UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLIII
2005

INFINITE SYSTEMS OF HYPERBOLIC DIFFERENTIAL-FUNCTIONAL INEQUALITIES

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Abstract. The paper deals with systems of hyperbolic differential-functional inequalities related to initial problem on the generalized Haar pyramid for equations

$$\partial_t z_\lambda(t,x) = f_\lambda(t,x,z,\partial_x z_\lambda(t,x)),\ \lambda \in \Lambda,$$

where $(t,x) = (t,x_1,\dots,x_n)$, $z = \{z_\lambda\}_{\lambda\in\Lambda}$ and Λ is a compact set of indices. A theorem on strong differential-functional inequalities is the main result of the paper. Extremal solutions of initial problems for infinite systems of ordinary differential-functional equations are used in the proof of a theorem on weak partial differential-functional inequalities.

1. Introduction. The classical theory of partial differential inequalities has been developed extensively in monographs [**6**], [**7**] and [**8**]. As it is well known, they apply in the investigation of different problems. The basic examples of such questions are: estimates of solutions of partial equations, estimates of the domain of the existence of solutions, criterion of uniqueness, estimates of the error of approximate solutions. Moreover, discrete versions of differential inequalities are used to prove the convergence of approximate methods. Differential-functional inequalities play a similar role in the theory of differential-functional equations with partial derivatives. Monograph [**3**] contains an exposition of hyperbolic differential-functional inequalities and their applications.

The aim of this paper is to contribute to the theory of first order partial differential-functional inequalities. We deal with infinite systems of hyperbolic differential-functional inequalities related to initial problems on the Haar pyramid.

Our results extend the results of paper [**4**] where comparison theorems for infinite systems were presented. Existence results for initial problems can be found in [**5**].

Infinite systems of parabolic differential-functional equations were investigated in [**1**] and [**2**].

For any metric spaces X and Y, by $C(X,Y)$ we denote the class of all continuous functions from X into Y.

Now we formulate the problem. Let $a > 0$, $r_0 \in R_+$, $R_+ = [0, +\infty)$ and let the functions $\alpha, \beta : [0,a) \to R^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $\tilde{\alpha}, \tilde{\beta} : [0,a) \to R^n$, $\tilde{\alpha} = (\tilde{\alpha_1}, \ldots, \tilde{\alpha_n})$, $\tilde{\beta} = (\tilde{\beta_1}, \ldots, \tilde{\beta_n})$ satisfy the conditions:

(i) α and β are of class C^1 on $[0,a)$ and $\alpha(t) < \beta(t)$ for $t \in [0,a)$,

(ii) $\tilde{\alpha}, \tilde{\beta} \in C([-r_0, 0], R^n)$ and $\tilde{\alpha}(t) \leq \tilde{\beta}(t)$ for $t \in [-r_0, 0]$,

(iii) $\beta(0) = \tilde{\beta}(0) = b$ where $b = (b_1, \ldots, b_n)$, $b_i > 0$ for $1 \leq i \leq n$ and $\alpha(0) = \tilde{\alpha}(0) = -b$.

Let E be the generalized Haar pyramid

$$E = \Big\{ (t,x) \in R^{1+n} : t \in (0,a), x = (x_1, \ldots, x_n) \in [\alpha(t), \beta(t)] \Big\}$$

and

$$E_0 = \Big\{ (t,x) \in R^{1+n} : t \in [-r_0, 0], x \in [\tilde{\alpha}(t), \tilde{\beta(t)}] \Big\}, \;\; \partial_0 E = \partial E \cap ((0,a) \times R^n),$$

where ∂E is the boundary of E.

