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A GEOMETRICAL VERSION OF THE MOORE THEOREM IN THE CASE OF INFINITE DIMENSIONAL BANACH SPACES

by Marcin Ziomek

Abstract. In this paper the Author shows that if one defines a triod in a suitable way, then it is possible to prove the Moore theorem in the infinite dimensional case.

1. Introduction. The classical Moore theorem is a certain refinement of the Suslin property of separable spaces (each family of pairwise disjoint open sets is countable). In [**4**] Moore has formulated the following property:

Each family of triods in $\mathbb{R}^2$ *is countable.*

A triod is a set homeomorphic with $[-1,1]\times\{0\}\cup\{0\}\times[0,1]$. The generalization of this theorem for $\mathbb{R}^n$ was proved by Young in [**5**]. By a "triod" in $\mathbb{R}^n$ one means a set which is homeomorphic to "an umbrella" (by an n*-dimensional umbrella* we understand the union of an n-ball Q and a simple arc L such that the set $Q \cap L$ contains exactly one point lying in the set $Q\backslash int\, Q$ and being an end point of L). Another version of such properties was proved by Bing and Borsuk in [**1**].

A direct generalization to the case of infinite dimensional Banach spaces is not true. Indeed, let us consider the space l_2. Let

$$B = \{x \in l_2 : x_1 = 0 \wedge \|x\| \leq 1\} \cup \{x \in l_2 : x_1 \in [0,1] \wedge \forall k \geq 2 \ x_k = 0\}.$$

If one understands a triod as a set, homeomorphic (or even isometric) to B, then the property from the Moore theorem does not hold. Indeed, consider the hyperplanes $H_c = \{x \in l_2 : x_1 = c\}$ and $c \in \mathbb{R}$. It follows from the Riesz theorem, that H_0 is isometric to l_2. Let $v = (c, 0, \dots)$, then $H_c = H_0 + v$ and thus $T_v(H_0) = H_c$, where $T_v : l_2 \to l_2$ and $T_v(x) = x+v$. Hence we have a triod in each hyperplane H_c. But these hyperplanes form an uncountable family of pairwise disjoint sets.

However, it is possible to prove a kind of Moore theorem in infinite dimensional case if one considers a more "rigid" notion of the triod.

2. The main theorem. Let $(E, ||\cdot||)$ be a real Banach space, let E^* be the conjugate of E and let $x, u \in E,\ r > 0,\ f \in E^*$ such that $f(x) \neq f(u)$ and $\|x - u\| = r$. Let $B(x, r)$ be an (open) ball with the center at x and the radius r, and let ab denote the segment with ends a and b.

Definition 1. The hyperplane defined by a functional f and a constant c is the set $\{y \in E : f(y) = c\}$. We will denote it by $H_{f,c}$ (clearly $H_{f,0} = \ker f$).

Definition 2. A triod given by the parameters x, r, f and u is the set

$$\big(B(x,r) \cap H_{f,f(x)}\big) \cup xu.$$

It will be denoted by $T(x, r, f, u)$. The point x will be called the emanation point, the number r will be called the radius of the triod and the segment joining the points x and u will be called a handle.

Clearly if $\lambda \neq 0$, then $H_{f,f(x)} = H_{\lambda f, \lambda f(x)}$, i.e. without loss of generality we may assume that the norm of f equals 1.

We will below use the following simple lemmas.

Lemma 3. *If A is an uncountable set and $h : A \to (0, +\infty)$ is an arbitrary function, then there exists a real positive number d such that* $\operatorname{card}\{a \in A : h(a) \geq d\} > \aleph_0$.

Proof. Since $A = \bigcup A_n$, where

$$A_n = \{a \in A : h(a) \geq \frac{1}{n}\}$$

then A_n must be uncountable for at least one n. □

Lemma 4. *Let $x, y, z \in E$ and $d > 0$ be such that*

$$\max\{\|x - y\|, \|x - z\|, \|y - z\|\} \leq \frac{d}{4} \tag{1}$$

and $f, g, h \in E^*$. If the sets $B(x,d) \cap H_{f,f(x)}$, $B(y,d) \cap H_{g,g(y)}$, $B(z,d) \cap H_{h,h(z)}$ are pairwise disjoint, then

$$y, z \notin H_{f,f(x)} \wedge x, z \notin H_{g,g(y)} \wedge x, y \notin H_{h,h(z)}, \tag{2}$$

$$H_{f,f(x)} \cap yz \neq \varnothing \vee H_{g,g(y)} \cap xz \neq \varnothing \vee H_{h,h(z)} \cap xy \neq \varnothing. \tag{3}$$

Proof. Property (2) follows from the fact that the centers are pairwise different and from the assumed inequality.

