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USEFUL PROPERTIES OF INDEX PAIRS FOR UPPERSEMICONTINUOUS MULTIVALUED DYNAMICAL SYSTEMS

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Abstract. We extend some properties of index pairs proved by Mrozek in a singlevalued setting in [**3**] to multivalued maps. These properties are crucial in proving the corectness of the definition and the homotopy property of Conley type index for multivalued maps, see [**6**], [**5**], [**7**].

1. Introduction. Any index of Conley type is a topological invariant defined for isolated invariant sets with use of index pairs. In this paper we consider pairs for isolated invariant sets of multivalued discrete dynamical systems. Here, an index is an equivalence class, in the sense of the Szymczak relation, of a pair consisting of some space – built with use of an index pair, and the homotopy class of the index map acting on this space. To prove that the index depends only on the isolated invariant set, one needs to show that it is independent of the choice of the specific index pair related to the invariant set considered.

There are two main ways of proving this independence of the choice of an index pair. Assume P and Q are two index pairs for the same isolated invariant set. In the method developed by Szymczak, the actual isomorphism between the equivalence classes of the Szymczak relation corresponding to P and Q is given. In the approach introduced by Mrozek, a sequence of index pairs between P and Q is built. The final isomorphism is a composition of the

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isomorphisms existing between the Szymczak equivalence classes corresponding to the intermediate pairs between P and Q.

There are many variations of the definition of an index pair. Here we deal with the slightly modified definition proposed by Mrozek and Kaczyński in [**2**]. Changes which we introduced in that definition are essential in defining the Conley index for multivalued maps, following the ideas of Mrozek, Reineck and Srzednicki [**4**] developed for single-valued flows (see [**6**] or [**5**] for details).

All properties stated here are used to prove that the definition of the index given in [**6**] is well posed and that it possesses a homotopy property (see [**7**] or [**5**]). The proof of the correctness of the definition of index defined in [**6**] is conducted in a way similar to that introduced by Mrozek (using 'midway pairs'). All properties proven in this paper are extensively used in [**5**], [**6**] and [**7**].

2. Multivalued Maps and Dynamical Systems. By $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}^-$, $\mathbb{R}$, I we denote integers, natural numbers (with zero), negative integers with zero, real numbers and the interval $[0,1]$, respectively.

Let X be a topological space. For any set $A \subset X$ by $\operatorname{int} A$, $\operatorname{bd} A$, $\operatorname{cl} A$ we denote the interior, boundary and closure of A, respectively. If $A \subset B \subset X$ by $\operatorname{int}_B A$ we understand a relative interior of A in B. By $P = (P_1, P_2)$ we denote a pair of subsets of X. Note that we *do not* require that $P_2 \subset P_1$. If $Q = (Q_1, Q_2)$ is another such a pair of subsets of X, then $P \subset Q$ means that $P_1 \subset Q_1$ and $P_2 \subset Q_2$. By $\operatorname{int} P$ we denote the pair $(\operatorname{int} P_1, \operatorname{int} P_2)$. Similarly we extend the notation of $\operatorname{bd} P$ and $\operatorname{cl} P$. By an *interval* in $\mathbb{Z}$ we understand a trace of a closed interval in $\mathbb{R}$ and denote it by $[m,n]$, for $m, n \in \mathbb{Z}$ or $m = -\infty$ or $n = +\infty$.

Let X and Y be topological spaces. We denote by

$$F : X \multimap Y \tag{2.1}$$

a *multivalued map*, that is a map defined on X with values being subsets of Y.

The set

$$\operatorname{graph}(F) = \{(x,y) \in X \times Y : y \in F(x)\} \tag{2.2}$$

is called the *graph* of the map F.

For $P = (P_1, P_2)$, by $F(P)$ we mean a pair of sets $(F(P_1), F(P_2))$.

Let also Z be a topological space and $G : Y \multimap Z$ a multivalued map. The *composition* of the maps F and G is a multivalued map $G \circ F : X \multimap Z$, defined as

$$G \circ F(x) := \bigcup \{G(y) : y \in F(x)\}, \text{ for } x \in X. \tag{2.3}$$

If $F : X \multimap X$, for $k \in \mathbb{N} \setminus \{0\}$, by F^k we understand k-times composition according to formula (2.3).

We say that $F : X \multimap Y$ is *upper semicontinuous at the point* x_0 if the set

$$F^{*-1}(A) := \{x \in X : F(x) \cap A \neq \emptyset\}, \tag{2.4}$$

called the *large counter image* of the set A is closed for any closed $A \subset Y$ such that $F(x_0) \cap A \neq \emptyset$. The above condition is equivalent to the fact that the set

$$F^{-1}(U) := \{x \in X : F(x) \subset U\}, \tag{2.5}$$

called a *small counter image* of the set U is open for any open $U \subset Y$ such that $F(x_0) \subset U$. If $F : X \multimap Y$ is upper semicontinuous at each point $x_0 \in X$ we say that it is an *upper semicontinuous map*.

Let us denote by $\mathcal{USC}^c$ the category whose objects are Hausdorff spaces and morphisms are upper semicontinuous maps with compact values. Composition of morphisms is defined by formula (2.3). More information on upper semicontinuous maps can be found in [**1**].

Throughout this paper we assume that (X, d_X) (or briefly X) is a locally compact metric space and $F \in \mathcal{USC}^c(X, X)$.

To simplify notation, we will write x instead of $\{x\}$.