Write $S_t = [\tilde{\alpha}(t), \tilde{\beta}(t)]$ for $t \in [-r_0, 0]$ and $S_t = [\alpha(t), \beta(t)]$ for $t \in [0,a)$. Put $E_t = (E_0 \cup E) \cap ([-r_0, t] \times R^n)$ for $t \in [-r_0, a)$. Let $\Lambda \subset R$ be a compact set of indices with an arbitrary number of elements and

$$\mathcal{X} = \Big\{ p = \{p_\lambda\}_{\lambda \in \Lambda} : p \in C(\Lambda, R) \Big\}.$$

For $p, \bar{p} \in \mathcal{X}$, $p = \{p_\lambda\}_{\lambda \in \Lambda}$, $\bar{p} = \{\bar{p}_\lambda\}_{\lambda \in \Lambda}$, we write $p < \bar{p}$ if $p_\lambda < \bar{p}_\lambda$ for $\lambda \in \Lambda$. We define the relation $p \leq \bar{p}$ in the similar way.

Write $\Gamma = E \times C(E_0 \cup E, \mathcal{X}) \times R^n$ and suppose that

$$f = \{f_\lambda\}_{\lambda \in \Lambda}, \; f_\lambda : \Gamma \to R, \;\; \varphi = \{\varphi_\lambda\}_{\lambda \in \Lambda}, \; \varphi_\lambda : E_0 \to R,$$

are given functions. We consider the initial value problem

$$\partial_t z_\lambda(t,x) = f_\lambda(t,x,z,\partial_x z_\lambda(t,x)), \; \lambda \in \Lambda, \tag{1}$$

$$z(t,x) = \varphi(t,x), \; (t,x) \in E_0, \tag{2}$$

where $z = \{z_\lambda\}_{\lambda \in \Lambda}$ and $\partial_x z_\lambda(t,x) = (\partial_{x_1} z_\lambda(t,x), \ldots, \partial_{x_n} z_\lambda(t,x))$.

Let $F[z] = \{F_\lambda[z]\}_{\lambda \in \Lambda}$ be the Nemytski operator corresponding to (1), i.e.

$$F_\lambda[z](t,x) = f_\lambda(t,x,z,\partial_x z_\lambda(t,x)), \; \lambda \in \Lambda.$$

A function $u : E_0 \cup E \to \mathcal{X}$, $u = \{u_\lambda\}_{\lambda \in \Lambda}$, will be called a function of class D if

(i) the function $\bar{u} : (E_0 \cup E) \times \Lambda \to R$ defined by $\bar{u}(t, x, \lambda) = u_\lambda(t, x)$ is continuous on $(E_0 \cup E) \times \Lambda$,

(ii) for each $\lambda \in \Lambda$, the function u_λ has the total differential on $\partial_0 E$ and partial derivatives of first order in an interior of E.

We will say that the function $f = \{f_\lambda\}_{\lambda\in\Lambda}$, $f_\lambda : \Gamma \to R$, satisfies the Volterra condition if for each $(t, x) \in E$, $q \in R^n$ and for $z, \bar{z} \in C(E_0 \cup E, \mathcal{X})$ such that $z(\tau, s) = \bar{z}(\tau, s)$ for $(\tau, s) \in E_t$ there is $f(t, x, z, q) = f(t, x, \bar{z}, q)$.

We will say that the function $f = \{f_\lambda\}_{\lambda\in\Lambda}$, $f_\lambda : \Gamma \to R$, satisfies the monotonicity condition W_+ if for $\lambda \in \Lambda$, $(t, x) \in E$, $q \in R^n$, $z, \bar{z} \in C(E_0 \cup E, \mathcal{X})$ such that $z(\tau, s) \leq \bar{z}(\tau, s)$, $(\tau, s) \in E_t$ and $z_\lambda(t, x) = \bar{z}_\lambda(t, x)$ there is $f_\lambda(t, x, z, q) \leq f_\lambda(t, x, \bar{z}, q)$.

Remark 1. Let a function $\tilde{f} = \{\tilde{f}_\lambda\}_{\lambda\in\Lambda}$, where

$$\tilde{f}_\lambda : E \times \mathcal{X} \times C(E_0 \cup E, \mathcal{X}) \times R^n \to R$$

in variables (t, x, p, z, q), $p = \{p_\lambda\}_{\lambda\in\Lambda}$, be given. We assume that for $\lambda \in \Lambda$ the function $\tilde{f}_\lambda$ is non-decreasing with respect to the functional variable z and non-decreasing with respect to each p_μ, $\mu \in \Lambda$ and $\mu \neq \lambda$. The function $f = \{f_\lambda\}_{\lambda\in\Lambda}$, $f_\lambda : \Gamma \to R$ given by

$$f(t, x, z, q) = \tilde{f}(t, x, z(t, x), z, q)$$

satisfies the monotonicity condition W_+.

