is upper semicontinuous [**4**], i.e., for every $x \in X$ and for any sequences $\{x_n\}$ in X with $x_n \to x$ and $\{t_n\}$ in $\mathbf{R}^+$ with $t_n \to t$, $\sup\{d(y, F(x, t)) : y \in F(x_n, t_n)\} \to 0$ as $n \to +\infty$.

A set $M \subset X$ is called:

- *positively invariant* if $\pi(x, t) \in M$ for any $x \in M$ and any $t \in \mathbf{R}^+$;
- *negatively strongly invariant* if $\sigma((-\infty, 0]) \subset M$ for any $x \in M$ and any negative semisolution σ through x;
- *negatively weakly invariant* if for every $x \in M$ there exists a negative semisolution σ through x such that $\sigma((-\infty, 0]) \subset M$.

A set $M \subset X$ is called *strongly* (*weakly*) *invariant* if it is positively invariant and negatively strongly (weakly) invariant.

It is easy to see that for any x the positive trajectory $\pi^+(x)$ is positively invariant and the set $\sigma((-\infty, 0])$ is negatively weakly invariant for any solution σ through x.

For any $\varepsilon > 0$ and $M \subset X$, we define:

$$B(M, \varepsilon) = \{x : x \in X \text{ and } d(x, M) < \varepsilon\},$$
$$B[M, \varepsilon] = \{x : x \in X \text{ and } d(x, M) \leq \varepsilon\},$$
$$S(M, \varepsilon) = \{x : x \in X \text{ and } d(x, M) = \varepsilon\}.$$

The boundary, closure and interior of a set $M \subset X$ are denoted ∂M, $\overline{M}$ and $\operatorname{int} M$, respectively.

The limit sets, prolongations and prolongational limit sets of a point $x \in X$ are defined as follows.

DEFINITION 2.1. By a *positive limit set* of $x \in X$ we mean

$$L^+(x) = \{y \in X : \text{there exists a sequence } \{t_n\} \text{ in } \mathbf{R} \\ \text{with } t_n \to +\infty \text{ and } \pi(x, t_n) \to y\}.$$

By a *negative limit set* of $x \in X$ with respect to a solution σ we define

$$L^-_\sigma(x) = \{y \in X : \text{there exists a sequence } \{t_n\} \text{ in } \mathbf{R} \\ \text{with } t_n \to -\infty \text{ and } \sigma(t_n) \to y\},$$

where σ is a negative semisolution through x.
For each $x \in X$, the set

$$D^+(x) = \{y \in X : \text{ there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence} \\ \{t_n\} \text{ in } \mathbf{R}^+ \text{ such that } x_n \to x \text{ and } \pi(x_n, t_n) \to y\}$$

is called the *first positive prolongation* of x.
The *first negative prolongation* of x we can defined in two ways.

$$d^-(x) = \{y \in X : \text{there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence } \{t_n\} \\ \text{in } \mathbf{R}^- \text{such that } x_n \to x \text{ and for each } x_n \text{there exists} \\ \text{a semisolution } \sigma_n \text{ through } x_n \text{ such that } \sigma_n(t_n) \to y\},$$

$$D^-(x) = \{y \in X : \text{there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence } \{t_n\} \\ \text{in } \mathbf{R}^- \text{ and there exists a semisolution } \sigma_x \text{ through } x \\ \text{and } t \leq 0 \text{ such that } x_n \to \sigma_x(t) \text{ and for each } x_n \text{ there} \\ \text{exists a semisolution } \sigma_n \text{ through } x_n \text{ such that } \sigma_n(t_n) \to y\}.$$

The *positive prolongational limit set* of $x \in X$ is defined as

$$J^+(x) = \{y \in X : \text{there are a sequence} \{x_n\} \text{ in} X \text{ and a sequence } \{t_n\} \\ \text{in } \mathbf{R}^+ \text{ such that } x_n \to x, t_n \to +\infty \text{ and } \pi(x_n, t_n) \to y\}.$$

We define the *negative prolongational limit set* of $x \in X$ as

$$j^-(x) = \{y \in X : \text{there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence } \{t_n\} \text{ in } \mathbf{R}^- \\ \text{such that } x_n \to x, t_n \to -\infty \text{ and for each } x_n \text{ there} \\ \text{exists a semisolution } \sigma_n \text{ through } x_n \text{ such that } \sigma_n(t_n) \to y\},$$

$$J^-(x) = \{y \in X : \text{there are a sequence } \{x_n\} \text{ in } X \text{ and a sequence } \{t_n\} \text{ in } \mathbf{R}^- \text{ and there exists a semisolution } \sigma_x \text{ through } x \text{ and } t \le 0 \text{ such that } x_n \to \sigma_x(t), t_n \to -\infty \text{ and for each } x_n \text{ there exists a semisolution } \sigma_n \text{ through } x_n \text{ such that } \sigma_n(t_n) \to y\}.$$

We know that $j^-(x)$ and $J^-(x)$ are equal (see [**3**]). We will prove that $d^-(x) = D^-(x)$. Obviously, for any $x \in X$, we have $L_\sigma^-(x) \subset j^-(x) \subset d^-(x)$ for any semisolution σ through x. This is an immediate consequence of the definitions.

Theorem 2.2. *Let $x \in X$. Then $D^-(x) = d^-(x)$.*

Proof. The property $d^-(x) \subset D^-(x)$ is obvious.