Definition 2.1. ([**2**], Definition 2.1) Let $\Phi \in \mathcal{USC}^c(X \times \mathbb{Z}, X)$. We call Φ a multivalued dynamical system if

(i) $\forall x \in X \ : \ \Phi(x, 0) = x$;

(ii) $\forall m, n \in \mathbb{Z},\ mn > 0\ \forall x \in X \ : \ \Phi(\Phi(x, n), m) = \Phi(x, n + m)$;

(iii) $\forall x, y \in X \ : \ y \in \Phi(x, -1) \ \Leftrightarrow \ x \in \Phi(y, 1)$.

For a given $F : X \multimap X$, we can define $\Phi_F : X \times \mathbb{Z} \multimap X$ as

$$\Phi_F(x, n) := \begin{cases} F^n(x), & \text{for } x \in X \text{ and } n > 0, \\ x, & \text{for } x \in X \text{ and } n = 0, \\ (F^{*-1})^{-n}(x), & \text{for } x \in X \text{ and } n < 0. \end{cases} \tag{2.6}$$

Obviously, Φ_F satisfies conditions of Definition 2.1. We say that F *induces a multivalued dynamical system* (2.6), or briefly that F is a dynamical system.

A *trajectory (solution)* of a dynamical system F passing through $x \in X$ is a (singlevalued) map $\sigma : J \to X$ such that $\sigma(n+1) \in F(\sigma(n))$, for $n, n+1 \in J$, and $\sigma(n_0) = x$, for some $n_0 \in J$, where J is an interval in $\mathbb{Z}$.

Assume $N \subset X$ is a compact subset and $F : X \multimap X$ is a dynamical system. Let us introduce the following notation

$$\text{Inv}^{+}_{F} N := \{x \in N : \text{there is a solution } \sigma : \mathbb{N} \to N \text{ of } F \text{ passing through } x\},$$

$$\text{Inv}^{-}_{F} N := \{x \in N : \text{there is a solution } \sigma : \mathbb{Z}^{-} \to N \text{ of } F \text{ passing through } x\},$$

$$\text{Inv}_{F} N := \{x \in N : \text{there is a solution } \sigma : \mathbb{Z} \to N \text{ of } F \text{ passing through } x\}.$$

The sets $\text{Inv}^{+}_{F} N$, $\text{Inv}^{-}_{F} N$ and $\text{Inv}_{F} N$ are called a *positive invariant*, a *negative invariant*, and an *invariant part of* N, respectively. Whenever the

underlying map is known from the context, we will omit the index F and write Inv ^{+}N, Inv ^{-}N and Inv N, respectively.

A compact set $N \subset X$ is called an *isolating neighborhood* for a dynamical system F iff

$$\text{Inv } N \cup F(\text{Inv } N) \subset \text{int } N. \tag{2.7}$$

A compact set $S \subset X$ is called an *isolated invariant set* for a dynamical system F iff there exists an isolating neighborhood N such that S is its invariant part. By virtue of definition (2.5) of a small counter image, condition (2.7) is equivalent to

$$\text{Inv } N \subset \text{int } N \cap F^{-1}(\text{int } N). \tag{2.8}$$

A *diameter* of a set $A \subset X$ is defined as follows

$$\text{diam } A := \sup\{d_X(y, y') : y, y' \in A\};$$

let us put

$$\text{diam } {}_N F := \sup\{\text{diam } F(x) : x \in N\},$$

$$\text{dist } (A, B) := \min\{d_X(x, y) : x \in A, y \in B\}, \text{ for } A, B \subset X.$$

Notice that if dist (Inv N, bd N) $>$ diam ${}_N F$, then condition (2.7) is satisfied.

For our purposes we need to slightly modify the definition of an index pair introduced in the multivalued context by Mrozek and Kaczyński [**2**].

Definition 2.2. Let N be an isolating neighborhood for a multivalued dynamical system F. Then the pair $P = (P_1, P_2)$ of compact subsets of N such that $P_1 \setminus P_2 \subset \text{int } N$ is called an index pair in the neighborhood N for a multivalued dynamical system F if

(a) $F(P_i) \cap N \subset P_i$, $i = 1, 2$;
(b) $F(P_1 \setminus P_2) \subset \text{int } N$;
(c) Inv $^{-}N \subset \text{int } {}_N P_1$ and Inv $^{+}N \subset N \setminus P_2$.

Not to mention other differences, notice that we here admit index pairs that are not topological pairs, i.e., we omit the condition $P_2 \subset P_1$ required in [**2**].

Only slightly modifying the proof of Theorem 2.6 ([**2**]) we obtain the existence of our index pairs. The detailed proof is given in [**5**].

Theorem 2.3. *Assume N is an isolating neighborhood for F and W is any neighborhood of* Inv N. *Then there exists an index pair P in the isolating neighborhood N, such that $P_1 \setminus P_2 \subset W$.*

The family of index pairs in the isolating neighborhood N for a dynamical system F is denoted by $IP(N, F)$.