2. Strong differential-functional inequalities. For each $(t, x) \in E$, there exist (possibly empty) sets of integers $I_0[t, x]$, $I_+[t, x]$, $I_-[t, x]$ such that

$$I_+[t, x] \cap I_-[t, x] = \emptyset, \;\; I_0[t, x] \cup I_+[t, x] \cup I_-[t, x] = \{1, \ldots, n\}$$

and $x_i = \beta_i(t)$ for $i \in I_+[t, x]$, $x_i = \alpha_i(t)$ for $i \in I_-[t, x]$, $\alpha_i(t) < x_i < \beta_i(t)$ for $i \in I_0[t, x]$.

Theorem 1. *Suppose that*

1) a function $f = \{f_\lambda\}_{\lambda\in\Lambda}$, $f_\lambda : \Gamma \to R$, in the variables (t, x, z, q), $q = (q_1, \ldots, q_n)$, satisfies the Volterra condition and the monotonicity condition W_+,

2) for $(t, x, z, q) \in \Gamma$, $\bar{q} \in R^n$ such that $q_j \leq \bar{q}_j$ for $j \in I_-[t, x]$, $q_j \geq \bar{q}_j$ for $j \in I_+[t, x]$ and $q_j = \bar{q}_j$ for $j \in I_0[t, x]$ there is

$$f_\lambda(t, x, z, q) - f_\lambda(t, x, z, \bar{q}) + \sum_{j\in I_-[t,x]} \alpha'_j(t)(q_j - \bar{q}_j) + \sum_{j\in I_+[t,x]} \beta'_j(t)(q_j - \bar{q}_j) \leq 0,$$

where $\lambda \in \Lambda$,

3) functions $u, v : E_0 \cup E \to \mathcal{X}$, $u = \{u_\lambda\}_{\lambda\in\Lambda}$, $v = \{v_\lambda\}_{\lambda\in\Lambda}$, are of class D and

$$u(t, x) \leq v(t, x), \;\; (t, x) \in E_0, \;\; u(0, x) < v(0, x), \;\; x \in [-b, b],$$

4) the differential-functional inequality

$$\partial_t u(t,x) - F[u](t,x) < \partial_t v(t,x) - F[v](t,x)$$

holds on E.

Under these assumptions,

$$u(t,x) < v(t,x) \text{ for } (t,x) \in E. \tag{3}$$

Proof. We define the function $r : E_0 \cup E \to \mathcal{X}$, $r = \{r_\lambda\}_{\lambda\in\Lambda}$, by $r(t,x) = u(t,x) - v(t,x)$ and the function $\bar{r} : (E_0 \cup E) \times \Lambda \to R$ by $\bar{r}(t,x,\lambda) = r_\lambda(t,x)$. We prove that

$$r(t,x) < 0 \text{ for } (t,x) \in E. \tag{4}$$

Let $\mathcal{J}$ denote the set

$$\mathcal{J} = \{t \in (0,a) : r(\tau,x) < 0 \text{ for } (\tau,x) \in E \text{ and } \tau \le t\}.$$

From assumption 3) it follows that

$$\bar{r}(0,x,\lambda) < 0$$

for every $x \in [-b,b]$ and $\lambda \in \Lambda$. The function $\bar{r}$ is continuous on $(E_0 \cup E) \times \Lambda$ thus there exists $t \in (0,a)$ such that for every $(\tau,x,\lambda) \in E \times \Lambda$, $\tau \le t$, there is $\bar{r}(\tau,x,\lambda)) < 0$ and the set $\mathcal{J}$ is not empty.

Let us put $t^* = \sup \mathcal{J}$. We prove that $t^* = a$.

