UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLIII
2005

CONDITIONALLY MINIMAL PROJECTIONS

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Abstract. Sets of projections which do not contain the orthogonal projection are considered. In such sets, projections of minimal norm are found. The research presented in this paper has been motivated by earlier results concerning oblique projections (see [**14**]).

1. Introduction. Let $(X, \|\cdot\|)$ denote a normed space and let S be a linear subspace of X. A linear bounded operator $P : X \longrightarrow S$ is called a projection if $P(s) = s$ for any $s \in S$. Projections play an important role in approximation, optimization, spectral theory and orthogonal decomposition.

Recently, applications of projections to signal processing [**4**], [**7**], [**18**], sampling [**5**], [**30**], information theory [**30**], wavelets [**1**] and least square approximation [**16**], [**17**], [**31**] have been found.

Results connected with projections can be found in papers [**8**], [**9**], [**28**], [**29**], [**33**].

Survey of some results concerning so called oblique projections can be found in [**14**].

Among all projections $P : X \longrightarrow S$, we will distinguish a minimal projection. A projection $P_0 : X \longrightarrow S$ is called minimal if

$$\|P_0\| = inf\{\|P\| : P \in \mathbb{L}(X, S) : P|_S = id_S\}.$$

Numerous papers have been devoted to minimal projections. Let us mention the following [**2**], [**3**], [**6**], [**10**], [**11**], [**12**], [**13**], [**15**], [**19**], [**20**], [**21**], [**22**], [**23**], [**24**], [**25**], [**26**], [**32**].

If X is a Hilbert space, then the orthogonal projection is minimal. So seeking minimal projection, we will focus our attention on sets of projections which do not contain the orthogonal projection.

In this paper we will assume that $X = \mathbb{R}^n$.

Key words and phrases. Projection, operator norm, extreme points.

Let us consider the space $\mathbb{R}^n$ with the standard inner product given by the following formula

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n,$$

where $x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.
By $\mathbb{L}(\mathbb{R}^n)$ we will denote a set of all bounded linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $S \subset \mathbb{R}^n$ be a closed linear subspace of $\mathbb{R}^n$. By $P(S)$ we will denote the set of all projections on $\mathbb{R}^n$ with the range S:

$$P(S) = \{Q \in \mathbb{L}(\mathbb{R}^n) :\ Q^2 = Q,\ Q(\mathbb{R}^n) = S\}.$$

Let P_S denote the orthogonal projection on $\mathbb{R}^n$. Of course $P_S \in P(S)$.
For a given operator $A \in \mathbb{L}(\mathbb{R}^n)$ and a subspace $S \subset \mathbb{R}^n$, set

$$P(A, S) = \{Q \in P(S) :\ AQ = Q^*A\}.$$

If we take under consideration non-emptiness of the set $P(A, S)$ only, we find that the operator A need not be symmetric. In this paper we will assume that in the bases of subspaces $S, S^\perp$ the operator A has such matrix representation that:
$A = [a_{ij}]_{i,j=1}^n$, where $a_{ij} = a_{ji},\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, n$, and $k = dimS$. This is a much more general situation when compared with that considered in [**14**].

2. Preliminary results. In this paper we present three types of results:

- for a fixed $S \subset \mathbb{R}^n$, establish the set of bounded operators on $\mathbb{R}^n$ such that, for any operator A from this set, $P(A, S) \neq \emptyset$ holds;
- describe nonempty sets $P(A, S)$ for which $P_S \notin P(A, S)$;
- for some special $P(A, S)$ such that $P_S \notin P(A, S)$ and for some norms on $\mathbb{R}^n$, we will find $Q_0 \in P(A, S)$ such that

$$\|Q_0\| = \inf_{Q \in P(A,S)} \|Q\|.$$

These results have their origin in earlier research concerning oblique projections (see [**14**]).
For a closed k-dimensional subspace $S \subset \mathbb{R}^n$ we consider

$$S^\perp = \{y \in \mathbb{R}^n :\ \langle x, y\rangle = 0,\ x \in S\}.$$

Let $v_1, v_2, \ldots, v_k$ be a basis of the subspace S and $v_{k+1}, v_{k+2}, \ldots, v_n$ a basis of $S^\perp$. For any $Q \in P(S)$, we obtain the following matrix representation in these bases:

$$
(1) \qquad Q = \begin{bmatrix} 1 & 0 & \dots & 0 & r_{1k+1} & r_{1k+2} & \dots & r_{1n} \\ 0 & 1 & \dots & 0 & r_{2k+1} & r_{2k+2} & \dots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & r_{kk+1} & r_{kk+2} & \dots & r_{kn} \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix},
$$