We exploit the notation introduced in [**2**]. Let N be a compact subset of X, $x \in X$ and $n \in \mathbb{Z}^+$. Let us define the maps $F_{N,n} : N \multimap N$, $F_{N,-n} : N \multimap N$, $F_N^+ : N \multimap N$ and $F_N^- : N \multimap N$ in the following way

$$F_{N,n}(x) := \{y \in N : \text{there exists } \sigma : [0,n] \to N, \text{ a solution of } F \text{ such that } \sigma(0) = x \text{ and } \sigma(n) = y\}, \tag{2.9}$$

$$F_{N,-n}(x) := \{y \in N : \text{there exists } \sigma : [-n,0] \to N, \text{ a solution of } F \text{ such that } \sigma(-n) = y \text{ and } \sigma(0) = x\}, \tag{2.10}$$

$$F_N^+(x) := \bigcup_{n \in \mathbb{Z}^+} F_{N,n}(x), \tag{2.11}$$

$$F_N^-(x) := \bigcup_{n \in \mathbb{Z}^+} F_{N,-n}(x). \tag{2.12}$$

For the Reader's convenience below we quote two lemmas from [**2**] which are extensively used in the proofs of some facts in this paper.

LEMMA 2.4. ([**2**], *Lemma 2.9*) *Let N be compact. Then*
(a) the sets $\operatorname{Inv}^+_F N$, $\operatorname{Inv}^-_F N$ *and* $\operatorname{Inv}_F N$ *are compact;*
(b) if A is compact with $\operatorname{Inv}^-_F N \subset A \subset N$, *then $F_N^+(A)$ is compact.*

LEMMA 2.5. ([**2**], *Lemma 2.10*) *Let K and N be compact subsets of X such that $K \subset N$ and $K \cap \operatorname{Inv}^+_F N = \emptyset$ ($K \cap \operatorname{Inv}^-_F N = \emptyset$, respectively). Then*
(a) $F_{N,n}(K) = \emptyset$ for all but finitely many $n > 0$ ($n < 0$, respectively);
(b) the mapping F_N^+ (F_N^-, respectively) is upper semicontinuous on K;
(c) $F_N^+(K) \cap \operatorname{Inv}^+_F N = \emptyset$ ($F_N^-(K) \cap \operatorname{Inv}^-_F N = \emptyset$, respectively).

3. Properties of index pairs. Lemmas 3.1 and 3.2 are the multivalued analogs of Lemma 5.8 and Lemma 5.13 in [**3**], respectively. They do not appear in [**2**], where multivalued maps are considered.

Although Lemma 3.1 does not differ much from Lemma 2.11 in [**2**], it is essential to prove the homotopy property of the index. Our lemma gives the set Z which possesses some properties that turn out to be significant in the proof of the homotopy property of the index. The analogous set in Lemma 2.11 in [**2**] does not need to have these properties.

More precisely, we apply Lemma 3.1 to prove Theorem 3.6, which is an important step in proving the homotopy property. The already mentioned Lemma 2.11 in [**2**] is not sufficient to prove this theorem. Moreover, Lemma 3.2 is used in the proof of Theorem 3.6.

LEMMA 3.1. *Let N and A be compact subsets of X, such that*

$$\operatorname{Inv}^- N \subset A \subset N \text{ and } F(A) \cap N \subset A. \tag{3.1}$$

Then for any open neighborhood V of A there exists a compact neighborhood Z of A in N, such that

$$F_N^+(Z) \subset V.$$

Proof. Assumption (3.1) implies that $(N \setminus V) \cap \operatorname{Inv}^- N = \emptyset$. Moreover, the set $N \setminus V$ is compact, because N is compact and V is open. Therefore, $N \setminus V$ satisfies assumptions of Lemma 2.5, which implies that

$$\exists m \in \mathbb{Z}^+ \; \forall k > m \; : \; F_{N,-k}(N \setminus V) = \emptyset. \tag{3.2}$$

From (2.9), (2.10), (3.1) and the assumption $A \subset N \cap V$, we infer that, for $k \in \mathbb{Z}$,

$$F_{N,k}(A) \subset A \subset V. \tag{3.3}$$

Moreover, from Proposition 2.7 in [**2**], for any $k \in \mathbb{Z}$, the map

$$F_{N,k} : N \multimap N \text{ is upper semicontinuous.} \tag{3.4}$$

Therefore, from (3.3) and (3.4) there follows that for any $k \in \mathbb{Z}$ and $x \in A$ there exists a compact neighborhood V_x^k of x in N such that

$$F_{N,k}(V_x^k) \subset V. \tag{3.5}$$

Let us fix any $k \in \mathbb{Z}$; owing to (3.5) and the compactness of A, we can select a finite subcover from $\{\operatorname{int}_N V_x^k\}_{x \in A}$, and therefore

$$A \subset V^k := V_{x_1^k}^k \cup V_{x_2^k}^k \cup \ldots \cup V_{x_{s_k}^k}^k,$$

where $\{\operatorname{int}_N V_{x_1^k}^k, \operatorname{int}_N V_{x_2^k}^k, \ldots, \operatorname{int}_N V_{x_{s_k}^k}^k\}$ is a finite cover of A.

Let us put

$$Z := V^0 \cap V^1 \cap \cdots \cap V^m, \tag{3.6}$$

where m is chosen from condition (3.2). The set Z is obviously a compact neighborhood of A in N. To complete the proof, it is enough to show that $F_N^+(Z) \subset V$. Let $y \in F_N^+(Z)$. Then

$$y \in F_{N,n}(x) \tag{3.7}$$

for some $x \in Z$ and some $n \in \mathbb{Z}^+$.

Let us consider the following cases.

• If $n > m$, then from (3.2) it follows that

$$F_{N,-n}(N \setminus V) = \emptyset. \tag{3.8}$$

Condition (3.7) implies that $x \in F_{N,-n}(y)$. Knowing (3.8) we receive $y \in V$.

