UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLIII
2005

ON THE DENSE TRAJECTORY OF LASOTA EQUATION

by Antoni Leon Dawidowicz and Najemedin Haribash

Abstract. In presented paper the dense trajectory of dynamical system given by Lasota equation is constructed.

1. Introduction. The equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = F(u)$$

was first introduced by McKendrick in 1926 [**8**] and von Foerster in 1959 [**2**]. It described the dynamics of population age structure. A classical system of equations with a non-local boundary condition has always been the subject of interest of the whole world of mathematics. The next stage of research work on similar type equations was the work of Lasota, Ważewska and Mackey [**5, 6**]. They used a similar type of equation, to be precise, equation

$$\frac{\partial u}{\partial t} + c(x)\frac{\partial u}{\partial x} = F(x, u)$$

to describe blood cell population. Appearance of interpretation for this equation inspired professor Lasota and his partners to study chaos and stability in dynamical systems given by this equation. The first impulse was given by professor Lasota [**3**] proving the existence of invariant measure, therefore calling this "Lasota equation" would be good and legitimate. Apart from Lasota and one of the authors, also Rudnicki [**9**] and Szarek [**4**] worked on invariant measures.Then Łoskot [**7**] analyzed them in the turbulence aspect in the Bass sense. The subject of this work is to prove the existence of a dense trajectory for Lasota equation. In the construction of a dense trajectory, a generalization of Avez method was used. Till this time this method has been used as a tool for invariant measure construction in the works of Lasota and his students. Construction of this trajectory may also be called an Avez construction, as right inverses are the basic tool here. However, this variant of Avez method is more interesting as it does not require discretisation of this system.

2. Formulation of the theorem. In paper [**1**], the existence of an invariant measure for the dynamical system generated on the space V of all Lipschitz functions on $[0,1]$ by the equation

$$\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial x}=\lambda u \tag{1}$$

is proved, where $\lambda>1$. The following theorem ensures the existence of a dense trajectory for the same system.

THEOREM 1. *Let us consider equation* (1) *in the domain*

$$x\in[0;1]\ t\geq 0$$

with the initial condition

$$u(0,x)=v(x).$$

Let $\{T_t\}_{t\geq 0}$ *be the semidynamical system generated by this problem, i.e.,*

$$(T_t v)(x)=e^{\lambda t}v(xe^{-t}). \tag{2}$$

If $\lambda>1$, *then there exists a dense trajectory of system* $\{T_t\}$ *at the space* V *of all Lipschitz functions on* $[0;1]$ *vanishing in* 0.

3. Auxiliary elementary lemma. To prove the theorem, the following elementary technical lemma is necessary

LEMMA 1. *Let* $\{s_n\},\{b_n\},\{c_n\}$ *be arbitrary sequences of positive numbers. Then, there exists the sequence* $\{a_n\}$ *such that for every positive integer* n

$$0<a_n<c_n \tag{3}$$

and

$$\sum_{k=n}^{\infty}a_k s_k\leq b_n. \tag{4}$$

PROOF. To prove the lemma, first for any positive integer p we construct the sequence $\{a_n^p\}$ satisfying (3) for any n and (4) for all $n\geq p$. The sequence $\{a_n^1\}$ is defined by the formula

$$a_n^1=\min\left\{c_n,\frac{b_1}{2^{n+1}s_n}\right\}.$$

It is obvious that $a_n^1\leq c_n$. Moreover,

$$\sum_{k=1}^{\infty}a_k^1 s_k\leq b_1\sum_{k=1}^{\infty}\frac{1}{2^{k+1}}=\frac{1}{2}b_1<b_1.$$

Now assume the existence of sequences $\{a_n^r\}$ for all $r = 1, \dots, p$. Assume also, that these sequences satisfy the following inequalities

$$a_n^r \leq a_n^{r-1} \text{ for } r = 2, \dots, p$$

and

$$\sum_{k=n}^{\infty} a_k^r s_r < b_r$$

for all $n \geq r$. From the convergence of the last series, there follows that there exists such $k(p)$, that

$$\sum_{k=n(p)}^{\infty} a_k^r s_r < \frac{1}{2} b_{p+1}. \tag{5}$$

Define

$$a_n^{p+1} = \begin{cases} \min\left\{\frac{b_{p+1}}{2\varphi(n,p)}, a_n^p\right\} & \text{for } p \leq n \leq n(p), \\ a_n^p & \text{otherwise,} \end{cases} \tag{6}$$