where $r_{1j}, r_{2j}, \dots, r_{kj} \in \mathbb{R},\ j = k+1, k+2, \dots, n$. Let us consider a sequence of variables $r = (r_{1k+1}, \dots, r_{kk+1}, r_{1k+2}, \dots, r_{kk+2}, \dots, r_{1n}, \dots, r_{kn})$ and let $A = [a_{ij}]_{i,j=1}^{n}$. The equation $AQ = Q^*A$ leads to a system of linear equations which can be written as follows:

$$
(2) \qquad Cr = b,
$$

where

$$
b = (a_{1k+1}, \dots, a_{1n}, a_{2k+1}, \dots, a_{2n}, \dots, a_{kk+1}, \dots, a_{kn}, 0, \dots, 0)^T \\ \in \mathbb{R}^{\frac{n(n-1)-k(k-1)}{2} \times 1}
$$

and

$$
(3) \qquad C^T = \begin{bmatrix} C_1^T & C_2^T & \dots & C_k^T & C_{k+1}^T & C_{k+2}^T & \dots & C_{n-1}^T \end{bmatrix},
$$

where

$$
C_1 = \begin{bmatrix} a_{11} & 0 & . & . & 0 \\ a_{12} & 0 & . & . & 0 \\ . & . & . & . & . \\ a_{1k} & 0 & . & . & 0 \\ 0 & a_{11} & 0 & . & 0 \\ 0 & a_{12} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & a_{1k} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & a_{11} \\ 0 & 0 & . & . & a_{12} \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & a_{1k} \end{bmatrix}, \quad C_2 = \begin{bmatrix} a_{12} & 0 & . & . & 0 \\ a_{22} & 0 & . & . & 0 \\ . & . & . & . & . \\ a_{2k} & 0 & . & . & 0 \\ 0 & a_{12} & 0 & . & 0 \\ 0 & a_{22} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & a_{2k} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & a_{12} \\ 0 & 0 & . & . & a_{22} \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & a_{2k} \end{bmatrix}, \quad \dots \quad ,
$$

$$C_k = \begin{bmatrix} a_{1k} & 0 & . & . & 0 \\ a_{2k} & 0 & . & . & 0 \\ . & . & . & . & . \\ a_{kk} & 0 & . & . & 0 \\ 0 & a_{1k} & 0 & . & 0 \\ 0 & a_{2k} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & a_{kk} & 0 & . & 0 \\ 0 & 0 & . & . & 0 \\ . & . & . & . & 0 \\ 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & a_{1k} \\ 0 & 0 & . & . & a_{2k} \\ 0 & 0 & . & . & . \\ 0 & 0 & . & . & a_{kk} \end{bmatrix},$$

$$C_{k+1} = \begin{bmatrix} -a_{1k+2} & -a_{1k+3} & . & . & -a_{1n} \\ . & . & . & . & . \\ -a_{kk+2} & -a_{kk+3} & . & . & -a_{kn} \\ a_{1k+1} & 0 & . & . & 0 \\ . & . & . & . & . \\ a_{kk+1} & 0 & . & . & 0 \\ 0 & a_{1k+1} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & a_{kk+1} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & 0 & a_{1k+1} \\ . & . & . & . & . \\ 0 & . & . & 0 & a_{kk+1} \end{bmatrix}, \; C_{k+2} = \begin{bmatrix} 0 & . & . & . & 0 \\ 0 & . & . & . & 0 \\ 0 & . & . & . & 0 \\ -a_{1k+3} & -a_{1k+4} & . & . & -a_{1n} \\ . & . & . & . & . \\ -a_{kk+3} & -a_{kk+4} & . & . & -a_{kn} \\ a_{1k+2} & 0 & . & . & 0 \\ . & . & . & . & . \\ a_{kk+2} & 0 & . & . & 0 \\ 0 & a_{1k+2} & 0 & . & 0 \\ . & . & . & . & . \\ 0 & a_{kk+2} & 0 & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & a_{1k+2} \\ . & . & . & . & . \\ 0 & . & . & . & a_{kk+2} \end{bmatrix},$$

$$\ldots, C_{n-1} = \begin{bmatrix} 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ -a_{1n} \\ . \\ -a_{kn} \\ a_{1n-1} \\ . \\ a_{kn-1} \end{bmatrix}.$$

There is $C_1, C_2, \ldots, C_k \in \mathbb{R}^{k(n-k)\times(n-k)},\ C_m \in \mathbb{R}^{k(n-k)\times(n-m)}$, $m \in \{k+1, k+2, \ldots, n-1\}$. $C \in \mathbb{R}^{\frac{n(n-1)-k(k-1)}{2}\times k(n-k)}$, where $k = dimS$ and $k(n-k)$ is the number of variables.