Let $y \in D^-(x)$. It means that there are a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in $\mathbf{R}^-$, a solution σ_x through x and $t \le 0$ such that $x_n \to \sigma_x(t)$ and for each x_n there exists a semisolution σ_n through x_n such that $\sigma_n(t_n) \to y$. We may assume that either $t_n \to -\infty$ or $t_n \to \tau \in \mathbf{R}^-$, taking subsequences if necessary. In the first case $y \in J^-(x)$ and so $y \in j^-(x) \subset d^-(x)$. In the second case $\pi(x_n, -t) \to \pi(\sigma_x(t), -t) = \sigma_x(0) = x$. Set $\tilde{x_n} = \pi(x_n, -t)$, then there exists a solution through $\tilde{x_n}$ which contains $\tilde{x_n}$ and x_n in its image. We denote this solution by $\tilde{\sigma_n}$. Hence $\tilde{x_n} \to x$ and $\tilde{\sigma_n}(t + t_n) = \sigma_n(t_n) \to y$. The sequence $\{t + t_n\} \in \mathbf{R}^-$ and $t + t_n \to t + \tau$. Consequently, $y \in d^-(x)$. $\square$

Lemma 2.3. ([**1**], 5.15.) *A negative limit set $L_\sigma^-(x)$ is closed, positively invariant and if X is locally compact, then it is weakly invariant and contains no start points.*

Lemma 2.4. *A negative prolongational limit set of x and first negative prolongation of x are closed, positively invariant and if X is locally compact, then they are weakly invariant.*

For the above results we refer to [**3**] and to S. Elaydi and S.K. Kaul [**7**]. Although they stated another definitions of $J^-(x)$ and $D^-(x)$, after easy verification we see that those definitions are equivalent to the presented here.

3. Dissipativity. In this section, we give some basic definitions of some types of negative dissipativity and their mutual relations. We assume that the semidynamical system π on a locally compact metric space X without start points has an infinite negative escape time $N(x) = +\infty$ for each point $x \in X$.

Definition 3.1. If for any negative semisolution σ_x through x the set $\overline{\sigma_x}$ is compact, then the semidynamical system π is said to be *negatively quasi-dissipative at x.*

DEFINITION 3.2. If there exists a negative semisolution σ_x through x such that the set $\overline{\sigma_x}$ is compact, then the semidynamical system π is said to be *negatively σ_x quasi-dissipative at x.*

Note that if π is negatively σ_x quasi-dissipative at x then there may exist other negative semisolution σ'_x through x for which $\overline{\sigma'_x}$ is not compact.

It is easy to see that if π is negatively quasi-dissipative at x then it is negatively σ_x quasi-dissipative at x for any semisolution σ_x through x.

If the semidynamical system π is negatively σ quasi-dissipative at x, then the negative limit set $L^-_\sigma(x)$ is nonempty, compact, connected and weakly invariant ([**1**], 5.5, 5.15).

DEFINITION 3.3. Let x be given point in X. If there exist a compact neighbourhood U of x and a compact set V such that there exists $t(U) > 0$ with $F(U, [t(U), +\infty)) \subset \operatorname{int} V$, then the semidynamical system π is said to be *locally negatively strongly dissipative at x.*

As an obvious consequence of this definition we get

PROPOSITION 3.4. *If the semidynamical system π is locally negatively strongly dissipative at x with corresponding sets U and V, then for any $y \in U$ there exists $t(y) > 0$ such that $F(y, [t(y), +\infty)) \subset \operatorname{int} V$.*

PROPOSITION 3.5. *If the semidynamical system π is locally negatively strongly dissipative at x then it is negatively quasi-dissipative at x.*

PROOF. The semidynamical system π is locally negatively strongly dissipative at x so there exist a compact neighbourhood U of x and a compact set V such that for any $y \in U$, there is a $t(y) > 0$ such that $F(y, [t(y), +\infty)) \subset \operatorname{int} V$. Since $x \in U$ there is a $t(x) > 0$ such that $F(x, [t(x), +\infty)) \subset \operatorname{int} V$. Thus for any semisolution σ_x through x we have

$$\overline{\sigma_x((-\infty, t(x)])} \subset \overline{\operatorname{int} V} \subset V.$$

Hence $\overline{\sigma_x((-\infty, t(x)])}$ is compact, as it is a closed subset of compact set, and so $\overline{\sigma_x}$ is compact. □

DEFINITION 3.6. The semidynamical system π is *pointwise negatively strongly dissipative over a nonempty set $M \subset X$* if there exists a compact set $N \subset X$ such that for any $y \in M$ there exists $t(y) > 0$ such that $F(y, [t(y), +\infty)) \subset \operatorname{int} N$.

If the semidynamical system π is pointwise negatively strongly dissipative over M then π may not be locally negatively strongly dissipative at x for any $x \in M$. For example, consider the planar differential system

$$x'_1(t) = -x_1 \text{ and } x'_2(t) = x_2$$

If $M = \{(0,1)\}$ then the semidynamical system is pointwise negatively strongly dissipative over M, but it is not locally negatively strongly dissipative at $x = (0,1)$. If the semidynamical system π is locally negatively strongly dissipative at x then it is pointwise negatively strongly dissipative over M for $M = \{x\}$ or $M = U_x$.

PROPOSITION 3.7. *If the semidynamical system π is pointwise negatively strongly dissipative over M then for any $x \in M$ it is negatively quasi-dissipative at x.*

The proof will be omitted because it is simple and similar to the proof of Proposition 3.5.

DEFINITION 3.8. A nonempty subset $M \subset X$ is called an *isolated set with* $\varepsilon > 0$ if for any weakly invariant set N contained in $B[M,\varepsilon]$ we have $N \subset M$.
We say that M is an *isolated* if it is isolated with ε for some $\varepsilon > 0$.

Note that in Definition 3.8 it is not required that there exists a weakly invariant set contained in M.

4. Semidynamical systems near a closed negatively strongly invariant set. In the following we consider a semidynamical system $(X, \mathbf{R}^+, \pi)$ on a locally compact metric space X and we assume that π has no start points and the infinite negative escape time $N(x) = +\infty$ for each point $x \in X$.
We discuss the behaviour of this semidynamical system near a closed, negatively strongly invariant set $E \subset X$.