where

$$\varphi(n,p) = \sum_{k=n+1}^{\infty} a_k^p s_n.$$

Now, having defined the sequence $\{a_n^p\}$ for all p the sequence $\{a_n\}$ defined by the classical diagonal formula

$$a_n = a_n^n$$

satisfies the condition of thesis. □

4. Proof of the theorem. The space V with the topology of uniform convergence is a separable metric space. Therefore, the topology has a countable basis. Let $\{\sigma_n\} \subset V$ and $\{\varepsilon_n\} \subset \mathbb{R}_+^*$ be such sequences, that the set $\{U_n\}_{n=1}^{\infty}$, where

$$U_n = U(\sigma_n, \varepsilon_n) = \{v \in V : |\sigma_n(x) - v(x)| < \varepsilon_n \forall x \in [0;1]\} \tag{7}$$

is a basis of uniform topology in V. Since $\sigma_n \in V$, for every n, there exists the optimal Lipschitz constant of σ_n i.e.

$$s_n = \sup_{x,y \in [0,1], x \neq y} \left| \frac{\sigma_n(x) - \sigma_n(y)}{x - y} \right|.$$

Obviously,

$$\sup_{x,y \in [0;1]} |\sigma_n(x) - \sigma_n(y)| \leq s_n |x - y| \leq s_n.$$

Since σ_n is continuous and vanishes at 0, one can define the sequence $\{\kappa_n\}$ by the following recurrence formula

$$\kappa_0 = 1,$$

$$\kappa_n = \sup\{x \in [0;1] : \forall \xi \in [0;x] |\sigma_n(x)| \leq \varepsilon_n\}.$$

Let

$$c_n = \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^{\frac{1}{\lambda}},$$

$$b_n = \varepsilon_{n-1}.$$

By Lemma 1 there exists such sequence $\{a_n\}$, that

$$\left(\prod_{k=1}^{n} a_k\right)^{\lambda} \leq \kappa_n,$$

$$\sum_{k=n}^{\infty} a_k s_k \leq \varepsilon_n.$$

Define

$$\theta_j = \max\{-\lambda \ln a_j, 0\}$$

and

$$t_n = \sum_{j=1}^{n} \theta_j.$$

By the last four formulae there is

$$e^{-t_n} = \prod_{k=1}^{n} e^{-\theta_k} = \prod_{k=1}^{n} e^{\lambda \ln a_k} = \left(\prod_{k=1}^{n} a_k\right)^{\lambda} \leq \kappa_n$$

and

$$\begin{aligned}\sum_{n=k}^{\infty} e^{-\lambda t_n} s_n &\leq \sum_{n=k}^{\infty} \exp\left(-\sum_{j=1}^{n} \lambda\theta_j\right) s_n \\ &\leq \exp\left(-\sum_{j=1}^{k-1} \lambda\theta_j\right) \sum_{n=k}^{\infty} e^{-\lambda\theta_n} s_n \\ &\leq \exp\left(-\sum_{j=1}^{k-1} \lambda\theta_j\right) \sum_{n=k}^{\infty} a_n s_n \leq e^{-\lambda t_{k-1}} \varepsilon_{k-1}.\end{aligned}$$

Continuing the proof, we have to construct a family of right inverses of T_t i.e., the family of the maps $S : V \to V$ satisfying the condition $T_t S v = v$ for every $v \in V$. Let $\sigma \in V$ Define $S_\sigma^t : V \to V$ as

$$(S_\sigma^t v)(x) = \begin{cases} e^{-\lambda t} x(xe^t) & \text{for } x \le e^{-t}, \\ \sigma(x) - \sigma(e^{-t}) + e^{-\lambda t} v(1) & \text{for } x > e^{-t}. \end{cases}$$

From this definition we conclude that for every $t > 0$ and for every $\sigma \in V$

$$T_t S_\sigma^t = \mathrm{id}_V \, .$$

Let now σ_n and θ_n be defined as above and let

$$v \in \bigcap_{n=1}^{\infty} S_{\sigma_1}^{\theta_1} \ldots S_{\sigma_n}^{\theta_n}(V).$$

We claim, that such v exists and is unique. The uniqueness of v follows from the continuity of v, v/s vanishing at zero and a natural condition that for every interval $[\exp(-t_n); \exp(-t_{n-1})]$

$$v(x) = e^{-\lambda \sum_{k=1}^{n} \theta_k} \sigma_n \left(x e^{\sum_{k=1}^{n} \theta_k} \right) = e^{-\lambda t_n} \sigma_n \left(x e^{-t_n} \right) \tag{8}$$

up to additive constant.