Example 1. Let $n = 3,\ k = dimS = 1$. Let v_1 and v_2, v_3 be base of the subspaces S and $S^{\perp}$, respectively. Suppose that the matrix representations of the projection Q and an operator A are as follows:

$$Q = \begin{bmatrix} 1 & r_{12} & r_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{32} & a_{33} \end{bmatrix}.$$

Then in the equation (2) there is

$$C = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{11} \\ -a_{13} & a_{12} \end{bmatrix} \quad \text{and} \quad r = (r_{12}, r_{13})^T, \quad b = (a_{12}, a_{13}, 0)^T.$$

3. Non-emptiness of the set $P(A, S)$.

Theorem 1. *Let $A = [a_{ij}]_{i,i=1}^{n},\ a_{ij} = a_{ji},\ i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, n$, and let $S \subset \mathbb{R}^n$ be a subspace of dimension k. Then*

$$P(A, S) \neq \emptyset \quad \text{if and only if} \quad r(C) = r(C_d),$$

where

$$C_d = \begin{bmatrix} C_1^T & A_1 \\ C_2^T & A_2 \\ \vdots & \vdots \\ C_k^T & A_k \\ C_{k+1}^T & \mathbb{O}_{k+1} \\ C_{k+2}^T & \mathbb{O}_{k+2} \\ \vdots & \vdots \\ C_{n-1}^T & \mathbb{O}_{n-1} \end{bmatrix},$$

and

$$A_1 = \begin{bmatrix} a_{1k+1} \\ \vdots \\ a_{1n} \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{2k+1} \\ \vdots \\ a_{2n} \end{bmatrix}, \quad \dots, \quad A_k = \begin{bmatrix} a_{kk+1} \\ \vdots \\ a_{kn} \end{bmatrix},$$

$$\mathbb{O}_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{(n-m)\times 1}, \ m \in \{k+1, \dots, n-1\}.$$

PROOF. It is an obvious consequence of the theory of linear equations. □

COROLLARY. *Let the matrix representation of an operator A be such that $a_{ij} = 0, \ i = 1, 2, \dots, k, \ \ j = k+1, k+2, \dots, n$ and let $dimS = k$. Then $P(A, S) \neq \emptyset$.*

Note that each operator A considered in this paper has a representation

$$A = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}, \tag{4}$$

where $a \in \mathbb{L}(S), \ a = a^*, \ b \in \mathbb{L}(S^\perp, S), \ c \in \mathbb{L}(S^\perp), \ S \subset \mathbb{R}^n, \ dimS = k$.

Then, for $A \in \mathbb{L}(\mathbb{R}^n)$ with a matrix representation $A = [a_{ij}]_{i,j=1}^n$, there is

$$\begin{gathered} a = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{1k} & \dots & a_{kk} \end{bmatrix}, \quad b = \begin{bmatrix} a_{1k+1} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{kk+1} & \dots & a_{kn} \end{bmatrix}, \\ c = \begin{bmatrix} a_{k+1k+1} & \dots & a_{k+1n} \\ \vdots & \ddots & \vdots \\ a_{nk+1} & \dots & a_{nn} \end{bmatrix}. \end{gathered} \tag{5}$$

THEOREM 2. *Let an operator A has the matrix representation given by formula (4), where operators a, b, c are given by (5). Let $S \subset \mathbb{R}^n$ be a k-dimensional subspace.*

Then

$$P(A,S) \neq \emptyset \quad \textit{if and only if} \quad R(b) \subset R(a).$$

PROOF. Let $v_1, v_2, \ldots, v_k$ be a basis of the subspace S and let v_{k+1}, v_{k+2}, $\ldots$, v_n be a basis of the subspace $S^{\perp}$.