• If $0 \leq n \leq m$, then from definition (3.6) there follows that $x \in V^n$. Because $V^n = V_{x_1^n}^n \cup V_{x_2^n}^n \cup \ldots \cup V_{x_{s_n}^n}^n$, there is

$$x \in V_{x_i^n}^n, \tag{3.9}$$

for some $i \in \{1, 2, \ldots, s_n\}$. Condition (3.5) implies that $F_{N,n}(V^n_{x^n_i}) \subset V$, and thus from (3.7) and (3.9) there follows that

$$y \in F_{N,n}(x) \subset F_{N,n}(V^n_{x^n_i}) \subset V,$$

which completes the proof. □

LEMMA 3.2. *Let N and K be compact subsets of X such that*

$$K \cap \text{Inv}^{+} N = \emptyset \text{ and } F(K) \cap N \subset K. \tag{3.10}$$

Then for any open neighborhood U of K there exists a compact neighborhood Z of K such that $F^+_N(Z) \subset U$.

PROOF. Assumption (3.10) implies that $F^+_N(K) \subset K \subset U$. Note that the set K satisfies the assumptions of Lemma 2.5, from which we learn that the map $F^+_N|_K : K \multimap N$ is upper semicontinuous. Therefore, for any $x \in K$ we can find a compact neighborhood V_x of x in K such that

$$F^+_N(V_x) \subset U. \tag{3.11}$$

From the compactness of K we can select a finite subcover $\{V_x : x \in K^{sk}\}$ from $\{V_x : x \in K\}$. Let us put

$$Z := \bigcup \{V_x : x \in K^{sk}\}.$$

The set Z is compact and condition (3.11) implies that $F^+_N(Z) \subset U$. □

Lemma 3.3, which is applied later to prove Theorem 3.5 is a multivalued equivalent of Lemma 5.9 in [**3**].

LEMMA 3.3. *Let U and V be open neighborhoods of* Inv ^{+}N *and* Inv ^{-}N*, respectively. Then there exists $P \in IP(N, F)$ such that*

$$P_1 \subset V \text{ and } N \setminus P_2 \subset U. \tag{3.12}$$

PROOF. Note that Inv $N =$ Inv $^{+}N \cap$ Inv $^{-}N \subset U \cap V$, and by definition (2.8) of the isolating neighborhood, Inv $N \subset$ int $N \cap F^{-1}$(int N). Therefore, without loss of generality we may assume that

$$U \cap V \subset \text{int } N \cap F^{-1}(\text{int } N). \tag{3.13}$$

Then by Lemma 3.1 there exists a compact neighborhood Z of Inv ^{-}N in N such that

$$F^+_N(Z) \subset V. \tag{3.14}$$

We want to show that

$$P_1 := F^+_N(Z),$$
$$P_2 := F^+_N(N \setminus U)$$

satisfy the requirements of the lemma.

First note that by Lemma 2.4 (b) the set P_1 is compact and as a conclusion from Lemma 2.5 also the set P_2 is compact.

Straight from definition (2.11) we infer that both sets P_1 and P_2 are forward invariant with respect to N, therefore condition (a) from the definition of the index pair is satisfied.

From (2.11) we get $N \setminus U \subset F_N^+(N \setminus U) = P_2$, therefore

$$N \setminus P_2 \subset U. \tag{3.15}$$

Using additionally (3.14) and (3.13), we obtain

$$P_1 \setminus P_2 \subset U \cap V \subset \text{int } N \cap F^{-1}(\text{int } N),$$

thus condition (b) from the definition of the index pair holds.

From (3.14) we know that

$$\text{Inv }^- N \subset \text{int }_N Z \subset \text{int }_N P_1. \tag{3.16}$$

By the assumption, $(N \setminus U) \cap \text{Inv }^+ N = \emptyset$, thus we can apply Lemma 2.5 and obtain that

$$F_N^+(N \setminus U) \cap \text{Inv }^+ N = \emptyset,$$

which implies that

$$\text{Inv }^+ N \subset N \setminus P_2. \tag{3.17}$$

Formulas (3.16) and (3.17) give condition (c) from the definition of the index pair.

Concluding, $P = (P_1, P_2)$ is an index pair, which by (3.14) and (3.15) satisfies (3.12). □

Let us slightly modify the definition of the related index pairs, stated originally by Mrozek ([**3**], Definition 5.10) for single-valued dynamical systems. In referred to Definition 5.10, in the condition analogous to (3.18), the closure cl $(Q_1 \setminus P_2)$ appears.

Definition 3.4. Let $P, Q \in IP(N, F)$ be such that $P \subset Q$. We say that the pair P is related to a pair Q if

$$Q_1 \setminus P_2 \subset \text{int } N \cap F^{-1}(\text{int } N), \tag{3.18}$$

where F^{-1} is a small counter image defined by (2.5).

Related index pairs play an important role in the proof of the homotopy property of the Conley index. Consider a homotopic family of multivalued dynamical systems F_ν for $\nu \in [0, 1]$. Assume that N is an isolating neighborhood for F_ν, where ν is some parameter in $[0, 1]$. Homotopy property states that for all λ sufficiently close to ν the set N is also an isolating neighborhood of Inv $_{F_\lambda} N$ (see [**7**], Theorem 3.1 (a) or [**2**], Theorem 4.1 (a)) and the indices of Inv $_{F_\nu} N$ and Inv $_{F_\lambda} N$ are equal (see [**7**], Theorem 3.1 (b) or [**2**], Theorem 4.1

(b)). To prove that the indices are equal one needs to refer to appropriate index pairs. Let us briefly describe the idea sacrificing some accuracy for simplicity. First, Theorems 3.5 and 3.6 enable us to find related index pairs for the dynamical system F_ν (e.g. P^ν related to Q^ν). Then, by Theorem 3.8, we can construct an index pair P^λ for F_λ such that $P^\nu \subset P^\lambda \subset Q^\nu$. These inclusions induce isomorphisms either between the Leray reduction of the Alexander–Spanier cohomologies in the case of the index defined in [**2**], or the Szymczak equivalence classes in case of the homotopy index defined in [**5**].