Suppose that it is not true and $0 < t^* < a$. Then $\bar{r}(\tau,x,\lambda) < 0$ for $(\tau,x,\lambda) \in E \times \Lambda$, $\tau < t^*$, $\bar{r}(t^*,x,\lambda) \le 0$ for $(x,\lambda) \in S_{t^*} \times \Lambda$ and there is a sequence of points $(x_k,\lambda_k) \in S_{t^*} \times \Lambda$ such that $\lim_{k\to\infty} \bar{r}(t^*,x_k,\lambda_k) = 0$. The points (x_k,λ_k) are in a compact set, thus there exists $\lim_{m\to\infty}(x_{k_m},\lambda_{k_m}) = (x^*,\lambda^*)$, $(x^*,\lambda^*) \in S_{t^*} \times \Lambda$, for some subsequence. By the continuity of the function $\bar{r}$, we obtain $\bar{r}(t^*,x^*,\lambda^*) = 0$ or equivalently $r_{\lambda^*}(t^*,x^*) = 0$.

There is $\partial_{x_j} r_{\lambda^*}(t^*,x^*) \ge 0$ for $j \in I_+[t^*,x^*]$, $\partial_{x_j} r_{\lambda^*}(t^*,x^*) \le 0$ for $j \in I_-[t^*,x^*]$ and $\partial_{x_j} r_{\lambda^*}(t^*,x^*) = 0$ for $j \in I_0[t^*,x^*]$.

Define $\eta : [0,t^*] \to R$, $\eta = (\eta_1,\ldots,\eta_n)$ in the following way:

$$\begin{aligned}\eta_j(t) &= \alpha_j(t) \text{ for } j \in I_-[t^*,x^*],\\ \eta_j(t) &= \beta_j(t) \text{ for } j \in I_+[t^*,x^*],\\ \eta_j(t) &= x_j^* \text{ for } j \in I_0[t^*,x^*].\end{aligned}$$

Put $\gamma(t) = r_{\lambda^*}(t,\eta(t))$. The function γ attains its maximum at $t = t^*$. Thus, $\gamma'(t^*) \ge 0$ or

$$\begin{aligned}\partial_t r_{\lambda^*}(t^*,x^*) + &\sum_{j\in I_-[t^*,x^*]} \alpha_j'(t^*)\partial_{x_j} r_{\lambda^*}(t^*,x^*)\\ &+ \sum_{j\in I_+[t^*,x^*]} \beta_j'(t^*)\partial_{x_j} r_{\lambda^*}(t^*,x^*) \ge 0.\end{aligned} \tag{5}$$

From the assumptions we deduce that

$$\partial_t r_{\lambda^*}(t^*, x^*) = \partial_t (u_{\lambda^*}(t^*, x^*) - v_{\lambda^*}(t^*, x^*)) <$$

$$< F_{\lambda^*}[u](t^*, x^*) - F_{\lambda^*}[v](t^*, x^*) \leq$$

$$\leq f_{\lambda^*}(t^*, x^*, v, \partial_x u_{\lambda^*}(t^*, x^*)) - f_{\lambda^*}(t^*, x^*, v, \partial_x v_{\lambda^*}(t^*, x^*)) \leq$$

$$\leq - \sum_{j \in I_-[t^*, x^*]} \alpha_j'(t^*) \partial_{x_j} r_{\lambda^*}(t^*, x^*) - \sum_{j \in I_+[t^*, x^*]} \beta_j'(t^*) \partial_{x_j} r_{\lambda^*}(t^*, x^*),$$

which contradicts (5). Thus inequality (4) holds and the assertion follows. □

REMARK 2. It is enough to assume that the differential-functional inequalities in assumption 4) of Theorem 1 hold for $(t, x) \in \Delta$, where $\Delta = \Big\{(t^*, x^*) \in E : u(t, x) \leq v(t, x)$ *on* E_{t^*} and there is $\lambda^* \in \Lambda$ such that $u_{\lambda^*}(t^*, x^*) = v_{\lambda^*}(t^*, x^*)\Big\}$.