Let

$$v_n = S_{\sigma_1}^{\theta_1} \ldots S_{\sigma_n}^{\theta_n}(0).$$

The function v_n is equal to every function belonging to $S_{\sigma_1}^{\theta_1} \ldots S_{\sigma_n}^{\theta_n}(V)$ up to an additive constant on the interval $[\exp(-t_n); 1]$. Moreover,

$$|v_n(x) - v_{n+1}(x)| \le e^{-\lambda t_{n+1}} s_{n+1}.$$

From the last inequality there follows that the sequence $\{v_n\}$ converges uniformly to some function $\bar{v}$. To complete the claim, it is sufficient to show that $\bar{v}$ satisfies the Lipschitz condition. From (8) there follows that on every interval of the form

$$[\exp(-t_n); \exp(-t_{n-1})] \, ,$$

the function $\bar{v}$ satisfies the Lipschitz condition with the constant

$$e^{-\lambda t_n} s_n e^{t_n}.$$

Moreover,

$$e^{-\lambda t_n} s_n e^{t_n} = s_n \prod_{k=1}^{n} e^{(1-\lambda)\theta_k} \le 1.$$

Whence there follows, that the function $\bar{v}$ satisfies the Lipschitz condition with constant 1, and in consequence $\bar{v} = v$. To complete the proof of the theorem, it is sufficient to prove that $\{T_t v\}_{t \ge 0}$ is dense in V. From the definition of c_n and

t_n there follows that on the interval $[0, \exp(-\sum_{n=k}^{\infty} \theta_n)]$ the absolute value of function v is less than $\sum_{n=k}^{\infty} e^{-\lambda t_n} s_n \le \exp(-\lambda t_{k-1})\, \varepsilon_{k-1}$ and on the interval

$$\left[\exp\left(-\sum_{n=k}^{\infty}\theta_n\right); \exp\left(-\sum_{n=k-1}^{\infty}\theta_n\right)\right],$$

the function v is equal to the function $e^{-\lambda t_{k-1}}\sigma_{k-1}\left(xe^{t_{k-1}}\right)$.

In consequence,

$$T_{t_{k-1}} v$$

is less than ε_{k-1} on the interval $[0, e^{-t_k}]$ and equal to σ_{k-1} up to an additive constant on the interval $[e^{-t_k}, 1]$. Thus

$$T_{t_{k-1}} v \in U_{k-1}.$$

Since $\{U_n\}$ is a basis of uniform topology in V, the last formula completes the proof.

References

1. Dawidowicz A.L., *On the existence of an invariant measure for the dynamical system generated by partial differential equation*, Ann. Polon. Math., **XLI** (1983), 129–137.
2. von Foerster J., *Some remarks on changing populations,* in: *The kinetics of cell proliferation*, Stohlman F. (ed.), Grune & Stratton, New York, 1959, 382–407.
3. Lasota A., Pianigiani G., *Invariant measures on topological spaces*, Boll. Un. Mat. Ital., **5(15–B)** (1977), 592–603.
4. Lasota A., Szarek T., *Dimension of measures invariant with respect to Ważewska partial differential equations*, J. Differential Equations, **196(2)** (2004), 448–465.
5. Lasota A., Ważewska-Czyżewska M., *Matematyczne problemy dynamiki układu krwinek czerwonych*, Mat. Stos., **6** (1976), 23–40.
6. Lasota A., Ważewska-Czyżewska M., Mackey M.C., *Minimazing therapeutically induced anemia*, J. Math. Biol., **13** (1981), 149–158.
7. Łoskot K., *Turbulent solutions of first order partial differential equation*, J. Differential Equations, **58(1)** (1985), 1–14.
8. McKendrick A.G., *Application of mathematics to medical problems*, Edinburgh Math. Soc., **44** (1926), 98–130.
9. Rudnicki R., *Invariant measures for the flow of a first order partial differential equation*, Ergodic Theory Dynam. Systems, **5(3)** (1985), 437–443.

Received March 1, 2004

Jagiellonian University
Institute of Mathematics
ul. Reymonta 4/510
30-059 Kraków
Poland
e-mail: `Antoni.Leon.Dawidowicz@im.uj.edu.pl`

e-mail: `najem45@hotmail.com`