Let us stress that the actual proof of the homotopy property requires a much finer choice of appropriate index pairs than that outlined above. A detailed proof can be found in [**5**] or [**7**].

Theorems 3.5 and 3.6 are extensions of Lemma 5.12 and Lemma 5.15 in [**3**], respectively, to a multivalued setting. None of these theorems appears in [**2**], however the authors write that the proof of the homotopy property of their index goes along the same way as in the singlevalued case in [**3**].

THEOREM 3.5. *If N is an isolating neighborhood, then there exists index pairs $P, Q \in IP(N, F)$ such that $P \subset \operatorname{int}_N Q$, and the pair P is related to Q.*

PROOF. By definition (2.8) of the isolating neighborhood, $\operatorname{Inv} N \subset \operatorname{int} N \cap F^{-1}(\operatorname{int} N)$. Therefore, from Theorem 2.3 we infer that there exists $(Q_1, P_2) \in IP(N, F)$ such that

$$Q_1 \setminus P_2 \subset \operatorname{int} N \cap F^{-1}(\operatorname{int} N). \tag{3.19}$$

From property (c) in the definition of the index pair, $P_2 \subset N \setminus \operatorname{Inv}^+ N$, therefore we can choose an open neighborhood U' of the compact set P_2 such that

$$\operatorname{cl} U' \subset N \setminus \operatorname{Inv}^+ N. \tag{3.20}$$

Due to (3.20), the set $N \setminus \operatorname{cl} U'$ is a neighborhood of $\operatorname{Inv}^+ N$ and by condition (c) in the definition of the index pair, $\operatorname{int}_N Q_1$ is a neighborhood of $\operatorname{Inv}^- N$. By Lemma 3.3 applied to the sets $N \setminus \operatorname{cl} U'$ and $\operatorname{int}_N Q_1$, there exists $(P_1, Q_2) \in IP(N, F)$ such that

$$P_1 \subset \operatorname{int}_N Q_1 \text{ and } N \setminus Q_2 \subset N \setminus \operatorname{cl} U'. \tag{3.21}$$

From the definition of U' and from the second inclusion in (3.21) we obtain that

$$P_2 \subset U' \subset \operatorname{cl} U' \subset Q_2,$$

therefore,

$$P_2 \subset \operatorname{int}_N Q_2. \tag{3.22}$$

Let us put

$$P := (P_1, P_2),$$
$$Q := (Q_1, Q_2).$$

We have shown that

$$P \subset \text{int }_N Q.$$

Because $P \subset Q$ and $(Q_1, P_2), (P_1, Q_2) \in IP(N, F)$, then by Proposition 2.12 in [**2**], also the intersection is an index pair:

$$(Q_1 \cap P_1, P_2 \cap Q_2) = (P_1, P_2) \in IP(N, F).$$

It remains to show that $(Q_1, Q_2) \in IP(N, F)$, as it is straightforward that the pair P is related to Q by (3.19).

Condition (a) in the definition of an index pair is obvious, because (Q_1, P_2) and (P_1, Q_2) are index pairs in the isolating neighborhood N. Condition (c) in the definition of an index pair for (Q_1, Q_2) is satisfied, because

$$\text{Inv }^- N \subset \text{int }_N Q_1 \text{ and Inv }^+ N \subset N \setminus Q_2,$$

as a consequence of the fact that (Q_1, P_2) is an index pair in the isolating neighborhood N and the second is true because (P_1, Q_2) is an index pair in N.

Note that from (3.22) and (3.19) we obtain

$$Q_1 \setminus Q_2 \subset Q_1 \setminus P_2 \subset \text{int } N \cap F^{-1}(\text{int } N), \tag{3.23}$$

therefore, $Q_1 \setminus Q_2 \subset \text{int } N$ and also condition (b) in the definition of an index pair holds for (Q_1, Q_2). □

THEOREM 3.6. *Assume that $P, R \in IP(N, F)$, P is related to R and*

$$P \subset \text{int }_N R. \tag{3.24}$$

Then there exists $Q \in IP(N, F)$ such that

$$P \subset \text{int }_N Q \quad \textit{and} \quad Q \subset \text{int }_N R, \tag{3.25}$$

P is related to Q, and Q is related to R.