REMARK 3. In Theorem 1, instead of assumption 4), we may assume that for $\lambda \in \Lambda$

$$\partial_x u_\lambda(t, x) \leq F_\lambda[u](t, x) \textit{ and } \partial_x v_\lambda(t, x) \geq F_\lambda[v](t, x),$$

where $(t, x) \in E$ and for each (t, x) at most one of the above inequalities turns out to be an equality.

REMARK 4. Let

$$E_0 = [-r_0, 0] \times [-b, b]$$

and

$$E = \Big\{(t, x) : \, t \in (0, a), \; -b + Mt \leq x \leq b - Mt\Big\},$$

where $M = (M_1, \ldots, M_n) \in R^n_+$. We assume that $b - aM > 0$. If a function $f = \{f_\lambda\}_{\lambda \in \Lambda}$, $f_\lambda : E \times C(E_0 \cup E, \mathcal{X}) \times R^n \to R$, satisfies the Lipschitz condition

$$|f_\lambda(t, x, z, q) - f_\lambda(t, x, z, \bar{q})| \leq \sum_{j=1}^{n} M_j |q_j - \bar{q}_j|, \; \lambda \in \Lambda$$

on $E \times C(E_0 \cup E, \mathcal{X}) \times R^n$, then assumption 2) of Theorem 1 holds with $\alpha(t) = -b + Mt$ and $\beta(t) = b - Mt$.

3. Comparison problems. Put $\mathcal{X}_+ = \Big\{p \in \mathcal{X} : p = \{p_\lambda\}_{\lambda\in\Lambda},\ p_\lambda \geq 0, \lambda \in \Lambda\Big\}$ and $\tilde{\Gamma} = (0,a) \times C([-r_0,a),\mathcal{X}_+)$.

Assumption $H[\sigma]$. Suppose that $\sigma = \{\sigma_\lambda\}_{\lambda\in\Lambda}$, $\sigma_\lambda : \tilde{\Gamma} \to R_+$ and

1) the function $\bar{\sigma} : \tilde{\Gamma} \times \Lambda \to R_+$ given by $\bar{\sigma}(t,w,\lambda) = \sigma_\lambda(t,w)$ is continuous on $\tilde{\Gamma} \times \Lambda$,

2) σ satisfies the Volterra condition and the monotonicity condition W_+,

3) there is $L \in \mathcal{X}_+$ such that

$$\sigma(t,w) \leq L \ on\ \tilde{\Gamma}.$$

A function $\varphi : [-r_0,a) \to \mathcal{X}$, $\varphi = \{\varphi_\lambda\}_{\lambda\in\Lambda}$, will be called a function of class D_0 if the function $\bar{\varphi} : [-r_0,a) \times \Lambda \to R$ defined by $\bar{\varphi}(t,\lambda) = \varphi_\lambda(t)$ is continuous on $[-r_0,a) \times \Lambda$.

Lemma 1. *Suppose that Assumption $H[\sigma]$ is satisfied and a function $\eta : [-r_0,0] \to \mathcal{X}_+$ is of class D_0. Then on $[-r_0,a)$ there exists the maximum solution $\tilde{\omega} = \{\tilde{\omega}_\lambda\}_{\lambda\in\Lambda}$ of the problem*

$$w'(t) = \sigma(t,w) \tag{6}$$
$$w(t) = \eta(t),\ t \in [-r_0,0]. \tag{7}$$

and $\tilde{\omega} : [-r_0,a) \to \mathcal{X}_+$ is of class D_0. Moreover, if $\varphi : [-r_0,a) \to \mathcal{X}_+$ is of class D_0 and satisfies the differential-functional inequality

$$D_-\varphi(t) \leq \sigma(t,\varphi) \tag{8}$$

and the initial estimate $\varphi(t) \leq \eta(t)$ holds for $t \in [-r_0,0]$, then

$$\varphi(t) \leq \tilde{\omega}(t)\ for\ t \in (0,a).$$

Proof. Let $\varepsilon > 0$ be an arbitrary fixed number and let the same symbol ε denote $\varepsilon : \Lambda \to R_+$, $\varepsilon = \{\varepsilon_\lambda\}_{\lambda\in\Lambda}$, where $\varepsilon_\lambda = \varepsilon$, $\lambda \in \Lambda$. Consider the problem

$$w'(t) = \sigma(t,w) + \varepsilon, \tag{9}$$
$$w(t) = \eta(t) + \varepsilon,\ t \in [-r_0,0]. \tag{10}$$

