UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLIII
2005

ON THE CHAPLYGHIN METHOD FOR GENERALIZED SOLUTIONS OF PARTIAL DIFFERENTIAL FUNCTIONAL EQUATIONS

by Wojciech Czernous

Abstract. Initial boundary value problems for semilinear first-order partial differential functional equations are considered. It is shown that under natural assumptions on given functions there exists a Chaplyghin sequence and it is convergent to the Carathéodory solution of the original problem. Error estimates for approximate solutions are given. It is proved that the Chaplyghin method for initial boundary value problems is equivalent to the Newton method for a suitable integral functional equation.

1. Introduction. For any metric spaces X and Y, by $C(X,Y)$, we denote the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components.

Let $a > 0$, $h_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, $b = (b_1,\dots,b_n) \in \mathbb{R}^n$ and $h = (h_1,\dots,h_n) \in \mathbb{R}^n_+$ be given, where $b_i > 0$ for $1 \le i \le n$. We define the sets

$$E = [0,a] \times [-b,b], \quad D = [-h_0,0] \times [-h,h].$$

Let $\bar{c} = (c_1,\dots,c_n) = b + h$ and

$$E_0 = [-h_0,0] \times [-\bar{c},\bar{c}],$$
$$\partial_0 E = [0,a] \times ([-\bar{c},\bar{c}]\setminus(-b,b)), \quad \Omega = E_0 \cup E \cup \partial_0 E.$$

Suppose that $z : \Omega \to \mathbb{R}$ and $(t,x) \in E$ are fixed. We define the function $z_{(t,x)} : D \to \mathbb{R}$ as follows

$$z_{(t,x)}(\tau,\xi) = z(t+\tau, x+\xi),\ (\tau,\xi) \in D.$$

2000 *Mathematics Subject Classification.* 35R10, 65J15.

Key words and phrases. Initial boundary value problems, Carathéodory solutions, Chaplyghin method, error estimates, characteristics.

The function $z_{(t,x)}$ is the restriction of z to the set $[t-h_0,t]\times[x-h,x+h]$ and this restriction is shifted to the set D. Elements of the space $C(D,\mathbb{R})$ will be denoted by $w,\bar{w}$ and so on. We denote by $\|\cdot\|_0$ the supremum norm in the space $C(D,\mathbb{R})$. Let

$$f=(f_1,\dots,f_n)\colon E\to\mathbb{R}^n,\;\; G\colon E\times C(D,\mathbb{R})\to\mathbb{R},\;\; \varphi\colon E_0\cup\partial_0E\to\mathbb{R},$$

$$\alpha_0\colon E\to\mathbb{R},\;\; \alpha'\colon E\to\mathbb{R}^n,\;\; \alpha'=(\alpha_1,\dots,\alpha_n)$$

be given functions. Write $\alpha(t,x)=(\alpha_0(t,x),\alpha'(t,x))$, $(t,x)\in E$. We require that $\alpha(t,x)\in E$ and $\alpha_0(t,x)\le t$ for $(t,x)\in E$. We consider the problem consisting of the functional differential equation

$$\partial_t z(t,x)+\sum_{i=1}^{n} f_i(t,x)\partial_{x_i}z(t,x)=G(t,x,z_{\alpha(t,x)}), \tag{1}$$

and the initial-boundary condition

$$z(t,x)=\varphi(t,x)\;\text{ on }\; E_0\cup\partial_0E. \tag{2}$$

A function $\tilde{z}:[-h_0,\xi]\times[-\bar{c},\bar{c}]\to\mathbb{R}$, where $0<\xi\le a$, is a Carathéodory solution of problem (1), (2) if it is continuous and

(i) the derivatives $\partial_t\tilde{z}$, $\partial_x\tilde{z}=(\partial_{x_1}\tilde{z},\dots,\partial_{x_n}\tilde{z})$ exist almost everywhere on $[0,\xi]\times[-b,b]$,

(ii) $\tilde{z}$ satisfies (1) almost everywhere on $[0,\xi]\times[-b,b]$ and condition (2) holds.

Note that our hereditary setting contains well known delay structures as particular cases.

In the paper, we give sufficient conditions for the existence of a sequence $\{u^{(m)}\}$ and a function u^* such that:

(i) u^* is a Carathéodory solution of (1), (2) and $\{u^{(m)}\}$ is a sequence of solutions of some linear functional differential equations obtained by linearization of (1),

(ii) $\{u^{(m)}\}$ uniformly converges to u^* and the convergence is of the Newton type.

This method of approximating solutions of differential equations was introduced by Chaplyghin in [**5**]. The Chaplyghin method was applied to systems of first-order partial differential equations in [**9**], [**11**] and it was extended in [**10**] onto the case of infinite systems. The method for partial differential-functional equations with another model of functional dependence was considered in [**7**], but linearization with respect to the classical argument only was allowed here. The Chaplyghin method for classical solutions of semilinear functional-differential equations with initial boundary conditions was investigated in [**6**]. The results presented in [**6**], [**7**] have the following properties: it is assumed that the Chaplyghin sequences exist and the proofs of the convergence are based on theorems on differential inequalities.

It is clear that the results and methods presented in [**6**] do not apply to Carathéodory solutions of (1), (2).

The theory of the Chaplyghin method for parabolic functional-differential problems was developed in [**1**]–[**3**].

The aim of this paper is to give a further contribution to the Chaplyghin method for functional-differential problems. We prove that under suitable assumptions on given functions there exists a sequence of approximate solutions and it uniformly converges to a Carathéodory solution of (1), (2). We establish estimates for the difference between exact and approximate solutions of (1), (2).

Note that our results are new also in the case when (1) reduces to the equation without the functional dependence.

Throughout the paper, we use these general ideas for the Chaplyghin method which were introduced in [**12**].