THEOREM 4.1. *Let E be a closed, negatively strongly invariant subset of X and x be a point in X with $d(x,E) > 0$. Suppose that the semidynamical system π is locally negatively strongly dissipative at x and $D^-(x) \cap E \neq \emptyset$. Then for any $0 < \varepsilon < d(x,E)$ there exist $y \in S(E,\varepsilon)$ and a negative semisolution σ through y such that $L^-_\sigma(y) \subset B[E,\varepsilon]$.*

PROOF. Take $z \in D^-(x) \cap E$. Then there exist sequences $\{x_n\} \subset X$ and $\{t_n\} \subset \mathbf{R}^-$ such that $x_n \to x$ and for each x_n there exists a semisolution σ_n through x_n such that $\sigma_n(t_n) \to z$ as $n \to +\infty$. Since π is locally negatively strongly dissipative at x, we can choose a closed neighbourhood U_x of x and a compact set V such that $U_x \cap B[E,\varepsilon] = \emptyset$ and $F(U_x, [t(U_x), +\infty)) \subset \operatorname{int} V$, where $0 < \varepsilon < d(x,E)$. Then we can enlarge the set V to the set V_x so that $F(U_x, \mathbf{R}^+) \subset \operatorname{int} \mathbf{V_x}$, where V_x is also a compact set. Also, we can choose a compact neighborhood U_z of z such that $U_z \subset B[E, \frac{\varepsilon}{2}]$. Without loss of generality, we may assume that $\{x_n\} \subset U_x$ and $\{\sigma_n(t_n)\} \subset U_z$. Let $z_n = \sigma_n(t_n)$. Then $\{z_n\} \subset U_z$ and $z_n \to z$. The negative semisolution σ_n through x_n must "meet" the set $S(E,\varepsilon)$ between $t = 0$ and $t = t_n$. Define τ_n as t which fulfils following properties:

$$t_n < t < 0,\ \sigma_n(t) \in S(E,\varepsilon),\ \sigma_n((t_n,t)) \in B(E,\varepsilon).$$

Note that for any n there exists exactly one t with this property. Clearly $t_n < \tau_n < 0$. Let $y_n = \sigma_n(\tau_n)$, then $\pi(z_n, \tau_n - t_n) = y_n$ and $y_n \in S(E,\varepsilon)$.

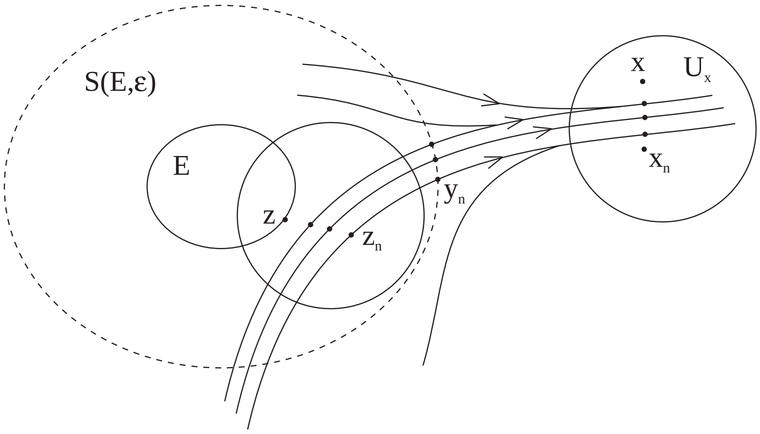

FIGURE 1.

If there exist a $y_n \in S(E,\varepsilon)$ and a negative semitrajectory σ through y_n such that $\sigma(t) \in B[E,\varepsilon]$ for all $t \in \mathbf{R}^-$, then let $y = y_n$ and $L^-_\sigma(y) \subset B[E,\varepsilon]$.

Assume that for every y_n all negative semitrajectories through y_n exit the set $B[E,\varepsilon]$. If there exist an x_n and a negative semisolution $\tilde{\sigma}$ through x_n and $\tilde{t} < 0$ such that $\tilde{\sigma}(\tilde{t}) \in S(E,\varepsilon)$, $\tilde{\sigma}(t_n) \neq z_n$ and for all $t < \tilde{t}$ we have $\tilde{\sigma}(t) \in B[E,\varepsilon]$, let $y = \tilde{\sigma}(\tilde{t})$. Then $L^-_{\tilde{\sigma}}(y) \subset B[E,\varepsilon]$.

For the cases above the theorem is proved.

Therefore we suppose that for any x_n every negative semitrajectory through x_n exits the set $B[E,\varepsilon]$. For any n we consider this negative semitrajectory which contains x_n, y_n and z_n in its image. Obviously, this negative semitrajectory also exits the set $B[E,\varepsilon]$. From the point x_n to the point z_n this trajectory is unequivocally determined. Then there exists precisely one such semitrajectory, however there may exist even infinitively many such negative semitrajectories. For every x_n we denote these semitrajectories as σ_{n_k}.

We know that for every point x_n, there is an $s_n < t_n$ satisfying

$$s_n = \max\{t : -\infty < t < t_n, \sigma_{n_k}(t) \in S(E,\varepsilon), \sigma_{n_k}((t,t_n)) \in B(E,\varepsilon)\},$$

where σ_{n_k} are negative semisolutions through x_n for which $\sigma_{n_k}(t_n) = z_n$.

For any x_n there exists a semitrajectory $\sigma_{n_{\tilde{k}}}$ such that $\sigma_{n_{\tilde{k}}}(s_n) \in S(E,\varepsilon)$. Denote this semitrajectory as σ_n. Notice that it is unequivocally determined from the point x_n to the point $\sigma_n(s_n)$.

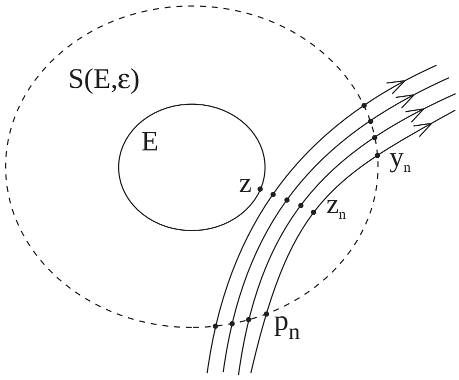

FIGURE 2.