Let us assume that $R(b) \subset R(a)$. Hence,

$$bv_{k+1}, bv_{k+2}, \ldots, bv_n \in R(a),$$

which means that there exist $x_{k+1}, x_{k+2}, \ldots, x_n \in S$ such that

$$bv_{k+1} = ax_{k+1},\ bv_{k+2} = ax_{k+2}, \ldots,\ bv_n = ax_n.$$

From the matrix representation of the operator b, (see (5)) we conclude

$$bv_{k+1} = \begin{bmatrix} a_{1k+1} \\ a_{2k+1} \\ \vdots \\ a_{kk+1} \end{bmatrix},\ bv_{k+2} = \begin{bmatrix} a_{1k+2} \\ a_{2k+2} \\ \vdots \\ a_{kk+2} \end{bmatrix}, \ldots,\ bv_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{kn} \end{bmatrix}.$$

Let $x_{k+1}, x_{k+2}, \ldots, x_n$ have the following representations in the given basis of S

$$x_{k+1} = (x_{1k+1}, \ldots, x_{kk+1}), \ldots, x_n = (x_{1n}, \ldots, x_{kn}).$$

Then

$$ax_{k+1} = \begin{bmatrix} a_{11}x_{1k+1} + \ldots + a_{1k}x_{kk+1} \\ \vdots \\ a_{1k}x_{1k+1} + \ldots + a_{kk}x_{kk+1} \end{bmatrix}, \ldots, ax_n = \begin{bmatrix} a_{11}x_{1n} + \ldots + a_{1k}x_{kn} \\ \vdots \\ a_{1k}x_{1n} + \ldots + a_{kk}x_{kn} \end{bmatrix}.$$

Hence, we obtain

$$\begin{gathered} a_{11}x_{1k+1} + \ldots + a_{1k}x_{kk+1} = a_{1k+1}, \\ \vdots \\ a_{1k}x_{1k+1} + \ldots + a_{kk}x_{kk+1} = a_{kk+1}, \\ \vdots \\ a_{11}x_{1n} + \ldots + a_{1k}x_{kn} = a_{1n}, \\ \vdots \\ a_{1k}x_{1n} + \ldots + a_{kk}x_{kn} = a_{kn}. \end{gathered}$$

Consequently, the system of linear equations given by the matrix C (see formula (3)) has a solution. This means that $P(A,S) \neq \emptyset$.

Now, let us assume that $P(A,S) \neq \emptyset$.

Hence, there exist $x_{k+1}, x_{k+2}, \ldots, x_n \in S$ such that

$$bv_{k+1} = ax_{k+1},\ bv_{k+2} = ax_{k+2}, \ldots,\ bv_n = ax_n.$$

This means that $bv_{k+1}, \ldots, bv_n \in R(a)$. Since $R(a)$ is a linear space,

$$\gamma_{k+1}bv_{k+1} + \ldots + \gamma_n bv_n = b(\gamma_{k+1}v_{k+1} + \ldots + \gamma_n v_n) \in R(a)$$

for any $\gamma_{k+1}, \ldots, \gamma_n \in \mathbb{R}$.

From this we conclude that $b(S^{\perp}) = R(b) \subset R(a)$. The proof is complete. $\square$

Remark. Note that, for the space $\mathbb{R}^n$, Theorem 2 is a generalization of the result obtained in [**14**].
In [**14**], a positive operator A has been considered, whereas an operator A considered in Theorem 2 need not even be symmetric.

Corollary. *Let the assumptions of Theorem 2 hold.*
If $b \equiv 0$ *then* $P(A, S) \neq \emptyset$.

4. The orthogonal projection. In what follows, we will assume that $P(A, S) \neq \emptyset$. If the orthogonal projection P_S belongs to $P(A, S)$, then P_S is an element of the minimum norm.
The theorem below characterizes operators $A \in \mathbb{L}(\mathbb{R}^n)$ such that $P_S \in P(A, S)$.

Theorem 3. *Let an operator* A *have the matrix representation* $A = [a_{ij}]_{i,j=1}^{n}$, $a_{ij} = a_{ji}$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, n$ *and let dimS* $= k$. *Then*

$$P_S \in P(A, S) \quad \textit{if and only if}$$
$$a_{ij} = a_{ji} = 0, \ i = 1, 2, \ldots, k, \ j = k+1, k+2, \ldots, n.$$

Proof. It suffices to use the results of the theory of linear homogenous equations. $\square$

Theorem 4. *Let the assumptions of Theorem 3 hold. Let* $P(A, S) \neq \emptyset$. *Then* $P_S \notin P(A, S)$ *if and only if there exist* $l \in \{1, 2, \ldots, k\}$ *and* $m \in \{k+1, k+2, \ldots, n\}$ *such that* $a_{lm} \neq 0$.
Additionally:
if $r(C) = k(n-k)$ *then* $\#P(A, S) = 1$;
if $r(C) < k(n-k)$ *then* $P(A, S)$ *is a non-trivial affine subspace of* $\mathbb{L}(\mathbb{R}^n)$,
where the matrix C *is given by* (2) *and* $r(C)$ *denotes the range of* C.