2. Characteristics. Write

$$\Delta_i^+ = \{x \in [-b, b] \colon\ x_i = b_i\}, \qquad \Delta_i^- = \{x \in [-b, b] \colon\ x_i = -b_i\},$$

where $1 \le i \le n$, and

$$\Delta = \bigcup_{i=1}^{n} \Big(\Delta_i^+ \cup \Delta_i^-\Big).$$

We will need the following function spaces in our considerations. Write $\Omega_t = \Omega \cap ([-h_0, t] \times \mathbb{R}^n)$ and $E_t = [0, t] \times [-b, b]$, where $0 \le t \le a$. We will denote by $\|\cdot\|_t$ the supremum norm in the spaces $C(\Omega_t, \mathbb{R})$. Analogously, we will use the symbol $\|\cdot\|_{(t)}$ to denote the supremum norm in $C(E_t, \mathbb{R})$. For $x \in \mathbb{R}^n$, where $x = (x_1, \ldots, x_n)$, we put

$$\|x\| = \sum_{j=1}^{n} |x_i|.$$

Let $p = (p_0, p_1) \in \mathbb{R}_+^2$. Denote by $J[p]$ the set of all functions $\varphi \in C(E_0 \cup \partial_0 E, \mathbb{R})$ such that $|\varphi(t, x)| \le p_0$ on $E_0 \cup \partial_0 E$ and

$$|\varphi(t, x) - \varphi(\bar{t}, \bar{x})| \le p_1[|t - \bar{t}| + \|x - \bar{x}\|] \ \text{ on }\ E_0 \cup \partial_0 E.$$

Assumption H$[f]$ Suppose that

1) the function $f \colon E \to \mathbb{R}^n$ is such that
   (i) $f(\cdot, x) \colon [0, a] \to \mathbb{R}^n$ is measurable for each $x \in [-b, b]$ and $f(t, \cdot) \colon [-b, b] \to \mathbb{R}^n$ is continuous for almost all $t \in [0, a]$,

(ii) there are $A_1, B_1 \in \mathbb{R}_+$ such that

$$\|f(t,x)\| \le A_1 \ \text{ on } \ E,$$

and

$$\|f(t,x) - f(t,\bar{x})\| \le B_1 \|x - \bar{x}\|$$

for $(t,x), (\bar{t},\bar{x}) \in E$,

2) there is $\kappa > 0$ such that for $1 \le i \le n$:

$$f_i(t,x) \le -\kappa \ \text{ for } \ x \in \Delta_i^+,$$

$$f_i(t,x) \ge \kappa \ \text{ for } \ x \in \Delta_i^-$$

for $t \in [0,a]$.

Suppose that $(t,x) \in E_c$. Consider the Cauchy problem

$$\eta'(\tau) = f(\tau, \eta(\tau)), \quad \eta(t) = x. \tag{3}$$

Denote by $g(\cdot,t,x) = (g_1(\cdot,t,x), \ldots, g_n(\cdot,t,x))$ a Carathéodory solution of (3). The function $g(\cdot,t,x)$ is the characteristic of equation (1).

Suppose that Assumption H$[f]$ is satisfied. Then $g(\cdot,t,x)$ exists on the interval $[0,c]$ and is unique. The existence and uniqueness of the above solution follow from classical theorems. Let $I_{(t,x)}$ be the domain of $g(\cdot,t,x)$ and $\delta(t,x)$ be the left endpoint of the maximal interval on which the characteristic $g(\cdot,t,x)$ is defined.

We prove a lemma on bicharacteristics.

Lemma 2.1. *Suppose that Assumption H[f] is satisfied and let $\varphi \in J[p]$, $c \in (0,a]$ be given. Then the solution $g(\cdot,t,x)$ exists on the interval $I_{(t,x)}$ such that for $\zeta = \delta(t,x)$ there is $\zeta = 0$ or $g(\zeta,t,x) \in \Delta$. The characteristics are unique on $I_{(t,x)}$. Moreover, the following estimate holds*

$$\|g(\tau,t,x) - g(\tau,\bar{t},\bar{x})\| \le \bar{C}\Big[|t - \bar{t}| + \|x - \bar{x}\|\Big] \tag{4}$$

for $\tau \in I_{(t,x)} \cap I_{(\bar{t},\bar{x})}$, $(t,x), (\bar{t},\bar{x}) \in E_c$ and

$$\bar{C} = \max\{1, A_1\} \exp\{cB_1\}.$$

Proof. The existence and uniqueness of solutions of (3) follow from the classical theorem on Carathéodory solutions of initial problems. The function $g(\cdot,t,x)$ satisfies the integral equation

$$g(\tau,t,x) = x + \int_t^\tau f(s, g(s,t,x))ds. \tag{5}$$

It follows from Assumption H[f] that the function $g(\cdot,t,x)-g(\cdot,\bar t,\bar x)$ satisfies the integral inequality

$$\|g(\tau,t,x)-g(\tau,\bar t,\bar x)\|\le\|x-\bar x\|$$
$$+A_1|t-\bar t|+\left|\int_{\bar t}^{\tau}B_1\|g(s,t,x)-g(s,\bar t,\bar x)\|ds\right|,\quad \tau\in I_{(t,x)}\cap I_{(\bar t,\bar x)}.$$

Then we obtain (4) by the Gronwall inequality. This proves Lemma 2.1. □

Now we give a lemma on a regularity of the function δ.

LEMMA 2.2. *Suppose that Assumption H[f] is satisfied and* $\varphi\in J[p]$, $c\in(0,a]$. *Then the function* δ *is continuous on* E_c. *Moreover, the following estimate holds*

$$|\delta(t,x)-\delta(\bar t,\bar x)|\le\frac{2\bar C}{\kappa}[|t-\bar t|+\|x-\bar x\|] \tag{6}$$

for $(t,x),(\bar t,\bar x)\in E_c$.