We denote $p_n = \sigma_n(s_n)$. Then $p_n \in S(E,\varepsilon)$ and $-\infty < s_n < t_n < \tau_n < 0$. Note that $\{y_n\} \subset V_x \cap S(E,\varepsilon)$ and $\{p_n\} \subset V_x \cap S(E,\varepsilon)$ and since $V_x \cap S(E,\varepsilon)$ is compact, we can choose a convergent subsequence of $\{y_n\}$ and a convergent subsequence of $\{p_n\}$ which we also rewrite as $\{y_n\}$ and $\{p_n\}$. Then there exist $y \in S(E,\varepsilon)$ and $p \in S(E,\varepsilon)$ such that

$$\lim_{n\to+\infty} y_n = y\ , \quad \lim_{n\to+\infty} p_n = p\ .$$

We know that $s_n - \tau_n < s_n - t_n$. Now we prove that $s_n - t_n \to -\infty$ as $n \to +\infty$, hence $s_n - \tau_n \to -\infty$. If this is not true, we could find a sequence of the form $\{s_n - t_n\}$ and a $T < 0$; without loss of generality we may assume that $s_n - t_n \to T$ as $n \to +\infty$. Then $t_n - s_n \to -T > 0$ as $n \to +\infty$ and $z = \lim_{n\to+\infty} z_n = \lim_{n\to+\infty} \pi(p_n, t_n - s_n) = \pi(p, -T)$. Therefore $p \in F(z, -T)$. This is impossible since $z \in E$, $p \notin E$ and E is negatively strongly invariant. So $s_n - \tau_n \to -\infty$.

Denote now as σ_{y_n} a semisolution through y_n for which $\sigma_{y_n}(0) = y_n = \sigma_n(\tau_n)$ and $\sigma_{y_n}(t) = \sigma_n(\tau_n + t)$ for any $t < 0$. Following, we notice that

$$p_n = \sigma_{y_n}(s_n - \tau_n) \ \text{ and } \ (s_n - \tau_n) \to -\infty\ .$$

Hence for any $t < 0$ there exists an $N_t > 0$ such that for any $n > N_t$ we have $\sigma_{y_n}(t) \in B[E,\varepsilon]$. On the other hand, the function $F(\cdot,\cdot)$ is upper semicontinuous, $y_n \to y$ and if we define $\tilde{t}_n$ as a constant sequence we have $\tilde{t}_n = m \to m$ for some $m > 0$. Then

$$\sup\{d(\chi, F(y,m)) : \chi \in F(y_n, m)\} \to 0 \text{ as } n \to +\infty\ .$$

Since $\sigma_{y_n}(-m) \in F(y_n, m)$, we obtain $d(\sigma_{y_n}(-m), F(y,m)) \to 0$ as $n \to +\infty$. Take $m = 1$. We have $d(\sigma_{y_n}(-1), F(y,1)) \to 0$ as $n \to +\infty$. Since $\sigma_{y_n}(-1) \in \operatorname{int} V_x$ for any $n \in \mathbf{N}$ we can choose a convergent subsequence of $\{\sigma_{y_n}(-1)\}$

which we rewrite as $\{\sigma_{y_n^1}(-1)\}$. So there exists $\tilde{y}^1$ such that $\sigma_{y_n^1}(-1) \to \tilde{y}^1$. We know that $\tilde{y}^1 \in V_x \cap F(y,1)$, so there exists a negative semisolution $\tilde{\sigma}_y$ through y such that $\tilde{y}^1 = \tilde{\sigma}_y(-1) \in B[E,\varepsilon]$, since $\sigma_{y_n^1}(-1) \in B[E,\varepsilon]$ for any $n > N_1$. When this reasoning is repeated once more we obtain

$$\sup\{d(\chi, F(\tilde{y}^1, m)) : \chi \in F(\sigma_{y_n^1}(-1), m)\} \to 0 \text{ as } n \to +\infty \ ,$$

and then $d(\sigma_{y_n^1}(-1-m), F(\tilde{y}^1, m)) \to 0$ as $n \to +\infty$, since $\sigma_{y_n^1}(-1-m) \in F(\sigma_{y_n^1}(-1), m)$. For $m = 1$ we have $d(\sigma_{y_n^1}(-2), F(\tilde{y}^1, 1)) \to 0$ as $n \to +\infty$. As previously $\sigma_{y_n^1}(-2) \in \operatorname{int} V_x$ and we can choose a convergent subsequence of $\{\sigma_{y_n^1}(-2)\}$ which we rewrite as $\{\sigma_{y_n^2}(-2)\}$. So there exists $\tilde{y}^2$ such that $\sigma_{y_n^2}(-2) \to \tilde{y}^2$. We know that $\tilde{y}^2 \in V_x \cap F(\tilde{y}^1, 1)$, so $\tilde{y}^2 = \tilde{\sigma}_y(-2) \in B[E,\varepsilon]$, since $\sigma_{y_n^2}(-2) \in B[E,\varepsilon]$ for any $n > N_2$. Repeating this reasoning again we obtain points $\tilde{y}^k = \tilde{\sigma}_y(-k) \in B[E,\varepsilon]$ for any $k \in \mathbf{N}$, where $\tilde{\sigma}_y$ is a negative semisolution through y. From the continuity of the function π we get $\pi(\tilde{y}^k, t) = \lim_{n\to+\infty} \pi(\sigma_{y_n^k}(-k), t)$ for any $t \in [0,1)$, and $\pi(\sigma_{y_n^k}(-k), t) \in B[E,\varepsilon]$. Hence $\pi(\tilde{y}^k, t) \in B[E,\varepsilon]$ for any $k \in \mathbf{N}$ and $t \in [0,1)$. Therefore $\tilde{\sigma}_y(-u) \in B[E,\varepsilon]$ for any $u > 0$ and then $L^-_{\tilde{\sigma}_y}(y) \subset B[E,\varepsilon]$. This completes the proof. $\square$