Suppose now that (3) does not hold. This implies in particular, that x, y, z are not co-linear and then $\dim \operatorname{lin}\{x-y, x-z\} = 2$. Denote $H = \operatorname{lin}\{x-y, x-z\} + x$, $L_x = H \cap H_{f,f(x)}$, $L_y = H \cap H_{g,g(y)}$, $L_z = H \cap H_{h,h(z)}$. Our hypothesis now implies that

$$(4) \qquad L_x \cap yz = \varnothing \wedge L_y \cap xz = \varnothing \wedge L_z \cap xy = \varnothing.$$

Because $x, y, z \in H$, then from (2) we obtain $\dim L_x = \dim L_y = \dim L_z = 1$. Then (4) implies that the lines L_x, L_y, L_z cannot be parallel. Hence one of them – say L_x – cuts L_y and L_z. We set: $L_x \cap L_y = a$, $L_x \cap L_z = b$. Since (4), then $a \neq b$. Because the considered sets are disjoint by the assumptions of the Lemma, there is $\|a-x\| \geq d$ or $\|a-y\| \geq d$ and $\|b-x\| \geq d$ or $\|b-z\| \geq d$. Since (1), then

$$(5) \qquad \min\{\|a-x\|, \|a-y\|, \|a-z\|, \|b-x\|, \|b-y\|, \|b-z\|\} \geq \frac{3d}{4}.$$

Now we will check that $L_y \cap L_z \neq \varnothing$. Suppose that $L_y \cap L_z = \varnothing$. Hence (4) implies $x \in ab$. In consequence, $(L_y + (x-y)) \cap yz \neq \varnothing$. This intersection is a single-point set; denote it by s. Because the lines L_y, L_z, $L_y + (x-y)$ are parallel and (4) and (5) hold, then $\|x-s\| \geq \frac{3d}{4}$. But this is impossible, since

$$\|x-s\| \leq \|x-y\| + \|y-s\| < \frac{d}{4} + \|y-z\| \leq \frac{d}{2}.$$

We denote the intersection $L_y \cap L_z$ by c. Clearly $c \neq a$ and $c \neq b$ as well as $\min\{\|c-x\|, \|c-y\|, \|c-z\|\} \geq \frac{3d}{4}$.

We observe that (4) implies $x \in ab$ or $y \in ac$ or $z \in bc$. Because of symmetry it is sufficient to consider the case of $x \in ab$. Without loss of generality we may assume that $\|a-x\| \geq \|b-x\|$, thus $2\|a-x\| \geq \|b-x\| + \|a-x\| = \|a-b\|$. In consequence

$$(6) \qquad \frac{\|a-b\|}{\|a-x\|} \leq 2.$$

Now we denote by c' and b' the points such that: $c' \in L_x$, $cc' || xy$ and $b' \in L_y$, $bb' || xy$.

We now observe that $c \notin ya$, hence it is sufficient to consider the following cases:

1. $y \in ca$.

   Now there are two possibilities:

   (a) $b \in c'x$.

   Thus $zy \cap bb' \neq \varnothing$. Let us denote the common point by t. Then from (1), (6) and the Tales theorem, we obtain

$$\frac{\|b-b'\|}{\frac{d}{4}} \leq \frac{\|b-b'\|}{\|x-y\|} = \frac{\|a-b\|}{\|a-x\|} \leq 2.$$

In consequence, $\|b - b'\| \leq \frac{d}{2}$. Thus by (5) there is

$$\frac{3d}{4} \leq \|b - y\| \leq \|b - t\| + \|t - y\| < \left\|b - b'\right\| + \|t - y\| \leq \frac{d}{2} + \|t - y\| .$$

This leads to the contradiction, since

$$\frac{d}{4} < \|t - y\| \leq \|z - y\| \leq \frac{d}{4}.$$

(b) $c' \in bx$.
Hence $zy \cap cc' \neq \varnothing$. To obtain a contradiction, it is sufficient to repeat the reasoning from 1(a) replacing b by c' and b' by c and using the fact that in this case $\|a - c'\| \leq \|a - b\|$.

2. $a \in yc$.
In this case $zy \cap bb' \neq \varnothing$ and we use the same argument as in 1(a).

□

We will also use the following theorem (its proof can be found in [**2**]).

Theorem 5. *If X is a topological space satisfying the second countability axiom, then for each set $A \subset X$ the set of points in A which are not its condensation points is countable.*

Theorem 6. *If E is a real separable Banach space, then any family of pairwise disjoint triods in E is countable.*

Proof. Suppose that E is a separable Banach space and let $\Im$ be an uncountable family of pairwise disjoint triods.

It follows from Lemma 3 that there exist $d > 0$ and an uncountable subset $\Im_1$ of $\Im$ such that all triods in $\Im_1$ have the radius d.

Without loss of generality we may assume that all triods in $\Im_1$ have the radius equal d and are still pairwise disjoint.