PROOF. The continuity of δ follows from classical theorems on continuous dependence on initial conditions for Carathéodory solutions of initial problems. Now we prove (6). This estimate is obvious in the case $\delta(t,x)=\delta(\bar t,\bar x)=0$ (i.e., in the case where solutions of problem (3) are defined on $[0,t]$ and $[0,\bar t]$). Suppose now that $0\le\delta(t,x)<\delta(\bar t,\bar x)$. Then for $\bar\zeta=\delta(\bar t,\bar x)$ there is $g(\bar\zeta,\bar t,\bar x)\in\Delta$ and there exists i, $1\le i\le n$, such that $|g_i(\bar\zeta,\bar t,\bar x)|=b_i$. Two possibilities can occur: either (i) $g_i(\bar\zeta,\bar t,\bar x)=b_i$ or (ii) $g_i(\bar\zeta,\bar t,\bar x)=-b_i$. Consider case (i). Let $x=(x_1,\dots,x_n)$, $\tilde x=(x_1,\dots,x_{i-1},b_i,x_{i+1},\dots,x_n)$. There is

$$|f_i(t,x)-f_i(t,\tilde x)|\le B_1(b_i-x_i), \tag{7}$$

for $(t,x)\in E_c$. Thus

$$f_i(t,x)\le-\frac{\kappa}{2}$$

for $(t,x)\in E_c$ such that $b_i-x_i\le\kappa(2B_1)^{-1}$. It follows from Lemma 2.1 that

$$b_i-g_i(\bar\zeta,t,x)=g_i(\bar\zeta,\bar t,\bar x)-g_i(\bar\zeta,t,x)\le\frac{\kappa}{2B_1}$$

for $(t,x),(\bar t,\bar x)\in E_c$ such that

$$|t-\bar t|+\|x-\bar x\|\le\frac{\kappa}{2B_1\bar C}. \tag{8}$$

Then we get

$$f_i(\bar\zeta,g(\bar\zeta,t,x))\le-\frac{\kappa}{2}<0$$

and, consequently,

$$\partial_t g_i(\delta(\bar t,\bar x),t,x)<0$$

for $(t,x),(\bar t,\bar x)\in E_c$ satisfying (8). From the above inequality and from the uniqueness of $g(\cdot,t,x)$, it can easily be seen that $g_i(\cdot,t,x)$ is decreasing on the interval $(\delta(t,x),\delta(\bar t,\bar x))$. Therefore,

$$b_i - g_i(\tau,t,x) \le \frac{\kappa}{2B_1}$$

and the estimate

$$f_i(\tau, g(\tau,t,x)) \le -\frac{\kappa}{2}$$

holds for $\tau\in(\delta(t,x),\delta(\bar t,\bar x))$ and $(t,x),(\bar t,\bar x)\in E_c$ such that (8) is satisfied. Then

$$\begin{aligned} &-\frac{\kappa}{2}[\delta(\bar t,\bar x)-\delta(t,x)] \\ &\ge \int_{\delta(t,x)}^{\delta(\bar t,\bar x)} f_i(\tau,g(\tau,t,x))d\tau = g_i(\delta(\bar t,\bar x),t,x) - g_i(\delta(t,x),t,x) \\ &\ge g_i(\delta(\bar t,\bar x),t,x) - g_i(\delta(\bar t,\bar x),\bar t,\bar x) \ge -\bar C\Big[|t-\bar t| + \|x-\bar x\|\Big]. \end{aligned}$$

Thus the proof of (6) for $(t,x),(\bar t,\bar x)\in E_c$, such that (8) holds, is complete in case (i). In a similar way we prove (6) in case (ii). Let $(t,x),(\bar t,\bar x)\in E_c$ be arbitrary. We put $M=\|x-\bar x\|+|t-\bar t|$. There exists $K\in\mathbf{N}$ such that

$$(K-1)\frac{\kappa}{2B_1\bar C} < M \le K\frac{\kappa}{2B_1\bar C}.$$

Let $\varepsilon\in\mathbf{R}$, $\varepsilon=\frac{1}{K}$. For $j=0,\dots,K$, we put

$$\bar x^{(j)} = j\varepsilon\bar x + (1-j\varepsilon)x, \qquad \bar t^{(j)} = j\varepsilon\bar t + (1-j\varepsilon)t.$$

Note that $(\bar t^{(0)},\bar x^{(0)})=(t,x)$, $(\bar t^{(K)},\bar x^{(K)})=(\bar t,\bar x)$ and

$$\|\bar x^{(j)}-\bar x^{(j+1)}\| + |\bar t^{(j)}-\bar t^{(j+1)}| = \frac{M}{K} \le \frac{\kappa}{2B_1\bar C}$$

for $j=0,\dots,K-1$. It is easy to see that

$$\|x-\bar x\| = \sum_{j=0}^{K-1}\|\bar x^{(j)}-\bar x^{(j+1)}\| \text{ and } \|t-\bar t\| = \sum_{j=0}^{K-1}\|\bar t^{(j)}-\bar t^{(j+1)}\|.$$

Then there is

$$\begin{aligned} &|\delta(t,x)-\delta(\bar t,\bar x)| \\ &\le \sum_{j=0}^{K-1}|\delta(\bar t^{(j)},\bar x^{(j)}) - \delta(\bar t^{(j+1)},\bar x^{(j+1)})| \\ &\le \sum_{j=0}^{K-1}\frac{2\bar C}{\kappa}\Big[|\bar t^{(j)}-\bar t^{(j+1)}| + \|\bar x^{(j)}-\bar x^{(j+1)}\|\Big] = \frac{2\bar C}{\kappa}\Big[|t-\bar t| + \|x-\bar x\|\Big]. \end{aligned}$$

Thus we see that (6) holds true for all $(t,x),(\bar t,\bar x)\in E_c$. $\square$

3. Chaplyghin sequences. We denote by $CL(D,\mathbb{R})$ the set of all continuous and real functions defined on $C(D,\mathbb{R})$ and by $\|\cdot\|_*$ the norm in $CL(D,\mathbb{R})$. We now formulate assumptions on G and α.