Corollary 4.2. *Adopt the assumptions of Theorem 4.1 and designations of x_n, y_n, z_n and τ_n, t_n, s_n as defined in the proof of Theorem 4.1. Additionally, we assume that for every y_n all negative semitrajectories through y_n exit the set $B[E,\varepsilon]$. Then for the limit point p of $\{p_n\}$ we have $L^+(p) \subset B[E,\varepsilon]$ and $p \in J^-(x)$. For the limit point y of $\{y_n\}$ we have $y \in D^-(x)$.*

Proof. We adopt the designations defined in the proof of Theorem 4.1. We know that $s_n - \tau_n \to -\infty$ as $n \to +\infty$, hence $\tau_n - s_n \to +\infty$. We have also $\pi(p_n, \tau_n - s_n) = y_n$. Hence for any $t > 0$ there exists an $N_t > 0$ such that for any $n > N_t$, we have $\pi(p_n, t) \in B[E,\varepsilon]$. Since $\lim_{n\to+\infty} \pi(p_n, t) = \pi(p, t)$, then $\pi(p, t) \in B[E,\varepsilon]$ for any $t > 0$. Therefore $\pi^+(p) \in B[E,\varepsilon]$ and then $L^+(p) \subset B[E,\varepsilon]$.

If $x_n \to x$ then for any x_n there is a semisolution σ_n through x_n such that $\sigma_n(s_n) = p_n$ and $p_n \to p$. It is clear that $s_n \to -\infty$ since $s_n < s_n - \tau_n$. Therefore $p \in J^-(x)$.

If $x_n \to x$ then for any x_n there is a semisolution σ_n through x_n such that $\sigma_n(\tau_n) = y_n$ and $y_n \to y$. It is clear that $\{\tau_n\} \subset \mathbf{R}^-$. Therefore $y \in D^-(x)$. $\square$

Remark 1. If the set E is isolated with $\alpha > 0$ in addition to the assumption of Theorem 4.1, then for any $0 < \varepsilon < \min\{\alpha, d(x,E)\}$ there are a $y \in S(E,\varepsilon)$ and a negative semisolution σ through y such that $L^-_\sigma(y) \subset E$.

Proof. It is true since for any $y \in S(E,\varepsilon)$ and for any negative semisolution σ through y the set $L^-_\sigma(y)$ is negatively weakly invariant. $\square$

THEOREM 4.3. *Let E be a nonempty closed subset of X and x be a point in X with $d(x, E) > 0$. Suppose that the semidynamical system π is locally negatively strongly dissipative at x and $D^-(x) \cap E \neq \emptyset$. Let $X \setminus E$ be negatively strongly invariant. Then for any $0 < \varepsilon < d(x, E)$, there exist $y \in S(E, \varepsilon)$ and a negative semisolution σ through y such that $L^-_\sigma(y) \subset B[E, \varepsilon]$.*

PROOF. The proof is similar to that of Theorem 4.1. In this case, after constructing sequences $\{\tau_n\}$, $\{t_n\}$ and $\{s_n\}$ similar to those constructed in the proof of Theorem 4.1, we can show that $s_n - \tau_n \to -\infty$. We know that $s_n - \tau_n < t_n - \tau_n$. Now we prove that $t_n - \tau_n \to -\infty$ as $n \to +\infty$, hence $s_n - \tau_n \to -\infty$. If this is not true, we could find a sequence of the form $\{t_n - \tau_n\}$ and a $T < 0$; without loss of generality we may assume that $t_n - \tau_n \to T$ as $n \to +\infty$. Then we have that $\tau_n - t_n \to -T > 0$ as $n \to +\infty$ and $y = \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \pi(z_n, \tau_n - t_n) = \pi(z, -T)$. Therefore $z \in F(y, -T)$. This is impossible since $z \in E$, $y \in X \setminus E$ and $X \setminus E$ is negatively strongly invariant. So $s_n - \tau_n \to -\infty$. The further part of the proof is similar to that of Theorem 4.1. □

If M is a closed, negatively strongly invariant subset of X with nonempty boundary ∂M and nonempty interior $\operatorname{int} M$, then $\operatorname{int} M$ is also negatively strongly invariant, but ∂M is in general not negatively strongly invariant. To prove this we need

LEMMA 4.4. *Let M be a subset of X. Then the following conditions are equivalent*

(i) *M is negatively strongly invariant;*
(ii) *$X \setminus M$ is positively invariant.*

PROOF. Assume (i). Let $x \in X \setminus M$. Suppose that there exists $t \in \mathbf{R}^+$ such that $\pi(x, t) \in M$. Then there exists a semisolution $\sigma_{\pi(x,t)}$ through $\pi(x, t)$ such that $\sigma_{\pi(x,t)}(-t) = x \in X \setminus M$. According to (i) this is impossible.
Now assume (ii). Let $x \in M$ and $t \in \mathbf{R}^-$. Suppose that there exists a semisolution σ_x through x such that $\sigma_x(t) \notin M$. Hence $\sigma_x(t) \in X \setminus M$ and $\pi(\sigma_x(t), -t) = \sigma_x(t - t) = x \in M$. This contradicts the positively invariance of $X \setminus M$. This completes the proof. □

LEMMA 4.5. ([**1**]; 3.4.1) *If M is positively invariant then $\overline{M}$ is also positively invariant.*

LEMMA 4.6. *If M is negatively strongly invariant then* $\operatorname{int} M$ *is also negatively strongly invariant.*

PROOF. Since M is negatively strongly invariant then $X \setminus M$ is positively invariant, so $\overline{X \setminus M}$ is positively invariant and finally $\operatorname{int} M = X \setminus (\overline{X \setminus M})$ is negatively strongly invariant. □