Proof. It suffices to use the results of the theory of linear equations. $\square$

Example 2. Let us consider the situation in Example 1. This situation leads to the following system of linear equations

$$\begin{cases} a_{11}r_{12} = a_{12}, \\ a_{11}r_{13} = a_{13}, \\ -a_{13}r_{12} + a_{12}r_{13} = 0. \end{cases}$$

If we assume that the solution of such linear system exists, we conclude that there is only one solution if and only if $a_{11} \neq 0$ and then $P(A, S)$ has one element only. The set $P(A, S)$ is infinite if and only if $a_{11} = a_{12} = a_{13} = 0$. In the second situation, r_{12}, r_{13} may be chosen arbitrarily, so $P_S \in P(A, S)$.

EXAMPLE 3. Let us consider the following situation: $n = 3$, $dim S = 2$, and

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & r_{13} \\ 0 & 1 & r_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

By (3) there is

$$\begin{cases} 4r_{13} + 2r_{23} = 2, \\ 2r_{13} + r_{23} = 1. \end{cases}$$

Hence, $r_{23} = 1 - 2r_{13},\ r_{13} \in \mathbb{R}$ and thus

$$P(A, S) = \{Q = \begin{bmatrix} 1 & 0 & r_{13} \\ 0 & 1 & 1 - 2r_{13} \\ 0 & 0 & 0 \end{bmatrix},\ r_{13} \in \mathbb{R}\}.$$

So $P(A, S)$ is not finite and $P_S \notin P(A, S)$.

5. Projections of minimum norms. Now we focus our attention on the case where $P(A, S)$ has more than one element and $P_S \notin P(A, S)$. In such a case, for the special class of operator A, we will find an element $Q \in P(A, S)$ of minimum norm. We will consider different norms in $\mathbb{R}^n$.

Let $S \subset \mathbb{R}^n$ be a k-dimensional subspace and let the operator A have the following matrix representation

$$A = \begin{bmatrix} a_{11} & 0 & . & . & a_{1k} & 0 & . & . & a_{1n} \\ 0 & a_{22} & 0 & . & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ a_{1k} & 0 & . & 0 & a_{kk} & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 & a_{k+1k+1} & . & . & a_{k+1n} \\ . & . & . & . & . & . & . & . & . \\ a_{1n} & 0 & . & . & 0 & a_{nk+1} & . & . & a_{nn} \end{bmatrix},$$

where $a_{1k}, a_{1n} \neq 0$.

Any projection Q admits a representation of form (1).
From the equation $AQ = Q^*A$ we obtain

$$\begin{cases} a_{11}r_{1n} + a_{1k}r_{kn} = a_{1n}, \\ a_{1k}r_{1n} + a_{kk}r_{kn} = 0, \\ a_{22}r_{2k+1} = \ldots = a_{22}r_{2n} = 0, \\ \ldots \\ a_{k-1k-1}r_{k-1k+1} = \ldots = a_{k-1k-1}r_{k-1n} = 0, \\ r_{kk+1} = \ldots = r_{kn-1} = 0. \end{cases}$$

Hence

- if there exists $l \in \{2, 3, \ldots, k-1\}$ such that $a_{ll} = 0$, then $r_{lk+1}, \ldots, r_{ln} \in \mathbb{R}$;
- if $a_{22}, \ldots, a_{k-1k-1} \neq 0$, then $r_{2k+1} = \ldots = r_{k-1n} = 0$.

  Additionally, from the equations

$$\begin{cases} a_{11}r_{1n} + a_{1k}r_{kn} = a_{1n}, \\ a_{1k}r_{1n} + a_{kk}r_{kn} = 0, \end{cases}$$

we obtain:

$$r_{kn} = \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}}, \quad r_{1n} = \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}}.$$

So $r_{kn} \neq 0$ and $r_{1n} = 0$ if and only if $a_{kk} = 0$.