Assumption H$[G]$. Suppose that

1) the function $G\colon E\times C(D,\mathbb{R})\to\mathbb{R}$ is such that
   (i) $G(\cdot,x,w):[0,a]\to\mathbb{R}$ is measurable for each $(x,w)\in[-b,b]\times C(D,\mathbb{R})$,
   (ii) $G(t,\cdot)\colon[-b,b]\times C(D,\mathbb{R})\to\mathbb{R}$ is continuous for almost all $t\in[0,a]$,
2) there are $A_2,B_2\in\mathbb{R}_+$ such that for $(t,x,w),(t,\bar{x},\bar{w})\in E\times C(D,\mathbb{R})$ there is

$$|G(t,x,w)|\le A_2$$

$$|G(t,x,w)-G(t,\bar{x},\bar{w})|\le B_2[\|x-\bar{x}\|+\|w-\bar{w}\|_0],$$

3) the Fréchet derivative $\partial_w G(t,x,w)\in CL(D,\mathbb{R})$ exists for $(t,x,w)\in E\times C(D,\mathbb{R})$,
4) there exist $A_3,B_3\in\mathbb{R}_+$ such that

$$\|\partial_w G(t,x,w)\|_*\le A_3,$$

$$\|\partial_w G(t,x,w)-\partial_w G(t,x,\bar{w})\|_*\le B_3\|w-\bar{w}\|_0$$

for $(t,x,w),(t,x,\bar{w})\in E\times C(D,\mathbb{R})$.

Assumption H$[\alpha]$. Suppose that

1) the function $\alpha\colon E\to E$ is continuous,
2) there are $r_0,r_1\in\mathbb{R}_+$ such that for $(t,x),(\bar{t},\bar{x})\in E$ there holds

$$|\alpha_0(t,x)-\alpha_0(\bar{t},\bar{x})|\le r_0[|t-\bar{t}|+\|x-\bar{x}\|],$$

$$\|\alpha'(t,x)-\alpha'(\bar{t},\bar{x})\|\le r_1[|t-\bar{t}|+\|x-\bar{x}\|].$$

Suppose that Assumptions H$[f]$, H$[G]$, H$[\alpha]$ are satisfied and $\varphi\in J[p]$. We consider a sequence $\{z^{(m)}\}$, $z^{(m)}\colon\Omega_c\to\mathbb{R}$, defined in the following way:

(i) $z^{(0)}\in C(\Omega_c,\mathbb{R})$, $z(t,x)=\varphi(t,x)$ for $(t,x)\in(E_0\cup\partial_0 E)\cap([-h_0,c]\times\mathbb{R}^n)$ and $z^{(0)}$ satisfies the Lipschitz condition with respect to (t,x) on Ω_c,

(ii) if $z^{(m)}\colon\Omega_c\to\mathbb{R}$ is known then $z^{(m+1)}$ is a Carathéodory solution of the linear equation

(9)
$$\partial_t z(t,x)+\sum_{i=1}^n f_i(t,x)\partial_{x_i}z(t,x)=G(t,x,z^{(m)}_{\alpha(t,x)})+\partial_w G(t,x,z^{(m)}_{\alpha(t,x)})(z-z^{(m)})_{\alpha(t,x)}$$

with the initial-boundary condition

(10) $$z(t,x)=\varphi(t,x)\ \text{ for }\ (t,x)\in(E_0\cup\partial_0 E)\cap([-h_0,c]\times\mathbb{R}^n).$$

We prove that there is $c \in (0, a]$ such that the sequence $\{z^{(m)}\}$ is well defined on Ω_c and converges to a Carathéodory solution of (1), (2).

Our considerations are based on the following idea. We transform problem (1), (2) into an abstract equation $F[z] = 0$ where $F : X \to X$ and X is a Banach space. Then we consider a Newton method for the above equation and we prove that $\{z^{(m)}\}$ is a Newton sequence for the abstract equation.

Suppose that $(X, \|\cdot\|_X)$ is a Banach space and

$$S = \{u \in X : \|u - u_0\|_X \leq \delta\},$$

where $u_0 \in X$ is arbitrary and $\delta > 0$. Let $\mathcal{F}\colon S \to X$ be a given operator such that $\mathcal{F}'(u)$ exists for $u \in S$. We consider the equation

$$\mathcal{F}(u) = 0 \tag{11}$$

and the Newton method

$$u_0, \qquad u_{m+1} = u_m - [\mathcal{F}'(u_m)]^{-1}\mathcal{F}(u_m), \ m \geq 0. \tag{12}$$

We will need the following theorem in our considerations.

Theorem 3.1. *Suppose that $\mathcal{F}\colon S \to X$ and*

1) *the Fréchet derivative $\mathcal{F}'(u)$ exists for $x \in S$,*
2) *there is $K \in \mathbb{R}_+$ such that*

$$\|\mathcal{F}'(u) - \mathcal{F}'(\bar{u})\|_* \leq K\|u - \bar{u}\|_X \ \textit{for} \ u, \bar{u} \in S,$$

3) *the operator $\mathcal{F}'(u_0)$ has the inverse $\Gamma_0 = [\mathcal{F}'(u_0)]^{-1}$ and there is $B \in \mathbb{R}_+$ such that $\|\Gamma_0\| \leq B$,*
4) *for the initial element u_0, the estimate $\|\Gamma_0\mathcal{F}(u_0)\|_X \leq \eta$ holds,*
5) *the constants B, K, η fulfil the inequality $\tilde{h} = BK\eta \leq 0,5$,*
6) *for δ, the following inequality holds:*

$$\frac{1 - \sqrt{1 - 2\tilde{h}}}{\tilde{h}}\eta \leq \delta.$$

Then

(i) *there exists a solution of equation (11),*
(ii) *the Newton sequence (12) exists and there is u^* such that*

$$u^* = \lim_{m\to\infty} u_m,$$

(iii) *$f(u^*) = 0$ and the following estimate holds*

$$\|u^* - u_m\|_X \leq \frac{1}{2^{m-1}}(2\tilde{h})^{2^m-1}\eta, \ \ m \geq 0.$$

The proof of the above theorem is given in [**8**].