We will prove that on $[-r_0,a)$ there exists the solution $\omega(\cdot,\varepsilon) = \{\omega_\lambda(\cdot,\varepsilon)\}_{\lambda\in\Lambda}$ of problem (9), (10) and it is of class D_0. Let $m \geq 1$ be a natural number and let $h = \frac{a}{m}$. Define the function $\omega^{(m)} : [-r_0,a) \to \mathcal{X}_+$, $\omega^{(m)} = \{\omega^{(m)}_\lambda\}_{\lambda\in\Lambda}$ in the following way:

$$\begin{aligned} \omega^{(m)}(t) &= \eta(t) + \varepsilon,\ t \in [-r_0,0], \\ \omega^{(m)}(t) &= \eta(0) + \varepsilon,\ t \in [0,h], \\ \omega^{(m)}(t) &= \eta(0) + \varepsilon + \int_0^{t-h} \Big(\sigma(s,\omega^{(m)}) + \varepsilon\Big) ds,\ t \in [h,a). \end{aligned} \tag{11}$$

The functions $\bar{\omega}^{(m)}(t,\lambda) = \omega_\lambda^{(m)}(t)$, $(t,\lambda) \in [-r_0, a) \times \Lambda$ form the class of uniformly bounded and equicontinuous functions. Thus there exists a subsequence uniformly convergent on $[-r_0, a) \times \Lambda$. Suppose that $\{\omega^{(m)}\}$ is convergent. Then

$$\omega(t,\varepsilon) = \lim_{m\to+\infty} \omega^{(m)}(t)$$

is a function of class D_0. It follows from (11) that

$$\begin{aligned} \omega(t,\varepsilon) &= \eta(t) + \varepsilon, \; t \in [-r_0, 0] \\ \omega(t,\varepsilon) &= \eta(0) + \varepsilon + \int_0^t \Big(\sigma(s, \omega(\cdot,\varepsilon)) + \varepsilon\Big) ds, \; t \in [0, a). \end{aligned}$$

Therefore $\omega(\cdot,\varepsilon)$ is the solution of (9), (10).

Assume that $\varphi : [-r_0, a) \to \mathcal{X}_+$ is of class D_0, satisfies inequality (8) and $\varphi(t) \le \eta(t)$, $t \in [-r_0, 0]$. We prove that

$$\varphi(t) < \omega(t,\varepsilon), \; t \in [-r_0, a). \tag{12}$$

The set

$$J = \{t \in (0,a) : \; \varphi(\tau) < \omega(\tau,\varepsilon), \; 0 \le \tau \le t\}$$

is nonempty. It is enough to prove that for $t^* = \sup J$ there is $t^* = a$. Suppose that it is not true and $t^* < a$. Then $\varphi_\lambda(\tau) < \omega_\lambda(\tau,\varepsilon)$ for $0 \le \tau < t^*$, $\lambda \in \Lambda$ and there exists $\lambda^* \in \Lambda$ such that $\varphi_\lambda(t^*) < \omega_\lambda(t^*,\varepsilon)$ for $\lambda \in \Lambda$ and $\varphi_{\lambda^*}(t^*) = \omega_{\lambda^*}(t^*,\varepsilon)$. In this situation,

$$D_-\Big(\varphi_{\lambda^*}(t^*) - \omega_{\lambda^*}(t^*,\varepsilon)\Big) \ge 0.$$

On the other hand, it follows from the assumptions that

$$\begin{gathered} D_-\Big(\varphi_{\lambda^*}(t^*) - \omega_{\lambda^*}(t^*,\varepsilon)\Big) \le \\ \le \sigma_{\lambda^*}(t^*, \varphi) - \sigma_{\lambda^*}(t^*, \omega(\cdot,\varepsilon)) - \varepsilon \le -\varepsilon < 0. \end{gathered}$$

The contradiction proves (12).