We have also

THEOREM 4.7. *Let E be a closed, negatively strongly invariant subset of X with $\partial E \neq \emptyset$ and $\operatorname{int} E \neq \emptyset$. Let $x \in \operatorname{int} E$ and the semidynamical system π be locally negatively strongly dissipative at x. If $D^-(x) \cap \partial E \neq \emptyset$, then for any $0 < \varepsilon < d(x, \partial E)$, there exists $y \in S(\partial E, \varepsilon)$ and a semisolution σ through y such that $L^-_\sigma(y) \subset B[\partial E, \varepsilon]$.*

PROOF. The proof is similar to that of Theorem 4.1. Since $x \in \operatorname{int} E$ we choose a neighborhood U_x of x such that $U_x \cap B[\partial E, \varepsilon] = \emptyset$, where $0 < \varepsilon < d(x, \partial E)$, $U_x \subset \operatorname{int} E$ and $F(U_x, \mathbf{R}^+) \subset \operatorname{int} \mathbf{V_x}$, where V_x is compact set. Also, we can choose a closed neighborhood U_z of z such that $U_z \subset B[\partial E, \frac{\varepsilon}{2}]$. In this case we construct sequences $\{\tau_n\}$, $\{t_n\}$, $\{s_n\}$, $\{y_n\}$ and $\{p_n\}$ similar to those constructed in the proof of Theorem 4.1. We consider the set ∂E instead of E. Note that $\{y_n\} \subset V_x \cap S(\partial E, \varepsilon)$ and $\{p_n\} \subset V_x \cap S(\partial E, \varepsilon)$ and since $V_x \cap S(\partial E, \varepsilon)$ is compact, we can choose a convergent subsequence of $\{y_n\}$ and a convergent subsequence of $\{p_n\}$ which we also rewrite as $\{y_n\}$ and $\{p_n\}$. Then there exist $y \in S(\partial E, \varepsilon)$ and $p \in S(\partial E, \varepsilon)$ such that

$$\lim_{n \to +\infty} y_n = y \ , \quad \lim_{n \to +\infty} p_n = p \ .$$

Observe that for any n we have $y_n \in \operatorname{int} E$ and $y \in \operatorname{int} E$. So as in the proof of Theorem 4.3 we show that $s_n - \tau_n \to -\infty$. We obtain that $z \in F(y, -T)$. This is impossible since $z \in \partial E$, $y \in \operatorname{int} E$ and the set $\operatorname{int} E$ is negatively strongly invariant. The further part of the proof is similar to that of Theorem 4.1. □

Note that for any $x \in X$ we have $L^-_\sigma(x) \subset J^-(x) \subset D^-(x)$, where σ is a semisolution through x. Hence the following corollaries hold.

COROLLARY 4.8. *The conclusions of Theorems 4.1, 4.3, 4.7 hold if the set $D^-(x)$ is replaced by $J^-(x)$.*

COROLLARY 4.9. *The conclusions of Theorems 4.1, 4.3, 4.7 hold if the set $D^-(x)$ is replaced by $L^-_{\sigma_x}(x)$, where σ_x is a semisolution through x.*

PROOF. The proof is similar to that of Theorem 4.1 (respectively 4.3, 4.7). The difference is that $z \in L^-_{\sigma_x}(x) \cap E$ (we consider the set ∂E instead of E when we prove the conclusion of Theorem 4.7), every point in $\{x_n\}$ is x and $z_n = \sigma_x(t_n) \to z$, where σ_x is a semisolution through x for which $\overline{\sigma_x((-\infty, 0])}$ is compact and $t_n \to -\infty$. In this case a semisolution σ_n from the proof of Theorem 4.1 is a semisolution σ_x. So $y_n = \sigma_x(\tau_n)$, where τ_n is defined as previously. We change also the definition of s_n. We define s_n as t which fulfills the following properties:

$$-\infty < t < t_n, \;\; \sigma_x(t) \in S(E, \varepsilon), \;\; \sigma_x((t, t_n)) \in B[E, \varepsilon] \ .$$

We denote $p_n = \sigma_x(s_n)$. In this case the points y_n and p_n belong to the semitrajectory σ_x. Hence we know that there exist $y \in S(E,\varepsilon)$ and $p \in S(E,\varepsilon)$ such that

$$\lim_{n\to+\infty} y_n = y\ ,\quad \lim_{n\to+\infty} p_n = p\ .$$

The further part of the proof is such as this of Theorem 4.1 (respectively 4.3, 4.7). □

By Theorem 4.1, Remark 1, Corollary 4.2 and 4.9, we also have the following.

Corollary 4.10. *Suppose E is a closed, negatively strongly invariant subset of X isolated with $\alpha > 0$. Let x be a point in X such that: $x \notin E$, there exists a solution σ_x through x such that $L^-_{\sigma_x}(x) \cap E \neq \emptyset$ and the semidynamical system is locally negatively strongly dissipative at x. If there exists $x_0 \in L^-_{\sigma_x}(x) \setminus E$ then for any $0 < \varepsilon < \min\{\alpha, d(x_0,E)\}$, there exist points $p \in S(E,\varepsilon) \cap L^-_{\sigma_x}(x)$, $y \in S(E,\varepsilon) \cap L^-_{\sigma_x}(x)$ and the solution σ_y through y such that $L^+(p) \subset E$ and $L^-_{\sigma_y}(y) \subset E$.*