4. Newton method for integral-functional equations. Suppose that Assumptions H[f], H[G], H[α] are satisfied and $\varphi \in J[p]$. Write

$$S(t,x) = (\delta(t,x), g(\delta(t,x),t,x)).$$

Let us consider the operator $F\colon C(\Omega_c,\mathbb{R}) \to C(\Omega_c,\mathbb{R})$ defined by

$$F[z](t,x) = z(t,x) - \varphi(S(t,x)) - \int_{\delta(t,x)}^{t} G(s, g(s,t,x), z_{\alpha(s,g(s,t,x))})ds,$$

$$F[z](t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in E_0 \cup \partial_0 E.$$

It is clear that the equation

$$F[z](t,x) = 0 \tag{13}$$

is obtained from (1), (2) by integrating (1) along characteristics. The existence result for (13) is based on the following method of successive approximations.

We consider a sequence $\{z^{(m)}\}$, $z^{(m)} : \Omega_c \to \mathbb{R}$, defined in the following way:

(A) $z^{(0)} \in C(\Omega_c,\mathbb{R})$, $z(t,x) = \varphi(t,x)$ for $(t,x) \in (E_0 \cup \partial_0 E) \cap ([-h_0,c] \times \mathbb{R}^n)$,

(B) if $z^{(m)}\colon \Omega_c \to \mathbb{R}$ is known, then $z^{(m+1)}$ is the solution of the equation

$$z(t,x) = \varphi(S(t,x)) + \int_{\delta(t,x)}^{t} G(P^{(m)}(s,t,x))ds$$

$$+ \int_{\delta(t,x)}^{t} \partial_w G(P^{(m)}(s,t,x))(z - z^{(m)})_{\alpha(s,g(s,t,x))}$$

with the initial-boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{for} \quad (t,x) \in (E_0 \cup \partial_0 E) \cap ([-h_0,c] \times \mathbb{R}^n),$$

where

$$P^{(m)}(s,t,x) = P[z^{(m)}](s,t,x)$$

and

$$P[z](s,t,x) = (s, g(s,t,x), z_{\alpha(s,g(s,t,x))}).$$

We prove that the above sequence is well defined and that it converges to a solution of (13).

Lemma 4.1. *Suppose that Assumptions H[f], H[G], H[α] are satisfied and $\varphi \in J[p]$. Then for $0 < c \le \tilde{c}$, where*

$$\tilde{c} = (2A_3)^{-1} \operatorname{arsinh}\{A_3(2B_3\tilde{\eta})^{-1}\}, \quad \tilde{\eta} = \|z^{(0)}\|_a + p_0 + aA_2,$$

there is:

1) *equation (13) has a solution $z^*\colon \Omega_c \to \mathbb{R}$,*

2) *the sequence* $\left\{z^{(m)}\right\}$ *is well defined by (A), (B) on* Ω_c *and*

$$\lim_{m\to\infty} z^{(m)}(t,x) = z^*(t,x) \quad on \;\; E_c,$$

3) *the following estimates hold*

$$\|z^* - z^{(m)}\|_c \le \frac{\widetilde{C}}{2^{m-1}}(2\tilde{h})^{2^m-1}\eta, \quad m \ge 0, \tag{14}$$

where

$$\widetilde{C} = \exp\{2A_3c\}, \quad \eta = 2(\|z^{(0)}\|_c + p_0 + cA_2), \quad \tilde{h} = B_3(A_3)^{-1}\sinh(2A_3c)\eta \le \frac{1}{2}. \tag{15}$$

Proof. We consider the Banach space $(C(\Omega_c,\mathbb{R}), \|\cdot\|_B)$, where

$$\|z\|_B = \max\{\|z\|_t e^{-2A_3t} \colon t\in[0,c]\}.$$

Note that

$$\|v\|_c e^{-2A_3c} \le \|v\|_B$$

and, consequently,

$$\|v\|_c \le \widetilde{C}\|v\|_B \tag{16}$$

for $v \in C(\Omega_c,\mathbb{R})$.

We apply Theorem 3.1 to prove the above properties of equation (13). It follows from Assumption H$[G]$ that the Fréchet derivative $F'[z]$ exists and

$$\left(F'[z]\,v\right)(t,x) = v(t,x) - \int_{\delta(t,x)}^{t} \partial_w G(P[z](s,t,x))v_{\alpha(s,g(s,t,x))}ds \;\text{ for }\; (t,x)\in E_c, \tag{17}$$

$$\left(F'[z]\,v\right)(t,x) = v(t,x) \;\text{ for }\; (t,x)\in (E_0\cup\partial_0E)\cap([-h_0,c]\times\mathbb{R}^n).$$

We consider the operator $U \colon C(\Omega_c,\mathbb{R}) \to C(\Omega_c,\mathbb{R})$ defined by

$$U[v](t,x) = \int_{\delta(t,x)}^{t} \partial_w G(P[z](s,t,x))v_{\alpha(s,g(s,t,x))}ds \;\text{ on }\; E$$

and

$$U[v](t,x) = 0, \;\; (t,x)\in E_0\cup\partial_0E.$$

Then

$$|U[v](t,x)| \le \int_{\delta(t,x)}^{t} \left|\partial_w G(P[z](s,t,x))v_{\alpha(s,g(s,t,x))}e^{-2A_3s}e^{2A_3s}\right| ds$$

$$\le \|v\|_B \int_0^t A_3e^{2A_3s}ds \le \frac{1}{2}\|v\|_B e^{2A_3t}$$

and, consequently,

$$|U[v](t,x)|e^{-2A_3t} \le \frac{1}{2}\|v\|_B$$

and

$$\|U[v]\|_B \le \frac{1}{2}\|v\|_B.$$

It follows that $\|U\|_* \le \frac{1}{2} < 1$ and $F'[z]$ exists. Moreover, there is

$$\| \left(F'[z]\right)^{-1} \| \le 2,$$

where $z \in C(\Omega_c, \mathbb{R})$. The Newton method for equation (13) with the starting function $z^{(0)}$ has the form

$$z^{(m+1)}(t,x) = z^{(m)}(t,x) - \left(F'[z^{(m)}]\right)^{-1} F[z^{(m)}](t,x), \quad m \ge 0.$$

It easily follows that the above relations are equivalent to

$$z^{(m+1)}(t,x) = \varphi(S(t,x)) + \int_{\delta(t,x)}^{t} G(P^{(m)}(s,t,x))ds$$

$$+ \int_{\delta(t,x)}^{t} \partial_w G(P^{(m)}(s,t,x))(z^{(m+1)} - z^{(m)})_{\alpha(s,g(s,t,x))}ds, \quad (t,x) \in E_c$$

and

$$z^{(m+1)}(t,x) = \varphi(t,x) \text{ for } (t,x) \in (E_0 \cup \partial_0 E) \cap ([-h_0, c] \times \mathbb{R}^n).$$

Then the sequence $\left\{z^{(m)}\right\}$ defined by relations (A), (B) is the Newton sequence for equation (13).