PROOF. We will show that

$$A := P_1,$$

$$V := \text{int }_N R_1$$

satisfy assumptions of Lemma 3.1. From properties (a) and (c) in the definition of an index pair:

$$\text{Inv }^- N \subset \text{int }_N P_1 \subset P_1 \text{ and } F(P_1) \cap N \subset P_1,$$

and so assumption (3.1) is satisfied and V is a neighborhood of A, because of (3.24). Therefore, from Lemma 3.1 we obtain that there exists a compact neighborhood Z of P_1 in N such that

$$F_N^+(Z) \subset \text{int }_N R_1. \tag{3.26}$$

Let us put

$$Q_1 := F_N^+(Z). \tag{3.27}$$

Note that from the definition of F_N^+ and from condition (3.26), the following inclusions hold

$$P_1 \subset \text{int }_N Q_1 \text{ and } Q_1 \subset \text{int }_N R_1. \tag{3.28}$$

Similarly,

$$K := P_2,$$
$$U := \text{int }_N R_2$$

satisfy assumptions of Lemma 3.2. As a consequence of properties (a) and (c) in the definition of an index pair, we obtain

$$\text{Inv }^+ N \cap P_2 = \emptyset \text{ and } F(P_2) \cap N \subset P_2,$$

and assumption (3.10) is satisfied. Moreover, U is a neighborhood of K by the assumption (3.24). Therefore, by applying Lemma 3.2 we infer that there exists a compact neighborhood Z' of P_2 in N such that

$$F_N^+(Z') \subset \text{int }_N R_2. \tag{3.29}$$

By putting

$$Q_2 := F_N^+(Z') \tag{3.30}$$

we immediately conclude that

$$P_2 \subset \text{int }_N Q_2 \text{ and } Q_2 \subset \text{int }_N R_2, \tag{3.31}$$

as a consequence of the definition of F_N^+ and condition (3.29).

It remains to show that $Q = (Q_1, Q_2)$ defined by formulas (3.27) and (3.30) is an index pair, P is related to Q and Q is related to R.

Let us first check that Q is an index pair.

• Condition (a) in the definition of an index pair follows from the definition of F_N^+.

• Let us check condition (c). From the assumption that P is an index pair in an isolating neighborhood N and from (3.28), we obtain that

$$\text{Inv }^- N \subset \text{int }_N P_1 \subset P_1 \subset \text{int }_N Q_1. \tag{3.32}$$

From condition (c) for the index pair R and from the inclusion $Q_2 \subset R_2$, which follows from (3.31), we obtain that

$$\text{Inv }^+ N \subset N \setminus R_2 \subset N \setminus Q_2. \tag{3.33}$$

Concluding, formulas (3.32) and (3.33) give condition (c) in the definition of an index pair for Q.

• Let us now prove condition (b) in the definition of an index pair and the inclusion

$$Q_1 \setminus Q_2 \subset \text{int } N. \tag{3.34}$$

The second inclusion in (3.28) and the first inclusion in (3.31) imply that

$$Q_1 \setminus Q_2 \subset R_1 \setminus Q_2 \subset R_1 \setminus P_2. \tag{3.35}$$

From the assumption that the pair P is related to R there follows that

$$R_1 \setminus P_2 \subset \text{int } N \text{ and } F(R_1 \setminus P_2) \subset \text{int } N. \tag{3.36}$$

As a consequence of (3.35) and (3.36), we obtain property (b) and (3.34) for a pair Q.

• Let us show that both Q_1 and Q_2 are compact.

Conditions (3.26) and (3.32) imply that

$$\text{Inv }^{-}N \subset Z \subset N,$$

and so the assumptions of Lemma 2.4 (b) are satisfied, therefore, Q_1 is a compact set.

From (3.33) we know that $\text{Inv }^{+}N \subset N \setminus Q_2 \subset N \setminus Z'$, because $Z' \subset Q_2$ by definition (3.30); thence

$$\text{Inv }^{+}N \cap Z' = \emptyset,$$

and Z' is compact, and therefore, as a conclusion from Lemma 2.5, the set Q_2 is compact.

Concluding, we proved that

$$Q \in IP(N, F).$$

To complete the proof it is enough to show that P is related to Q and Q is related to R. Due to (3.28) and (3.31), we know that

$$P \subset Q \subset R, \tag{3.37}$$

and the assumption that the pair P is related to R implies that

$$R_1 \setminus P_2 \subset \text{int } N \cap F^{-1}(\text{int } N).$$

Therefore, using (3.37), we obtain that

$$Q_1 \setminus P_2 \subset R_1 \setminus P_2 \subset \text{int } N \cap F^{-1}(\text{int } N), \tag{3.38}$$

and thus P is related to Q. Moreover,

$$R_1 \setminus Q_2 \subset R_1 \setminus P_2 \subset \text{int } N \cap F^{-1}(\text{int } N), \tag{3.39}$$

and as a consequence Q is related to R. □

The following simple fact will be used in the proof of the next theorem.

LEMMA 3.7. ([**2**], *Lemma 4.2*) *For a compact subset $N \subset X$, the maps $\lambda \to \text{Inv }^{+}_{F_\lambda}N$, $\lambda \to \text{Inv }^{-}_{F_\lambda}N$ and $\lambda \to \text{Inv }_{F_\lambda}N$, for $\lambda \in I$ are upper semicontinuous.*

THEOREM 3.8. *Consider* $F \in \mathcal{USC}^c(X \times [0,1], X)$ *and by* $F_\nu \in \mathcal{USC}^c(X, X)$ *for* $\nu \in [0,1]$ *denote the following multivalued map*

$$F_\nu(x) := F(x, \nu), \text{ for } x \in X. \tag{3.40}$$

Let N *be an isolating neighborhood for* F_ν*, for some parameter* $\nu \in [0,1]$*. Moreover, assume that* P^ν *and* Q^ν *are index pairs for* $\operatorname{Inv}_{F_\nu} N$ *such that* P^ν *is related to* Q^ν *and*

$$P^\nu \subset \operatorname{int}_N Q^\nu. \tag{3.41}$$

Then there exists a neighborhood Λ_0 *of* ν *in* $[0,1]$ *such that for any* $\lambda \in \Lambda_0$ *there exists an index pair* P^λ *for* $\operatorname{Inv}_{F_\lambda} N$ *such that*

$$P^\nu \subset P^\lambda \subset Q^\nu.$$

PROOF. We will show that for λ sufficiently close to ν a pair of sets

$$P^\lambda := (F_\lambda)^+_N(P^\nu) \tag{3.42}$$

is an index pair which satisfies requirements of the theorem.