Summarizing the above, we obtain the following description of $P(A, S)$:

- if there exists $l \in \{2, 3, \ldots, k-1\}$ such that $a_{ll} = 0$ then

$$P(A,S) = \left\{ Q = \begin{bmatrix} 1 & 0 & . & 0 & 0 & . & . & 0 & \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & 1 & . & . & . & . & . & . & . \\ . & . & . & . & r_{lk+1} & . & . & . & r_{ln} \\ . & . & . & 0 & . & . & . & . & . \\ 0 & . & . & 1 & 0 & . & . & 0 & \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 \end{bmatrix}, \right. \tag{6}$$

$$\left. r_{lk+1}, \ldots, r_{ln} \in \mathbb{R} \right\},$$

– if $a_{22}, \ldots, a_{kk} \neq 0$ then

$$
(7) \qquad P(A,S) = \left\{ Q = \begin{bmatrix} 1 & 0 & . & 0 & 0 & . & . & 0 & \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & 1 & . & . & . & . & . & . & 0 \\ 0 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 0 \\ 0 & . & . & 1 & 0 & . & . & 0 & \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \\ 0 & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . \\ 0 & . & . & . & . & . & . & . & 0 \end{bmatrix} \right\}.
$$

In (7), $P(A,S)$ has one element only, which is not interesting while looking for an element of minimum norm.

From now on, we focus our attention on the case given by (6). In such a case, the set $P(A,S)$ is not finite.

Example 4. Let us assume that an operator Q has the matrix representation $Q = [q_{ij}]_{i,j=1}^n$ and let us consider the following norm

$$
\|Q\| = \sqrt{\sum_{i,j=1}^{n} \mid q_{ij} \mid^2}.
$$

For any $Q \in P(A,S)$, where $P(A,S)$ is given by (6), there is

$$
\|Q\|^2 = k + r_{lk+1}^2 + \ldots + r_{ln}^2 + (\frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}})^2 + (\frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}})^2.
$$

The above expression attains its minimum if $r_{lk+1} = \ldots = r_{ln} = 0$.

Now, for a projection $Q \in P(A,S)$, we will consider different types of operator norms. These norms will be given by different norms in the space $\mathbb{R}^n$. For an operator $Q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the operator norm is defined by the formula

$$
(8) \qquad \|Q\|_{op} = \sup_{\|x\|=1} \|Qx\|.
$$

Let D be a set in $\mathbb{R}^n$. An element $d \in D$ will be called an extreme point of D if for any $d_1, d_2 \in D$, if there exists $\alpha \in (0,1)$ such that $d = \alpha d_1 + (1-\alpha)d_2$ then $d = d_1 = d_2$.

Let us denote by $E \subset \mathbb{R}^n$ the set of extreme points of the unit sphere. It is well-known (see, e.g., [**27**]) that

$$
\|Q\|_{op} = \sup_{y \in E} \|Qy\|.
$$

EXAMPLE 5. Consider $\mathbb{R}^n$ with the l_1-norm, that is

$$\|x\|_1 = \sum_{i=1}^{n} \mid x_i \mid, \quad x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n. \tag{9}$$

Then

$$E = \{(\pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\}, \quad \#E = 2n.$$

Because of the definition of $\| \cdot \|_1$, it is enough to consider norms $\|Qe_i\|_1$, $i = 1, 2, \dots, n,\ e_i = (0, 0, \dots, 1, \dots, 0)$ (a 1 on i-th position).

Because of the matrix representation of any $Q \in P(A, S)$, where $P(A, S)$ is given by (6), we have

$$Qe_1 = (1, 0, \dots, 0), \dots, Qe_k = (0, \dots, 0, 1, 0 \dots 0), \text{(a 1 on } k\text{-th position)},$$
$$Qe_{k+1} = (r_{1k+1}, \dots, r_{kk+1}, 0, \dots, 0), \dots, Qe_n = (r_{1n}, \dots, r_{kn}, 0, \dots, 0),$$

where

$$r_{1n} = \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}},\ r_{kn} = \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}}.$$

From the above, for any $Q \in P(A, S)$, we obtain

$$\|Q\|_{op} = max\{1, \sum_{i=1}^{k} \mid r_{ik+1} \mid, \dots, \sum_{i=1}^{k} \mid r_{in-1} \mid,$$
$$\mid \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \mid + \mid r_{2n} \mid + \dots + \mid r_{k-1n} \mid + \mid \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \mid\}.$$