We will write an estimate of the Lipschitz constant for the Fréchet derivative F'. According to (17), there is

$$\left|(F'[z]v - F'[\bar{z}]v)(t,x)\right|$$

$$= \left| \int_{\delta(t,x)}^{t} \left[\partial_w G(P[z](s,t,x)) - \partial_w G(P[\bar{z}](s,t,x)\right] v_{\alpha(s,g(s,t,x))} ds \right|$$

$$\le \int_{\delta(t,x)}^{t} \|\partial_w G(P[z](s,t,x)) - \partial_w G(P[\bar{z}](s,t,x)\|_* \|v_{\alpha(s,g(s,t,x))}\|_0 ds$$

$$\le \int_{\delta(t,x)}^{t} B_3 \|(z-\bar{z})_{\alpha(s,g(s,t,x))}\|_0 \|v_{\alpha(s,g(s,t,x))}\|_0 ds$$

$$\le \int_{\delta(t,x)}^{t} B_3 \|z-\bar{z}\|_s e^{-2A_3 s} \ \|v\|_s e^{-2A_3 s} \ e^{4A_3 s} ds$$

$$\le \|z-\bar{z}\|_B \|v\|_B \int_0^t B_3 e^{4A_3 s} ds = \|z-\bar{z}\|_B \|v\|_B \ B_3 (4A_3)^{-1}(e^{4A_3 t} - 1).$$

Then

$$\left|(F'[z]v - F'[\bar{z}]v)(t,x)\right| e^{-2A_3 t} \le \|z-\bar{z}\|_B \|v\|_B \ B_3 (2A_3)^{-1} \sinh(2A_3 t)$$

and, consequently,

$$\|(F'[z]-F'[\bar z])v\|_B \le \|z-\bar z\|_B\|v\|_B\, B_3(2A_3)^{-1}\sinh(2A_3t)$$

and

$$\|F'[z]-F'[\bar z]\|_* \le \|z-\bar z\|_B\, B_3(2A_3)^{-1}\sinh(2A_3t).$$

Suppose that $\eta>0$ is such a constant that

$$\|z^{(1)}-z^{(0)}\|_B \le \eta,$$

and that c is a constant small enough for the following inequality to hold:

$$\tilde h = B_3(A_3)^{-1}\sinh(2A_3c)\eta \le \frac{1}{2}. \tag{18}$$

We put

$$\delta = \frac{1-\sqrt{1-2\tilde h}}{\tilde h}\eta.$$

Now by Theorem 3.1 and (16), all the assertions of the Lemma 4.1 follow. $\square$

Now we prove that differential-functional equations (9) with the initial-boundary condition

$$z(t,x)=\varphi(t,x) \ \text{ for } \ (t,x)\in(E_0\cup\partial_0E)\cap([-h_0,c]\times\mathbb{R}^n) \tag{19}$$

are equivalent to integral equations

$$z(t,x)=\varphi(S(t,x))+\int_{\delta(t,x)}^t G(P^{(m)}(s,t,x))ds \tag{20}$$

$$+\int_{\delta(t,x)}^t \partial_w G(P^{(m)}(s,t,x))(z-z^{(m)})_{\alpha(s,g(s,t,x))}ds$$

with initial-boundary condition (19).

Suppose that $z^{(m+1)}\colon\Omega_c\to\mathbb{R}$ is a Carathéodory solution of (9), (19) and there is $b^{(m+1)}$ such that

$$|z^{(m+1)}(t,x)-z^{(m+1)}(\bar t,\bar x)|\le b^{(m+1)}[|t-\bar t|+\|x-\bar x\|].$$

It follows easily that $z^{(m+1)}$ satisfies (20). We prove that classical solutions of (20) satisfy (9) almost everywhere on E_c. We state a lemma on regularity of the Newton sequence $\{z^{(m)}\}$.

Suppose that $z\colon\Omega\to\mathbb{R}$ is continuous . For $t\in[0,a]$, we define

$$[|z|]_{L.t}=\sup\left\{\frac{|z(s,y)-z(\bar s,\bar y)|}{|s-\bar s|+\|y-\bar y\|} : (s,y),(\bar s,\bar y)\in\Omega_t,(s,y)\ne(\bar s,\bar y)\right\}.$$

Lemma 4.2. *Suppose that Assumptions H[f], H[G] and H[α] are satisfied and that*

1) $\varphi\in J[p]$, $c\in(0,a]$,

2) $z^{(0)} \in C(\Omega, \mathbb{R})$ *is given such that* $\left[|z^{(0)}|\right]_{L.c} < \infty$,

3) *the sequence* $z^{(m)}$ *is well defined by (A), (B) on* Ω_c *for* $m \geq 0$.