• We first show that for λ sufficiently close to ν the following condition holds:

$$P_1^\nu \subset P_1^\lambda \subset Q_1^\nu. \tag{3.43}$$

The first inclusion is obvious. To prove the second inclusion, consider a compact set

$$K := N \setminus \operatorname{int}_N Q_1^\nu, \tag{3.44}$$

for which we will show that

$$(F_\nu)^-_N(K) \cap P_1^\nu = \emptyset. \tag{3.45}$$

Let us assume that for some $x \in K$ there exists $y \in (F_\nu)^-_N(x) \cap P_1^\nu$. Then, for some $n \geq 0$,

$$x \in (F_\nu)_{N,n}(y) \subset (F_\nu)_{N,n}(P_1^\nu) \subset P_1^\nu, \tag{3.46}$$

where the last inclusion is a consequence of property (a) in the definition of an index pair. Formula (3.46) is in contradiction with the following fact

$$x \in K = N \setminus \operatorname{int}_N Q_1^\nu \subset N \setminus P_1^\nu, \tag{3.47}$$

and so we have proved (3.45).

From Theorem 3.1 (a) in [**7**] or Theorem 4.1 (a) in [**2**], we know that there exists a compact neighborhood Δ of ν in $[0,1]$ such that

$$N \text{ is an isolating neighborhood for } F_\lambda, \text{ for all } \lambda \in \Delta. \tag{3.48}$$

From Lemma 4.2 in [**2**] and property (c) in the definition of an index pair for Q^ν by diminishing if needed the neighborhood Δ, we obtain

$$\text{Inv}\ {}^{-}_{F_\lambda}N \subset \text{int}\ {}_{N}Q_1^\nu, \text{ for } \lambda \in \Delta. \tag{3.49}$$

Let us define a map

$$G : X \times \Delta \ni (x,\lambda) \multimap F_\lambda(x) \times \{\lambda\} \subset X \times \Delta; \tag{3.50}$$

it is upper semicontinuous. It is easy to see that

$$M := N \times \Delta \tag{3.51}$$

is an isolating neighborhood of the invariant set

$$\text{Inv}\ {}_{G}M = \bigcup\{\text{Inv}\ {}_{F_\lambda}N \times \{\lambda\} : \lambda \in \Delta\}.$$

From (3.49) we know that $K \subset N \setminus \text{Inv}\ {}^{-}_{F_\lambda}N$ for $\lambda \in \Delta$. Moreover, from (3.51) we know that

$$M \setminus \text{Inv}\ {}^{-}_{G}M = \bigcup\{(N \setminus \text{Inv}\ {}^{-}_{F_\lambda}N) \times \{\lambda\} : \lambda \in \Delta\}, \tag{3.52}$$

therefore,

$$K \times \Delta \subset M \setminus \text{Inv}\ {}^{-}_{G}M. \tag{3.53}$$

Notice that G^-_M can be expressed by the formula

$$G^-_M(x,\lambda) = (F_\lambda \times id_I)^-_{N\times\Delta}(x,\lambda) = (F_\lambda)^-_N(x) \times \{\lambda\}, \tag{3.54}$$

where $(x,\lambda) \in X \times \Delta$.

For any $x \in K$, the following equality holds due to (3.54) and (3.45):

$$G^-_M(x,\nu) \cap (P_1^\nu \times \Delta) = ((F_\nu)^-_N(x) \cap P_1^\nu) \times \{\nu\} = \emptyset. \tag{3.55}$$

From uppersemicontinuity of

$$G^-_M|_{M\setminus \text{Inv}\ {}^{-}_{G}M} : M \setminus \text{Inv}\ {}^{-}_{G}M \multimap M$$

(see Conclusion 4.2 in [**6**]) and (3.55), for any $x \in K$, we can find V_x, an open neighborhood of x in $N \setminus \text{Inv}\ {}^{-}_{F_\lambda}N$, and Δ_x, an open neighborhood of ν in Δ, such that

$$G^-_M(y,\lambda) \cap (P_1^\nu \times \Delta) = \emptyset, \text{ for any } (y,\lambda) \in V_x \times \Delta_x. \tag{3.56}$$

By compactness of K, there exist $x_1, \ldots, x_n$ such that $K \subset V_{x_1} \cup \ldots \cup V_{x_n}$ and from (3.56) and (3.54), we obtain

$$(F_\lambda)^-_N(y) \cap P_1^\nu = \emptyset, \text{ for } (y,\lambda) \in K \times \Delta_0, \tag{3.57}$$

where $\Delta_0 := \Delta_{x_1} \cap \ldots \cap \Delta_{x_n}$. Obviously, condition (3.57) is equivalent to

$$K \cap (F_\lambda)^+_N(x) = \emptyset, \text{ for } (x,\lambda) \in P_1^\nu \times \Delta_0. \tag{3.58}$$

From definition (3.42), from (3.58) and (3.44), we obtain

$$P_1^\lambda = (F_\lambda)_N^+(P_1^\nu) \subset N \setminus K = \text{int }_N Q_1^\nu \subset Q_1^\nu, \text{ for } \lambda \in \Delta_0.$$

Thus we proved that $P_1^\nu \subset P_1^\lambda \subset Q_1^\nu$, for λ close to ν.