So, for any $Q \in P(A, S)$, there is $\|Q\|_{op} \geq 1$ and $\|Q\|_{op} = 1$ if, in its matrix representation, Q has elements such that

$$\sum_{i=1}^{k} \mid r_{ik+1} \mid \leq 1, \dots, \sum_{i=1}^{k} \mid r_{in-1} \mid \leq 1,$$
$$\mid \frac{-a_{kk}a_{1n}}{a_{1k}^2 - a_{11}a_{kk}} \mid + \mid r_{2n} \mid + \dots + \mid r_{k-1n} \mid + \mid \frac{a_{1n}a_{1k}}{a_{1k}^2 - a_{11}a_{kk}} \mid \leq 1.$$

EXAMPLE 6. Consider $\mathbb{R}^n$ with the norm $\| \cdot \|_{max}$, where

$$\|x\|_{max} = max\{\mid x_1 \mid, \mid x_2 \mid, \dots, \mid x_n \mid\},\ x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n. \tag{10}$$

Then

$$E = \{e_i = (e_{i1}, \dots, e_{in}),\ e_{il} \in \{1, -1\}, l = 1, 2, \dots, n,\ i = 1, 2, .., 2^n\}.$$

So there is

$$\|Q\|_{op} = max\{\mid 1 + \sum_{i=k+1}^{n} r_{1i} \mid, \mid 1 + \sum_{i=k+1}^{n} r_{2i} \mid, \dots, \mid 1 + \sum_{i=k+1}^{n} r_{ki} \mid\},$$

where
$$r_{ml} \geq 0,\ m = 1, 2, \ldots, k,\ l = k+1, \ldots, n.$$
Thus the minimum norm is attained for such an operator Q whose matrix representation fulfils the following
$$r_{1k+1} = \ldots = r_{1n-1} = r_{2k+1} = \ldots = r_{2n} = \ldots = r_{kk+1} = \ldots = r_{kn-1} = 0.$$

EXAMPLE 7. Consider
$$Q : (\mathbb{R}^n, \|\cdot\|_1) \longrightarrow (\mathbb{R}^n, \|\cdot\|_{max}),$$
where $\|\cdot\|_1,\ \|\cdot\|_{max}$ are given by (9) and (10), respectively.

Then
$$E = \{(\pm 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, \pm 1)\}, \quad \#E = 2n,$$
and
$$\|Q\|_{op} = max\{1, \mid r_{1k+1} \mid, \ldots, \mid r_{kk+1} \mid, \ldots, \mid r_{1n} \mid, \ldots, \mid r_{kn} \mid\}.$$
So $\|Q\| \geq 1$ and $\|Q\| = 1$ if $\mid r_{lm} \mid \leq 1,\ l = 1, 2, \ldots, k,\ m = k+1, k+2, \ldots, n.$

EXAMPLE 8. We consider
$$Q : (\mathbb{R}^n, \|\cdot\|_{max}) \longrightarrow (\mathbb{R}^n, \|\cdot\|_1),$$
where $\|\cdot\|_1,\ \|\cdot\|_{max}$ are given by (9) and (10), respectively.

Then
$$E = \{e_i = (e_{i1}, \ldots, e_{in}),\ e_{il} \in \{1, -1\}, l = 1, 2, \ldots, n,\ i = 1, 2, .., 2^n\},$$
and
$$\|Q\|_{op} = \mid 1 + r_{1k+1} + \ldots + r_{1n} \mid + \ldots + \mid 1 + r_{kk+1} + \ldots + r_{kn} \mid,$$
where
$$r_{lm} \geq 0,\ l = 1, 2, \ldots, k,\ m = k+1, k+2, \ldots, n.$$
So $Q \in P(A, S)$ is of minimum norm if its matrix representation satisfies
$$r_{1k+1} = \ldots = r_{1n-1} = \ldots = r_{kk+1} = \ldots = r_{kn-1} = 0.$$

References

1. Aldroubi A., *Oblique and hierarchical multiwavelet bases*, Appl. Comput. Harmon. Anal., **4**, No. **3** (1997), 231–263.
2. Baronti M., Papini P.L., *Norm one projections onto subspaces of l_p*, Ann. Mat. Pura Appl., **4** (1988), 53–61.
3. Baronti M., Lewicki G., *Strongly unique minimal projections on hyperplanes*, J. Approx. Theory, **78** (1994), 1–18.
4. Behrens R.T., Scharf L.L., *Signal processing applications of oblique projection operators*, IEEE Trans. Signal Processing, **42** (1994), 1413–1424.
5. Benedetto J., *Frames, sampling and seizure prediction. Advances in wavelets (Hong Kong 1997)*, Springer, Singapore, 1999, 1–25.