Then there is

$$\left[|z^{(m)}|\right]_{L.c} < \infty \quad \textit{for } m \geq 0. \tag{21}$$

Proof. We prove (21) by induction. It follows from assumption 2) that (21) is satisfied for $m = 0$. Supposing now that condition (21) holds for a given $m \geq 0$, we will prove that the function $z^{(m+1)}$ given by (A), (B) satisfies (21). We put

$$L^{(m)} = \left[|z^{(m)}|\right]_{L.c}. \tag{22}$$

From assumption 3) it follows that there exist constants $d^{(m)}, d^{(m+1)} \in \mathbb{R}_+$ such that

$$|z^{(m)}(t,x)| \leq d^{(m)} \text{ and } |z^{(m+1)}(t,x)| \leq d^{(m+1)} \text{ on } E_c. \tag{23}$$

Put

$$\psi(\tau) = \left[|z^{(m+1)}|\right]_{L.\tau} \quad \text{for } \tau \in [0,c].$$

We will write an integral inequality for function $\psi\colon [0,c] \to \mathbb{R}_+$. Let $\tau \in [0,c]$ be arbitrary and $(t,x), (\bar{t},\bar{x}) \in E_\tau$, $(t,x) \neq (\bar{t},\bar{x})$. According to (22), (23), Assumptions H$[f]$, H$[G]$ and H$[\alpha]$, and Lemmas 2.1, 2.2, the following integral inequality holds:

$$\begin{aligned} |z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},\bar{x})| &\leq \bar{A}[|t-\bar{t}| + \|x-\bar{x}\|] \\ &+ \int_{\delta(\bar{t},\bar{x})}^{\bar{t}} A_3 \|(z^{(m+1)})_{\alpha(s,g(s,t,x))} - (z^{(m+1)})_{\alpha(s,g(s,\bar{t},\bar{x}))}\|_0 ds \end{aligned} \tag{24}$$

where

$$\bar{A} = \bar{A}_1 + \bar{A}_2$$

and

$$\bar{A}_1 = p_1 \left[(1+A_1)\widetilde{C} + \bar{C} \right] + \left[A_2 + A_3\tilde{d}\right] \left(1 + \widetilde{C}\right),$$

$$\bar{A}_2 = c\bar{C}\left[B_2 + B_3\tilde{d}\right]\left(1 + \widetilde{L}\right) + c\bar{C}A_3\widetilde{L}$$

and

$$\widetilde{C} = \frac{2\bar{C}}{\kappa}, \quad \widetilde{L} = L^{(m)}(r_0 + r_1), \quad \tilde{d} = d^{(m)} + d^{(m+1)}.$$

Note that

$$\|(z^{(m+1)})_{\alpha(s,g(s,t,x))} - (z^{(m+1)})_{\alpha(s,g(s,\bar{t},\bar{x}))}\|_0$$

$$= \sup_{(\tau,y)\in D} \left| z^{(m+1)}(\alpha(s+\tau, g(s,t,x)+y)) - z^{(m+1)}(\alpha(s+\tau, g(s,\bar{t},\bar{x})+y)) \right|$$

$$\leq \left[|z^{(m+1)}|\right]_{L.s} (r_0+r_1)\|g(s,t,x) - g(s,\bar{t},\bar{x})\|$$

$$\leq \left[|z^{(m+1)}|\right]_{L.s} (r_0+r_1)\bar{C}[|t-\bar{t}| + \|x-\bar{x}\|].$$

Then we get

$$|z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},\bar{x})| \leq \bar{A}[|t-\bar{t}| + \|x-\bar{x}\|]$$

$$+A_3(r_0+r_1)\bar{C}[|t-\bar{t}| + \|x-\bar{x}\|] \int_{\delta(\bar{t},\bar{x})}^{\bar{t}} \left[|z^{(m+1)}|\right]_{L.s} ds$$

and, consequently,

$$\frac{|z^{(m+1)}(t,x) - z^{(m+1)}(\bar{t},\bar{x})|}{[|t-\bar{t}| + \|x-\bar{x}\|]} \leq \bar{A} + A_3(r_0+r_1)\bar{C} \int_{\delta(\bar{t},\bar{x})}^{\bar{t}} \left[|z^{(m+1)}|\right]_{L.s} ds$$

for $(t,x), (\bar{t},\bar{x}) \in E_\tau$, $(t,x) \neq (\bar{t},\bar{x})$. According to the definition of $[|\cdot|]_{L.\tau}$, there is

$$\left[|z^{(m+1)}|\right]_{L.\tau} \leq \bar{A} + A_3(r_0+r_1)\bar{C} \int_{\delta(\bar{t},\bar{x})}^{\bar{t}} \left[|z^{(m+1)}|\right]_{L.s} ds.$$

Then

$$\psi(\tau) \leq \bar{A} + A_3(r_0+r_1)\bar{C} \int_0^\tau \psi(s) ds.$$

Now from the Gronwall inequality it follows that

$$\psi(\tau) \leq \bar{A} \exp\left\{\tau A_3(r_0+r_1)\bar{C}\right\}$$

and, consequently,

$$\left[|z^{(m+1)}|\right]_{L.c} < \infty,$$

which completes the proof of the Lemma 4.2. □

THEOREM 4.3. *Suppose that Assumptions H[f], H[G], H[α] are satisfied and* $\varphi \in J[p]$.

Then there is $c > 0$ *such that*

1) *the sequence* $\{z^{(m)}\}$ *is well defined by* (i), (ii) *on* Ω_c *and there exists* $z^* \in C(\Omega_c, \mathbb{R})$ *such that*

$$\lim_{m\to\infty} z^{(m)}(t,x) = z^*(t,x) \ \textit{ uniformly on } E_c,$$

2) *the following estimates hold:*

$$\|z^* - z^{(m)}\|_c \leq \frac{\widetilde{C}}{2^{m-1}}(2\tilde{h})^{2^m-1}\eta, \quad m \geq 0 \tag{25}$$

where $\widetilde{C}, \tilde{h}, \eta$ *are given by (15).*

PROOF. It follows from Lemma 4.1 that the sequence $\{z^{(m)}\}$ defined by (20) and (19) exists and converges on Ω_c for some $c > 0$. Furthermore, from Lemma 4.2 and the chain rule differentiation Lemma (cf. [**4**]), there follows:

$$\frac{d}{d\tau} z^{(m+1)}(\tau, g(\tau, t, x)) \tag{26}$$

$$= \partial_t z^{(m+1)}(\tau, g(\tau, t, x)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, t, x)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, t, x))$$

for almost all $\tau \in I_{(t,x)}$ and almost all $(t, x) \in E_c$, where $m \geq 0$.