• Let us proceed to the proof of the second inclusion. We want to show that for λ sufficiently close to ν

$$P_2^\nu \subset P_2^\lambda \subset Q_2^\nu. \tag{3.59}$$

As in the previous case, inclusion $P_2^\nu \subset P_2^\lambda$ is obvious. To prove the right-hand side inclusion, first notice that

$$(F_\nu)_N^+(x) \subset P_2^\nu, \text{ for } x \in P_2^\nu. \tag{3.60}$$

Assumption (3.41) and (3.60) imply that

$$(F_\nu)_N^+(P_2^\nu) \subset \text{int }_N Q_2^\nu. \tag{3.61}$$

It is easy to check that

$$G_M^+(x,\lambda) = (F_\lambda \times id_I)_{N\times\Delta}^+(x,\lambda) = (F_\lambda)_N^+(x) \times \{\lambda\}, \tag{3.62}$$

for $(x,\lambda) \in X \times \Delta$.

Due to the upper semicontinuity of

$$G_M^+|_{M\setminus \text{Inv }_G^+ M} : M \setminus \text{Inv }_G^+ M \multimap M$$

and (3.61), as previously, for any $x \in P_2^\nu$ we can find V_x', an open neighborhood of x in $N \setminus \text{Inv }_{F_\lambda}^+ N$, and Δ_x', an open neighborhood of ν in Δ, such that

$$G_M^+(y,\lambda) \subset \text{int }_N Q_2^\nu \times \Delta_x', \text{ for any } (y,\lambda) \in V_x' \times \Delta_x'. \tag{3.63}$$

By the compactness of P_2^ν, there exist $x_1, \ldots, x_m$ such that $P_2^\nu \subset V_{x_1}' \cup \ldots \cup V_{x_m}'$; by (3.62) and (3.63)

$$(F_\lambda)_N^+(y) \subset \text{int }_N Q_2^\nu \subset Q_2^\nu, \text{ for } (y,\lambda) \in P_2^\nu \times \Delta_1,$$

where $\Delta_1 := \Delta_{x_1}' \cap \ldots \cap \Delta_{x_n}'$, which completes the proof of (3.59).

Let us prove now that P^λ is an index pair.

• We first show that P_1^λ and P_2^λ are compact.

From condition (c) in the definition of an index pair and Lemma 3.7, we infer that

$$\text{Inv }_{F_\lambda}^- N \subset \text{int }_N P_1^\nu \subset P_1^\nu \subset N, \text{ for } \lambda \text{ close to } \nu,$$

and P_1^ν is compact, therefore, by Lemma 2.4 (b) the set

$$P_1^\lambda = (F_\lambda)_N^+(P_1^\nu) \text{ is compact.}$$

From condition (c) in the definition of an index pair and Lemma 3.7, we infer that

$$\text{Inv }_{F_\lambda}^+ N \subset N \setminus P_2^\nu, \text{ for } \lambda \text{ close to } \nu,$$

therefore, $\operatorname{Inv}^{+}_{F_\lambda} N \cap P_2^\nu = \emptyset$ and P_2^ν; thus from Lemma 2.5 (b) we infer that the map $(F_\lambda)^+_N$ is upper semicontinuous on the set P_2^ν and has compact values hence the set

$$P_2^\lambda = (F_\lambda)^+_N(P_2^\nu) \text{ is compact.}$$

• To prove condition (a) in the definition of an index pair, it is enough to notice that for $i = 1, 2$ there is

$$F_\lambda(P_i^\lambda) \cap N = F_\lambda((F_\lambda)^+_N(P_i^\nu)) \cap N \subset (F_\lambda)^+_N(P_i^\nu) = P_i^\lambda.$$

• We want to show that

$$P_1^\lambda \setminus P_2^\lambda \subset \operatorname{int} N,$$

and that condition (b) in the definition of an index pair is satisfied.

Since

$$P_1^\lambda \subset Q_1^\nu \text{ and } P_2^\nu \subset P_2^\lambda,$$

then

$$P_1^\lambda \setminus P_2^\lambda \subset Q_1^\nu \setminus P_2^\nu,$$

and the assumption that P^ν is related to Q^ν implies that

$$Q_1^\nu \setminus P_2^\nu \subset \operatorname{int} N \cap F_\nu^{-1}(\operatorname{int} N),$$

thus

$$P_1^\lambda \setminus P_2^\lambda \subset \operatorname{int} N \text{ and } F_\nu(P_1^\lambda \setminus P_2^\lambda) \subset \operatorname{int} N.$$

Because $F : X \times I \multimap X$ is upper semicontinuous, also

$$F_\lambda(P_1^\lambda \setminus P_2^\lambda) \subset \operatorname{int} N,$$

for λ close to ν.

• In order to prove condition (c) in the definition of an index pair, it is enough to notice that using property (c) for an index pair P^ν, Lemma 3.7 and (3.43), we obtain

$$\operatorname{Inv}^{-}_{F_\lambda} N \subset \operatorname{int}{}_N P_1^\nu \subset \operatorname{int}{}_N P_1^\lambda,$$

for λ sufficiently close to ν. Similarly, exploiting property (c) for an index pair Q^ν, Lemma 3.7 and (3.59), we obtain

$$\operatorname{Inv}^{+}_{F_\lambda} N \subset N \setminus Q_2^\nu \subset N \setminus P_2^\lambda,$$

for λ close to ν. This completes the proof. □

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