Observe that the set $\Im_1$ can be written as the union of two sets $\{T(x, d, f, u) \in \Im_1 : f(x) < f(u)\}$ and $\{T(x, d, f, u) \in \Im_1 : f(x) > f(u)\}$. Hence, at least one of them (without loss of generality we assume that the first one) is uncountable. It follows from Lemma 3 that there exists $\delta > 0$ such that the set $\Im_2 = \{T(x, d, f, u) \in \Im_1 : f(u - x) \geq \delta\}$ is uncountable. Since the triods are pairwise disjoint, then the set of their emanation points $G = \{x \in E : T(x, d, f, u) \in \Im_2\}$ is uncountable. By Theorem 5, there exists (in G) an emanation point which is its condensation point. Consider the ball with the center at this point and with the radius $\frac{\delta}{8}$. It follows from Lemma 4 that there exist the triods $T(\theta, d, g, w)$ and $T(x, d, f, u)$ in $\Im_2$ (using a translation if necessary, we may assume that the origin is the first emanation point) such that $g(x) > 0$. Hence $g(w) \geq \delta$ and $0 < \|x\| < \frac{\delta}{4}$.

Notice that $\delta \leq d$. Indeed, since $\|g\| = 1$, $g(w) \geq \delta$, by the definition of the radius it follows that

$$\delta \leq g(w) \leq \|w\| = d. \tag{7}$$

Moreover,

$$g(x) \leq \|x\| < \frac{\delta}{4}, \tag{8}$$

$$\frac{g(x)}{g(w)} < \frac{1}{4}. \tag{9}$$

Observe that $x \notin \mathbb{R}w$ and consider the following cases:

1. $\mathbb{R}w$ and $H_{f,f(x)}$ have exactly one common point.
We denote this point by $\overline{w}$. Then there exists $\lambda \in \mathbb{R}$ such that

$$\overline{w} = \lambda w. \tag{10}$$

Hence $\overline{w} \in H_{f,f(x)}$ and

$$\|\overline{w}\| = |\lambda|\, d, \tag{11}$$

$$g(\overline{w}) = \lambda g(w). \tag{12}$$

(a) $0 < \lambda < \frac{1}{2}$.
In consequence, using (8), (11) and (7), we obtain

$$\|x - \overline{w}\| \leq \|x\| + \|\overline{w}\| < \frac{\delta}{4} + \frac{d}{2} < d.$$

But $\overline{w} \in H_{f,f(x)}$, hence $\overline{w} \in T(x, d, f, u)$. This is impossible, since the triods are pairwise disjoint (clearly, $\overline{w} \in T(\theta, d, g, w)$).

(b) $\frac{1}{2} \leq |\lambda|$.
Since $g(\overline{w}) \neq g(x)$, hence $\mathbb{R}(\overline{w} - x) + x$ and $\ker g$ have exactly one common point; let us denote it by t. Let $\alpha \in \mathbb{R}$ be such that $\overline{w} = t + \alpha(x - t)$. Hence $\|\overline{w} - t\| = |\alpha|\, \|x - t\|$ and $g(\overline{w}) = \alpha g(x)$. In consequence,

$$|g(\overline{w})| = \frac{\|\overline{w} - t\|}{\|x - t\|} g(x).$$

In consequence, using (10), (12) and (9), we obtain

$$\begin{aligned} \|x - t\| &= \frac{g(x)\, \|\overline{w} - t\|}{|g(\overline{w})|} = \frac{g(x)\, \|\lambda w - t\|}{|\lambda|\, g(w)} < \frac{\|\lambda w - t\|}{4\,|\lambda|} \\ &\leq \frac{\|w\|}{4} + \frac{\|t\|}{4\,|\lambda|} \leq \frac{d}{4} + \frac{\|t\|}{2}. \end{aligned}$$

Therefore, using (8) and (7), we obtain

$$\|t\| \le \|x\| + \|x - t\| < \frac{\delta}{4} + \frac{d}{4} + \frac{\|t\|}{2} \le \frac{d}{2} + \frac{\|t\|}{2}$$
$$\|t\| < d.$$

In consequence, since $t \in \ker g$, we obtain $t \in T(\theta, d, g, w)$. Moreover,

$$\|x - t\| < \frac{d}{4} + \frac{\|t\|}{2} < d.$$

Then, since $t \in H_{f,f(x)}$, we obtain $t \in T(x, d, f, u)$. This is impossible, since the triods are pairwise disjoint.

(c) $-\frac{1}{2} < \lambda \le 0$.
Hence $\|\overline{w}\| < \frac{d}{2}$ and $g(\overline{w}) \le 0$. Let t be the intersection point of the segment $x\overline{w}$ and $\ker g$. Since $\|\overline{w}\| < \frac{d}{2}$, by (8) there also is $\|t\| < \frac{d}{2}$. Therefore $t \in T(x, d, f, u) \cap T(\theta, d, g, w)$.