By integrating the above equation on $[\delta(t, x), t]$ with respect to τ, we get

$$z^{(m+1)}(t, x) = \varphi(S(t, x)) \tag{27}$$

$$+ \int_{\delta(t,x)}^{t} \left[\partial_t z^{(m+1)}(\tau, g(\tau, t, x)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, t, x)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, t, x)) \right] d\tau$$

almost everywhere on E_c.

Suppose now that $z^{(m+1)}$ satisfies (20) almost everywhere on E_c and condition (19). Then from the above equation, we get

$$\int_{\delta(t,x)}^{t} \left[\partial_t z^{(m+1)}(\tau, g(\tau, t, x)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, t, x)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, t, x)) \right] d\tau$$

$$= \int_{\delta(t,x)}^{t} \left[G(P^{(m)}(\tau, t, x)) + \partial_w G(P^{(m)}(\tau, t, x))(z^{(m+1)} - z^{(m)})_{\alpha(\tau, g(\tau, t, x))} \right] d\tau$$

almost everywhere on E_c. We put $\zeta = \delta(t, x)$. For a given $x \in [-b, b]$, let us put $y = g(\zeta, t, x)$. It follows from Lemma 2.1 that $g(\tau, t, x) = g(\tau, \zeta, y)$ for almost all $\tau \in I_{(t,x)}$, $t \in [0, c]$ and $x = g(t, \zeta, y)$. Then

$$\int_{\zeta}^{t} \left[\partial_t z^{(m+1)}(\tau, g(\tau, \zeta, y)) + \sum_{i=1}^{n} f_i(\tau, g(\tau, \zeta, y)) \partial_{x_i} z^{(m+1)}(\tau, g(\tau, \zeta, y)) \right] d\tau$$

$$= \int_{\zeta}^{t} \left[G(P^{(m)}(\tau, \zeta, y)) + \partial_w G(P^{(m)}(\tau, \zeta, y))(z^{(m+1)} - z^{(m)})_{\alpha(\tau, g(\tau, \zeta, y))} \right] d\tau$$

almost everywhere on E_c. Differentiating the above relation with respect to t, we get

$$\partial_t z^{(m+1)}(t, g(t,\zeta,y)) + \sum_{i=1}^{n} f_i(t, g(t,\zeta,y))\partial_{x_i} z^{(m+1)}(t, g(t,\zeta,y))$$
$$= G(P^{(m)}(t,\zeta,y)) + \partial_w G(P^{(m)}(t,\zeta,y))(z^{(m+1)} - z^{(m)})_{\alpha(t,g(t,\zeta,y))}$$

almost everywhere on E_c. Then we use the relation $x = g(t,\zeta,y)$ and we infer that $z^{(m+1)}$ satisfies (9) almost everywhere on E_c.

Now we prove that relations (9) and (19) imply (20) and (19). Suppose that $z^{(m+1)}$ is a Carathéodory solution of (9), (19) and that $z^{(m+1)}$ satisfies the Lipschitz condition with respect to (t,x). Then (27) holds true almost everywhere on E_c. Now, taking relation (9) along characteristics with $z = z^{(m+1)}$ and applying it to the right hand side of (27), we get (20).

We proved that the sequences $\{z^{(m)}\}$ defined on the one hand by (i), (ii) and on the other by (A), (B) are equivalent on Ω_c for $c \in (0,a]$. From Lemma 4.1 there follows that there exists $c \in (0,a]$ such that the second sequence exists and estimate (14) holds. The assertion of Theorem 4.3 follows from the equivalence of the above sequences. This completes the proof. $\square$

Remark 4.4. The above result can be extended on initial boundary value problems for the following systems

$$\partial_t z_i(t,x) = \sum_{j=1}^{n} f_{ij}(t,x)\partial_{x_j} z_i(t,x) = G_i(t,x,z_{(t,x)}),\ i = 1,\dots,k,$$

where $z = (z_1,\dots,z_k)$.

It is important in our considerations that we do not assume monotonicity conditions for given functions with respect to the functional variable.

References

1. Brzychczy S., *Chaplyghin's method for a system of nonlinear functional-differential equations of parabolic type* (in Russian), Differentsialnye Uravneniya, **22** (1986), 705–708.
2. Brzychczy S., *Chaplyghin's method for a system of nonlinear functional-differential equations of parabolic type* (in Russian), Differentsialnye Uravneniya, **25**, No. **6** (1989), 1050–1052.
3. Brzychczy S., *Chaplyghin's method for infinite systems of parabolic differential-functional equations*, Univ. Iagel. Acta Math., **38** (2000), 153–162.
4. Cesari L., *A Boundary Value Problem for Quasi Linear Hyperbolic Systems*, Riv. Mat. Univ. Parma, **3** (1974), 107–131.
5. Chaplyghin S. A., *Collected papers on mechanics and mathematics*, Moscow, 1954 (in Russian).
6. Człapiński T., *On the Chaplyghin method for partial differential-functional equations of the first-order*, Univ. Iagel. Acta Math., **35** (1997), 137–149.

7. Kamont Z., *On the Chaplygin method for partial differential-functional equations of the first-order*, Ann. Polon. Math., **38** (1980), 27–46.
8. Krylov V.I., Bobkov V.V., Monastyrnyi P.I., *Vychislitelnye metody vysshey matematiki* (in Russian), Vysshaya Shkola, Minsk, 1972.
9. Mlak W., Schechter E., *On the Chaplyghin method for partial differential equations of the first-order*, Ann. Polon. Math., **22** (1969/1970), 1–18.
10. Nowotarska M., *Chaplyghin method for an infinite system of first-order partial differential-functional equations*, Univ. Iagel. Acta Math., **22** (1981), 125–142.
11. Nowotarska M., *Remark on the Chaplyghin method for a system of first-order partial differential-functional equations*, Univ. Iagel. Acta Math., **25** (1985), 47–52.
12. Vidossich G., *Chaplyghin's method is Newton's method*, J. Math. Anal. Appl., **66** No. **1** (1978), 188–206.

Received May 23, 2005

Gdańsk University of Technology
Department of Differential Equations
ul. Gabriela Narutowicza 11–12
80-952 Gdańsk
Poland