Proof. From Theorem 4.1 we know that for any $0 < \varepsilon_1 < d(x,E)$, there exist $y \in S(E,\varepsilon_1)$ and a negative semisolution σ through y such that $L^-_\sigma(y) \subset B[E,\varepsilon_1]$. From the proof of Corollary 4.9 we know also that $y = \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} \sigma_x(\tau_n)$ and since $\tau_n \to -\infty$ we have $y \in L^-_{\sigma_x}(x)$. The existence of the point $x_0 \in L^-_{\sigma_x}(x) \setminus E$ ensure that the semitrajectory $\sigma_x \not\subset B[E,\varepsilon_2]$, where $\varepsilon_2 < d(x_0,E)$. We define the points p_n as in the proof of Collorary 4.9. Hence there exists $p \in S(E,\varepsilon_2)$ such that $p = \lim_{n\to+\infty} p_n = \lim_{n\to+\infty} \sigma_x(s_n)$ and since $s_n \to -\infty$ we have $p \in L^-_{\sigma_x}(x)$. From Corollary 4.2 we know that $L^+(p) \subset B[E,\varepsilon_2]$. From Remark 1 we know that $L^-_\sigma(y) \subset E$ and $L^+(p) \subset E$ if $L^-_\sigma(y) \subset B[E,\alpha]$ and $L^+(p) \subset B[E,\alpha]$, since $L^-_\sigma(y)$ and $L^+(p)$ are weakly invariant and the set E is isolated with $\alpha > 0$. So the Collorary is true with $\varepsilon < \min\{\alpha, d(x_0,E)\}$ if $d(x_0,E) < d(x,E)$. If $d(x_0,E) > d(x,E)$ then the Collorary is also true with $\varepsilon < \min\{\alpha, d(x_0,E)\}$. This holds since $x_0 \in L^-_{\sigma_x}(x)\setminus E$ and $L^-_{\sigma_x}(x) \cap E \neq \emptyset$ so the semitrajectory σ_x leaves the set $B[E,\varepsilon]$ at the points p_n and enters at the points y_n infinitely often (where the points p_n and y_n are defined so as in Collorary 4.9). In this case we can find the points y and p in the same way as in Theorem 4.1 which completes the proof. □

Theorem 4.11. *Let E be a closed, negatively strongly invariant set. Suppose that there exists $\alpha > 0$ such that semidynamical system π is locally negatively strongly dissipative at each point of $B[E,\alpha]\setminus E$. Then one of the following statements holds*

(i) *The set E is not isolated, that is, for any $\varepsilon > 0$, there exists a weakly invariant set $K \subset B[E,\varepsilon]$ and $K \not\subset E$.*

(ii) *There exists* $y \in B[E,\alpha] \setminus E$ *and there exists a semisolution* σ_y *through* y *such that* $L^-_{\sigma_y}(y) \subset E$.

(iii) *There is an* $\varepsilon > 0$ *such that for any* $x \in B[E,\alpha] \setminus E$ *and for any semisolution* σ_x *through* x, $\lim_{t\to-\infty} d(\sigma_x(t), E) \geq \varepsilon$.

Proof. Assume that (i) and (ii) do not hold. We show that in such case (iii) holds.

We can choose $0 < \delta < \alpha$ such that for any weakly invariant set K, if $K \subset B[E,\delta]$ then $K \subset E$.

If there exist $x \in B[E,\alpha] \setminus E$ and a semisolution σ_x through x such that $L^-_{\sigma_x}(x) \cap E \neq \emptyset$, then take $0 < \varepsilon_0 < \min\{d(x,E), \delta\}$. From Corollary 4.9 and Remark 1, there exist $y \in S(E,\varepsilon_0)$ and a semisolution σ_y through y, such that $L^-_{\sigma_y}(y) \subset E$, which is impossible since $y \in B[E,\alpha] \setminus E$ and (ii) is not true. Hence for any $x \in B[E,\alpha] \setminus E$ and for any semisolution σ_x through x we have $L^-_{\sigma_x}(x) \cap E = \emptyset$. Moreover, for any $x \notin E$ and for any semisolution σ_x through x we have $L^-_{\sigma_x}(x) \cap E = \emptyset$.

Since the semidynamical system π is locally negatively strongly dissipative at each point of $B[E,\alpha] \setminus E$, we can find a compact set N such that for any $y \in B[E,\alpha] \setminus E$, there exist $T_y > 0$ and a neighbourhood U_y of y such that $F(U_y, [T_y, +\infty)) \subset \operatorname{int} N$. We may choose U_y such that $U_y \subset B[E,\alpha] \setminus E$.

Choose a sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n < \delta$ such that $\lim_{n\to+\infty} \varepsilon_n = 0$. If (iii) is not true, then for any ε_n we can find $x_n \in B[E,\alpha] \setminus E$ and we can find a semisolution σ_{x_n} through x_n, such that $L^-_{\sigma_{x_n}}(x_n) \cap S(E,\varepsilon_n) \neq \emptyset$. In this case, we must have $L^-_{\sigma_{x_n}}(x_n) \cap S(E,\delta) \neq \emptyset$. Otherwise $L^-_{\sigma_{x_n}}(x_n) \subset B[E,\delta]$ and then $L^-_{\sigma_{x_n}}(x_n) \subset E$, which is impossible. So we have

$$\inf\{d(y,E), y \in L^-_{\sigma_{x_n}}(x_n)\} < \varepsilon_n,$$

$$\sup\{d(y,E), y \in L^-_{\sigma_{x_n}}(x_n)\} > \delta.$$

Choose sufficiently small $\tau_n < 0$, $t_n < 0$ with $t_n - \tau_n < 0$, such that

$$y_n = \sigma_{x_n}(\tau_n) \in S(E,\delta),$$

$$z_n = \sigma_{x_n}(t_n) = \sigma_{y_n}(t_n - \tau_n) \in S(E,\varepsilon_n),$$

and $y_n \in N$, $z_n \in N$, where σ_{y_n} is a semisolution through y_n and $\sigma_{x_n}(\tau_n + t) = \sigma_{y_n}(t)$ for any $t < 0$. Since N is compact, we can choose two convergent subsequences $\{y_{n_k}\}$ and $\{z_{n_k}\}$. Then there exist $y \in S(E,\delta)$ and $z \in E$ such that

$$\lim_{k\to+\infty} y_{n_k} = y \ , \ \lim_{k\to+\infty} z_{n_k} = z \ .$$

Since $z_n = \sigma_{y_n}(t_n - \tau_n)$ and $\{t_n - \tau_n\} \subset \mathbf{R}^-$ we know that $z \in D^-(y)$. So $D^-(y) \cap E \neq \emptyset$.