6. Blatter J., Cheney E.W., *Minimal projections onto hyperplanes in sequence spaces*, Ann. Mat. Pura Appl., **101** (1974), 215–227.
7. Blu T., Unser M., *Quantitative Fourier analysis of approximation techniques. I. Interpolators and projections, Signal Processing*, IEEE Trans. Signal Processing, **47** (1999), 2783–2795.
8. Chalmers B.L., Lewicki G., *Symmetric spaces with maximal projection constans*, J. Funct. Anal., **200** (2003), 1–22.
9. Chalmers B.L., Lewicki G., *Symmetric subspaces of l_1 with large projection constants*, Studia Math., **134**, No. **2** (1999), 119–133.
10. Chalmers B.L., Metcalf F.T., *The determination of minimal projections and extensions in L_1*, Trans. Amer. Math. Soc., **329** (1992), 289–305.
11. Cheney E.W., Light W.A., *Approximation Theory in Tensor Product Spaces*, Lecture Notes in Math., Dold A. and Eckman B. (eds.), Vol. **1169**, Springer-Verlag, Berlin, 1985.
12. Cheney E.W., Morris P.D., *On the existence and characterization of minimal projections*, J. Reine Angew. Math., **270** (1974), 61–76.
13. Cheney E.W., Price K.H., *Minimal projections*, in: *Approximation Theory*, Talbot A. (ed.), Proc. Symp. Lancaster, July 1969, Academic Press, London/New York, 1970, 261–289.
14. Corach G., Maestripieri A., Stojanoff D., *A Classification of Projectors*, Banach Center Publ., Polish Acad. Sci. (to appear).
15. Franchetti C., *Projections onto hyperplanes in Banach spaces*, J. Approx. Theory, **38** (1983), 319–333.
16. Forsgren A., *On linear least-squares problems with diagonally dominant weight matrices*, SIAM J. Matrix Anal. Appl., **17** (1996), 763–788.
17. Forsgren A., Sporre G., *On weighted linear least-squares problems related to interior methods for convex quadratic programming*, SIAM J. Matrix Anal. Appl., **23** (2001), 42–56.
18. Kayalar S., Weinert H.L., *Oblique projections: formulas, algorithms and error bounds*, Math. Control Signal Systems, **2** (1989), 33–45.
19. Kilgore T.A., *A characterization of Lagrange interpolating projection with minimal Tchebycheff norm*, J. Approx. Theory, **24** (1978), 273–288.
20. Lewicki G., *Kolmogorov's type criteria for spaces of compact operators*, J. Approx. Theory, **64** (1991), 181–202.
21. Lewicki G., *Minimal Projections onto Two Dimensional Subspaces of $l_\infty^{(4)}$*, J. Approx. Theory, **88**, No. **1** (1997), 92–108.
22. Lewicki G., *Minimal projections onto subspaces of $l_\infty^{(n)}$ of codimension two*, Collect. Math., **44** (1993), 167–179.
23. Lewicki G., *Best approximation in spaces of bounded, linear operators*, Dissertationes Math., **330** (1994).
24. Odinec V.P. (Odyniec W.), *Codimension one minimal projections in Banach spaces and a mathematical programming problem*, Dissertationes Math., **254** (1986).
25. Odyniec W., Lewicki G., *Minimal Projections in Banach Spaces*, Lecture Notes in Math., Vol. **1449**, Springer-Verlag, Berlin/Heidelberg/New York, 1990.
26. Rolewicz S., *On projections on subspaces of codimension one*, Studia Math., **44** (1990), 17–19.
27. Rudin W., *Functional Analysis*, McGraw-Hill Book Company, New York, 1973.
28. Steinberg J., *A class of integral transforms which are projections*, Integral Equations Operator Theory, **34**, No. **1** (1999), 65–90.

29. Steinberg J., *Oblique projections in Hilbert spaces*, Integral Equations Operator Theory, **38**, No. **1** (2000), 81–119.
30. Unser M., *Quasi-orthogonality and quasi-projections*, Applied Comp. Harmonic Anal., **3** (1996), 201–214.
31. Wei M.S., *Upper bound and stability of scaled pseudoinverses*, Numer. Math., **7**2 (1995), 285–293.
32. Wojtaszczyk P., *Banach spaces for analysts*, Cambridge Studies in Advanced Math., Vol. **25**, Cambridge Univ. Press, Cambridge, 1991.
33. Wulbert D.E., *Some complemented function spaces in $C(X)$*, Pacific J. Math., **24**, No. **3** (1968), 586–602.

Received October 25, 2004

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