2. $\mathbb{R}w$ and $H_{f,f(x)}$ are disjoint.

Denote $\widehat{w} = \frac{g(x)}{g(w)} w$; then $g(\widehat{w}) = g(x)$. Observe that

$$x - \widehat{w} \in (\mathbb{R}w + x) \cap \ker g \wedge \mathbb{R}w + x \subset H_{f,f(x)}. \tag{13}$$

Moreover,

$$\|\widehat{w}\| = \frac{g(x)}{g(w)} d < \frac{d}{4}, \tag{14}$$

$$\|x - \widehat{w}\| < \frac{\delta}{4} + \frac{d}{4} \le \frac{d}{2}. \tag{15}$$

It follows from (13) and (14) that $x - \widehat{w} \in T(x, d, f, u)$, but from (13) and (15) there follows $x - \widehat{w} \in T(\theta, d, g, w)$. This is impossible, since the triods are pairwise disjoint.

3. $\mathbb{R}w$ contains in $H_{f,f(x)}$.

In this situation, $\theta \in H_{f,f(x)}$. Because $\operatorname{dist}(x, \theta) < \frac{\delta}{4} \le \frac{d}{4}$, then $\theta \in T(x, d, f, u)$. This is impossible, since the triods are pairwise disjoint.

□

Remark 7. In the proof of Theorem 6, the form of "the handle" (a segment) is used in the case 1(a) only, i.e. when $\mathbb{R}w$ and $H_{f,f(x)}$ have exactly one point in common and this point is of the form λw for a $\lambda \in (0, \frac{1}{2})$.

We can slightly generalize the definition of the triod.

Let $(E, ||\cdot||)$ be a real Banach space, let E^* be the conjugate of E and let $x, u \in E$, $r > 0$, $f \in E^*$, $\varphi \in E^{[a,b]}$ for some $a, b \in \mathbb{R}$, $a < b$ such that $f(x) \neq f(u)$, $\|x - u\| = r$, φ is continuous and $\varphi(a) = x$, $\varphi(b) = u$, $f(\varphi(t)) \neq f(x)$ ($\varphi(t) \notin H_{f,f(x)}$) for $t \in (a, b]$.

Definition 8. A generalized triod given by the parameters $x,\ r,\ f,\ u$ and φ is the set

$$\big(B(x,r)\cap H_{f,f(x)}\big)\cup\{\varphi(t):t\in[a,b]\}\,.$$

It will be denoted by $T(x,r,f,u,\varphi)$.

The main theorem of this paper is the following one.

Theorem 9. *If E is a real separable Banach space, then each family of pairwise disjoint generalized triods in E is countable.*

Proof. Suppose that E is a separable Banach space and let $\Im$ be an uncountable family of pairwise disjoint generalized triods.

As before in Theorem 6, by Lemma 3, we may then without loss of generality assume that all generalized triods in $\Im$ have radii greater or equal to d for some $d>0$. Fix an arbitrary triod $T(x,r,f,u,\varphi)$ and consider the sphere $S(x,\frac{d}{2})$. Then

$$\exists c\in(a,b)\left\{\|x-\varphi(c)\|=\frac{d}{2}\wedge\forall t\in(a,c)\ \|x-\varphi(t)\|<\frac{d}{2}\right\}.$$

Consider

$$\Im'=\{T(x,r',f,u',\varphi):T(x,r,f,u,\varphi)\in\Im\wedge r'=\frac{d}{2}\wedge u'=\varphi(c)\}.$$

This is a family of pairwise disjoint generalized triods.

Repeating the reasoning from the proof of Theorem 6, without loss of generality, we may assume that for all generalized triods $T(x,r',f,u',\varphi)$ in $\Im'$ the inequality $f(u'-x)\geq\delta$ holds for some fixed $0<\delta\leq d$. It follows from Lemma 4 and Theorem 5 that there exist triods $T(\theta,\frac{d}{2},g,w',\psi)$ and $T(x,\frac{d}{2},f,u',\varphi)$ in $\Im'$ such that $g(x)>0$ and $0<\|x\|<\frac{\delta}{4}$.

Notice that it is sufficient to consider the case of $\mathbb{R}w'\cap H_{f,f(x)}=\{\lambda w'\}$ for $\lambda\in(0,\frac{1}{2})$. In all other cases (Remark 7), we can repeat the reasoning from the proof of Theorem 6. Since θ and w' lie on opposite sides of the hyperplane $H_{f,f(x)}$, hence the curve joining θ and w' has the common point, say t, with this hyperplane. Since the entire curve is contained in the closed ball $B(\theta,\frac{d}{2})$, hence

$$\|x-t\|<\frac{\delta}{4}+\frac{d}{2}<d$$

which is impossible, since the generalized triods are pairwise disjoint. $\square$

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