By Theorem 4.1, for any $0 < \delta_0 < \delta$, there exist $y_0 \in S(E, \delta_0)$ and a negative semisolution σ_{y_0} through y_0 such that $L^-_{\sigma_{y_0}}(y_0) \subset E$. This is a contradiction to our assumption, and then the proof is completed. □

In order to illustrate the above theorem we shall consider two dynamical systems (every dynamical system is obviously also a semidynamical system).

First consider the differential system defined in $\mathbf{R^2}$ by the differential equations (in polar coordinates)

$$\frac{dr}{dt} = -r(1-r)\ ,\ \ \frac{d\theta}{dt} = 1.$$

The trajectories of the system are: a stationary point $(0,0)$, a periodic trajectory coinciding with the unit circle, spiralling trajectories through each point $P = (r, \theta)$ with $r \neq 0$, $r \neq 1$. Take as a set E a stationary point or a ball centred at $(0,0)$ of radius 1 containing a periodic trajectory, respectively. By properly choosing point x one can easily create examples which illustrate (ii) and (iii) of the above theorem.

On the other hand, we can build an example illustrating point (i) of the theorem by considering a semidynamical system defined on $\mathbf{R^2}$, given by the formula $\pi(z,t) = |z|e^{i(t+\alpha)}$, where $z \in \mathbf{C}$, $\alpha \in arg z$ and $t \in \mathbf{R}^+$. The trajectories of the system are concentric circles. We take as E the ball centred at $(0,0)$ of radius 1.

After an easy verification one can find that (ii) and (iii) exclude each other. If there exists $y \in B[E,\alpha] \setminus E$ and there exists a semisolution σ_y through y such that $L^-_{\sigma_y}(y) \subset E$ then there does not exist an $\varepsilon > 0$ such that for any $x \in B[E,\alpha] \setminus E$ and for any semisolution σ_x through x $\lim_{t\to-\infty} d(\sigma_x(t), E) \geq \varepsilon$, and conversly. It can be easily demonstrated that (i) and (iii) exclude each other as well. If the set E is not isolated then there does not exist $\varepsilon > 0$ such that for any $x \in B[E,\alpha] \setminus E$ and for any semisolution σ_x through x, $\lim_{t\to-\infty} d(\sigma_x(t), E) \geq \varepsilon$. However, one can build an example of a semidynamical system, which fulfills the requirements (i) and (ii) simultaneously.

Corollary 4.12. *The conclusions of Theorem 4.11 hold if we assume that $X \setminus E$ is negatively strongly invariant instead of assuming that E is negatively strongly invariant.*

The proof of Corollary 4.12 is similar to that of Theorem 4.11. The only difference is that we must use Theorem 4.3 instead of Theorem 4.1.

Theorem 4.13. *Let E be a closed, negatively strongly invariant set with $\operatorname{int} E \neq \emptyset$ and $\partial E \neq \emptyset$. Suppose there exists $\alpha > 0$ such that semidynamical system π is locally negatively strongly dissipative at each point of $B[\partial E, \alpha] \cap \operatorname{int} E$. Then one of the following statements holds*

(i) *The boundary* ∂E *is not isolated, that is, for any* $\varepsilon > 0$, *there exists a weakly invariant set* $K \subset B[\partial E, \varepsilon]$ *and* $K \not\subset \partial E$.

(ii) *There exists* $y \in \operatorname{int} E$ *and there exists a semisolution* σ_y *through* y, *such that* $L^-_{\sigma_y}(y) \subset \partial E$.

(iii) *There is an* $\varepsilon > 0$ *such that for any* $x \in \operatorname{int} E$ *and for any semisolution* σ_x *through* x, $\lim_{t \to -\infty} d(\sigma_x(t), \partial E) \geq \varepsilon$.

The proof of this Theorem is analogous to the proof of Theorem 4.11. We must only use the results of Theorem 4.7 instead of those of Theorem 4.1 and make the same discussion using Remark 1 and Corollary 4.9. The difference is that we consider the set $B[\partial E, \alpha] \cap \operatorname{int} E$ instead of the set $B[E, \alpha] \setminus E$.

Examples illustrating the conditions of the above Theorem can be constructed in a similar way as those referring to Theorem 4.11. Set E is to be defined as a ball centered at $(0, 0)$ of radius 1. The boundary of E will be the unit circle. One can notice that (ii) and (iii) of Theorem 4.13 exclude each other. Conditions defined in (i) and (ii) as well as (i) and (iii) can be fulfilled simultaneously.

References

1. Bhatia N.P., Hajek O., *Local Semidynamical Systems*, Lecture Notes in Mathematics **90**, Springer-Verlag, 1969.
2. Bhatia N.P., Szegö G.P., *Stability Theory of Dynamical Systems*, Springer-Verlag, 2000.
3. Bistroń A., *On minimal and invariant sets in semidynamical systems*, Univ. Iagel. Acta. Math., **41** (2003), 189–204.
4. Ciesielski K., *Continuity in semidynamical systems*, Ann. Polon. Math., **46** (1985), 61–70.
5. Ciesielski K., *On negative escape time in semidynamical systems*, Univ. Iagel. Acta. Math., **41** (2003), 181–188.
6. McCann R.C., *Negative escape time in semidynamical systems*, Funkcal. Ekvac., **20** (1977), 39–47.
7. Elaydi S., Kaul S.K., *Semiflows with global extensions-II. Notions of negative stability*, Nonlinear Anal., **12** (1988), 733–746.
8. Freedman H.I., Ruan S., Tang M., *Uniform Persistence and Flows Near a Closed Positively Invariant Set*, J. Dynam. Differential Equations, Vol. **6**, No. **4**, (1994), 583–600.
9. Pelczar A., *General Dynamical Systems*, (in Polish), Lecture Notes of the Jagiellonian University, **293**, Kraków 1978.
10. Saperstone S.H., *Semidynamical Systems in Infinite Dimensional Spaces*, Applied Mathematical Sciences, **37**, Springer-Verlag, 1981.

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