Now let the sequence $\{\varepsilon^{(k)}\}$ be such that $\varepsilon^{(k+1)} < \varepsilon^{(k)}$ for each natural k and $\lim_{k\to+\infty} \varepsilon^{(k)} = 0$. It is easy to see that

$$\omega(t, \varepsilon^{(k+1)}) < \omega(t, \varepsilon^{(k)})$$

and there exists $\tilde{\omega} : [-r_0, a) \to \mathcal{X}_+$ of class D_0 such that

$$\tilde{\omega}(t) = \lim_{k\to+\infty} \omega(t, \varepsilon^{(k)}).$$

It follows from the relations

$$\begin{gathered} \omega(t, \varepsilon^{(k)}) = \eta(0) + \varepsilon^{(k)} + \int_0^t \Big(\sigma(s, \omega(\cdot, \varepsilon^{(k)})) + \varepsilon^{(k)}\Big) ds, \; t \in [0, a), \\ \omega(t, \varepsilon^{(k)}) = \eta(t) + \varepsilon^{(k)}, \; t \in [-r_0, 0] \end{gathered}$$

that

$$\tilde{\omega}(t) = \eta(0) + \int_0^t \sigma(s, \tilde{\omega})ds,\ t \in [0, a),$$
$$\tilde{\omega}(t) = \eta(t),\ t \in [-r_0, 0]$$

or that $\tilde{\omega}$ is a solution of the problem (6), (7). Moreover, every other solution $\varphi : [-r_0, a) \to \mathcal{X}_+$ of class D_0 of (6), (7) satisfies

$$\varphi(t) < \omega(t, \varepsilon^{(k)}) \leq \tilde{\omega}(t),\ t \in [0, a).$$

Thus $\tilde{\omega}$ is the maximum solution of problem (6), (7). □

4. Weak differential-functional inequalities.

For a function $z = \{z_\lambda\}_{\lambda\in\Lambda}$, $z \in C(E_0 \cup E, \mathcal{X})$ and for $t \in [-r_0, a)$ let $Vz = \{V_\lambda z\}_{\lambda\in\Lambda}$ where

$$(V_\lambda z)(t) = \max\{|z_\lambda(t, x)| : x \in S_t\}.$$

The following property of V is important in our considerations.

Lemma 2. *If $z \in C(E_0 \cup E, \mathcal{X})$ then*

$$Vz \in C([-r_0, a), \mathcal{X}_+).$$

The lemma may be proved with use of methods developed in the proof of Theorem 33.1 in [**8**].

Theorem 2. *Suppose that Assumption $H[\sigma]$ is satisfied and let $\omega(t) = 0$, $t \in [-r_0, a)$ be the unique solution of the problem $w'(t) = \sigma(t, w)$, $w(t) = 0$ for $t \in [-r_0, 0]$. We assume that*

1) for $z, \bar{z} \in C(E_0 \cup E, \mathcal{X})$ such that $z(\tau, x) \leq \bar{z}(\tau, x)$ on E_t there is

$$f(t, x, \bar{z}, q) - f(t, x, z, q) \leq \sigma(t, V(\bar{z} - z)),\ (t, x, q) \in E \times R^n,$$

2) the function f satisfies the Volterra condition and monotonicity condition W_+,

3) for $(t, x, z, q) \in \Gamma$, $\bar{q} \in R^n$ such that $q_j \leq \bar{q}_j$ for $j \in I_-[t, x]$, $q_j \geq \bar{q}_j$ for $j \in I_+[t, x]$ and $q_j = \bar{q}_j$ for $j \in I_0[t, x]$ there is

$$f_\lambda(t, x, z, q) - f_\lambda(t, x, z, \bar{q}) + \sum_{j\in I_-[t,x]} \alpha_j'(t)(q_j - \bar{q}_j) + \sum_{j\in I_+[t,x]} \beta_j'(t)(q_j - \bar{q}_j) \leq 0,$$

where $\lambda \in \Lambda$,

4) functions $u, v : E_0 \cup E \to \mathcal{X}$, $u = \{u_\lambda\}_{\lambda\in\Lambda}$, $v = \{v_\lambda\}_{\lambda\in\Lambda}$, are of class D and

$$u(t, x) \leq v(t, x),\ (t, x) \in E_0,$$

5) the differential-functional inequality

$$\partial_t u(t, x) - F[u](t, x) \leq \partial_t v(t, x) - F[v](t, x)$$

holds on E.

Under these assumptions

$$u(t,x) \leq v(t,x) \; for \; (t,x) \in E. \tag{13}$$

PROOF. We consider the problem

$$w'_\lambda(t) = \sigma_\lambda(t,w) + \varepsilon, \; \lambda \in \Lambda, \tag{14}$$

$$w_\lambda(t) = \varepsilon, \; t \in [-r_0, 0], \; \lambda \in \Lambda, \tag{15}$$

where $\varepsilon \in R_+$ is chosen arbitrarily. From Lemma 1 it follows that on $[-r_0, a)$ there exists the maximum solution $\omega^\varepsilon = \{\omega^\varepsilon_\lambda\}_{\lambda\in\Lambda}$ of (14), (15).

Define $u^\varepsilon(t,x) = u(t,x) - \omega^\varepsilon(t)$. It follows that $u^\varepsilon(t,x) < v(t,x)$ on E_0 and

$$\partial_t u^\varepsilon_\lambda(t,x) - F_\lambda[u^\varepsilon](t,x) =$$

$$= \partial_t u_\lambda(t,x) - F_\lambda[u](t,x) - \sigma_\lambda(t,\omega^\varepsilon) - \varepsilon +$$

$$+ f_\lambda(t,x,u,\partial_x u_\lambda(t,x)) - f_\lambda(t,x,u^\varepsilon,\partial_x u_\lambda(t,x)) \leq$$

$$\leq \partial_t u_\lambda(t,x) - f_\lambda(t,x,u,\partial_x u_\lambda(t,x)) - \sigma_\lambda(t,\omega^\varepsilon) - \varepsilon + \sigma(t, V(u - u^\varepsilon)) =$$

$$= \partial_t u_\lambda(t,x) - f_\lambda(t,x,u,\partial_x u_\lambda(t,x)) - \varepsilon <$$

$$< \partial_t u_\lambda(t,x) - f_\lambda(t,x,u,\partial_x u_\lambda(t,x)) \leq \partial_t v_\lambda(t,x) - f_\lambda(t,x,v,\partial_x v_\lambda(t,x)).$$

It follows from Theorem 1 that

$$u^\varepsilon(t,x) < v(t,x) \text{ on } E.$$

Since for each $\lambda \in \Lambda$ the function $\omega^\varepsilon_\lambda : [-r_0, a) \to R$ is non-decreasing with respect to ε, there is

$$\lim_{\varepsilon \to 0} \omega^\varepsilon_\lambda(t) = 0$$

uniformly for $t \in [-r_0, a)$ and $\lambda \in \Lambda$. The proof is finished. □

As an application of Theorem 2, the following uniqueness result can be derived.

LEMMA 3. *Suppose conditions 1)–3) of Theorem 2 hold.*

Then Cauchy problem (1), (2) admits at most one solution of class D on $E_0 \cup E$.

PROOF. For two solutions z and $\bar{z}$ of (1), (2), there is

$$\partial_t z(t,x) - F[z](t,x) \leq \partial_t \bar{z}(t,x) - F[\bar{z}](t,x)$$

and

$$\partial_t \bar{z}(t,x) - F[\bar{z}](t,x) \leq \partial_t z(t,x) - F[z](t,x).$$

Thus $z(t,x) = \bar{z}(t,x)$ on $E_0 \cup E$. □

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Received June 13, 2005

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