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FIRST-ORDER DIFFERENTIAL INVARIANTS OF THE SPLITTING SUBGROUPS OF THE POINCARÉ GROUP $P(1,4)$

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Abstract. The functional bases of the first-order differential invariants for the splitting subgroups of the Poincaré group $P(1,4)$ are constructed. Some of the results obtained are presented.

The differential invariants of Lie groups of point transformations play an important role in geometry (see, for example, [14]), group analysis of differential equations (see, for example, [12, 14, 15]), etc. In particular, with the help of these invariants we can construct differential equations with non-trivial symmetry groups. Differential invariants have been studied in many works (see, for example, [8, 9, 11–15, 17–19]). The present paper is devoted to the construction of functional bases of the first-order differential invariants for the splitting subgroups of the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics (see, for example, [5, 7, 10]). In order to present some of the results obtained, we consider the Lie algebra of the group $P(1,4)$.

1. The Lie algebra of the group $P(1,4)$ and its non-conjugate subalgebras.

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P'_{μ} $(\mu = 0,1,2,3,4)$, satisfying the commutation relations

$$\left[P'_{\mu}, P'_{\nu}\right] = 0, \qquad \left[M'_{\mu\nu}, P'_{\sigma}\right] = g_{\mu\sigma}P'_{\nu} - g_{\nu\sigma}P'_{\mu},$$

$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

Let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \quad P'_1 = -\frac{\partial}{\partial x_1}, \quad P'_2 = -\frac{\partial}{\partial x_2}, \quad P'_3 = -\frac{\partial}{\partial x_3},$$

$$P'_4 = -\frac{\partial}{\partial x_4}, \quad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right).$$

Further, we will use the following basis elements:

$$G = M'_{40}, \quad L_1 = M'_{32}, \quad L_2 = -M'_{31}, \quad L_3 = M'_{21},$$

$$P_a = M'_{4a} - M'_{a0}, \quad C_a = M'_{4a} + M'_{a0}, \quad (a = 1,2,3),$$

$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \quad X_k = P'_k \quad (k = 1,2,3), \quad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

For the study of the subgroup structure of the group $P(1,4)$, we used the method proposed in [16]. Splitting subgroups of the group $P(1,4)$ have been found in [1, 2, 4].

One of the important consequences of the study of the non-conjugate subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$, and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (see also [7]). The Lie algebra of the group $\widetilde{G}(1,3)$ is generated by the following basis elements:

$$L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4.$$

2. The first-order differential invariants of splitting subgroups of the group $P(1,4)$.

For all splitting subgroups of the group $P(1,4)$, the functional bases of the first-order differential invariants are constructed. In the construction of the differential invariants, it has turned out that different splitting subalgebras of the Lie algebra of the group $P(1,4)$ may have the same functional basis of the first-order differential invariants. Consequently, there is no one-to-one correspondence between non-conjugate splitting subalgebras of the Lie algebra of the group $P(1,4)$ and their respective functional bases of the first-order differential invariants. Moreover, some of the functional bases (which are of the same dimension) may be equivalent. Our aim is to obtain non-equivalent functional bases only. Let $\{J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}\}$ be the functional bases of the first-order differential invariants which correspond to the splitting subalgebras L^1 and L^2 of the Lie algebra of the group $P(1,4)$.

Lemma. *Two functional bases* $\{J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}\}$ *and* $\{J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}\}$ *are equivalent if and only if they satisfy the following conditions:*

$$(*)\qquad \begin{array}{c} \widetilde{X}_1^{(1)} J_1^{(2)} = 0, \widetilde{X}_1^{(1)} J_2^{(2)} = 0, \ldots, \widetilde{X}_{r_1}^{(1)} J_t^{(2)} = 0 \\ \widetilde{X}_1^{(2)} J_1^{(1)} = 0, \widetilde{X}_1^{(2)} J_2^{(1)} = 0, \ldots, \widetilde{X}_{r_2}^{(2)} J_t^{(1)} = 0, \end{array}$$

where $\{\widetilde{X}_1^{(1)}, \widetilde{X}_2^{(1)}, \ldots, \widetilde{X}_{r_1}^{(1)}\}$, $\{\widetilde{X}_1^{(2)}, \widetilde{X}_2^{(2)}, \ldots, \widetilde{X}_{r_2}^{(2)}\}$ *are the first-prolonged bases operators of the Lie subalgebra* L^1 *and* L^2, *respectively;* r_1, r_2 *are the dimensions of the subalgebras* L^1 *and* L^2.

Proof. *The necessity.*

Let functional bases $\{J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}\}$ and $\{J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}\}$ be equivalent. Then there exist smooth functions $f_1, f_2, \ldots, f_t$ and $g_1, g_2, \ldots, g_t$ such that

$$(**)\qquad \begin{array}{lcl} J_1^{(2)} = f_1(J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}) & & J_1^{(1)} = g_1(J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}) \\ J_2^{(2)} = f_2(J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}) & & J_2^{(1)} = g_2(J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}) \\ \dots\dots\dots\dots\dots\dots & \text{and} & \dots\dots\dots\dots\dots\dots \\ J_t^{(2)} = f_t(J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}) & & J_t^{(1)} = g_t(J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}). \end{array}$$

Since $f_1(J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)})$, $f_2(J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)})$, $\ldots, f_t(J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)})$ are the first-order differential invariants for the subalgebra L^1, therefore, we obtain (see, for example, [14, 15])

$$\widetilde{X}_1^{(1)} J_1^{(2)} = 0, \quad \widetilde{X}_1^{(1)} J_2^{(2)} = 0, \quad \ldots \quad \widetilde{X}_{r_1}^{(1)} J_t^{(2)} = 0.$$

Since $g_1(J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)})$, $g_2(J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)})$, $\ldots, g_t(J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)})$ are the first-order differential invariants for the subalgebra L^2, therefore, we obtain

$$\widetilde{X}_1^{(2)} J_1^{(1)} = 0, \quad \widetilde{X}_1^{(2)} J_2^{(1)} = 0, \quad \ldots, \quad \widetilde{X}_{r_2}^{(2)} J_t^{(1)} = 0.$$

Thus, the conditions $(*)$ are satisfied.

The necessity is proved.

The sufficiency.

Let the conditions $(*)$ be satisfied.

The condition

$$\widetilde{X}_1^{(1)} J_1^{(2)} = 0, \quad \widetilde{X}_1^{(1)} J_2^{(2)} = 0, \quad \ldots, \quad \widetilde{X}_{r_1}^{(1)} J_t^{(2)} = 0$$

give us (see, for example, [14, 15]) that the functions $J_1^{(2)}, J_2^{(2)}, \ldots, J_t^{(2)}$ are the first-order differential invariants for the subalgebra L^1 and, therefore, have the following form:

$$\begin{aligned} J_1^{(2)} &= f_1(J_1^{(1)}, J_2^{(1)}, \dots, J_t^{(1)}) \\ J_2^{(2)} &= f_2(J_1^{(1)}, J_2^{(1)}, \dots, J_t^{(1)}) \\ &\dots\dots\dots\dots\dots\dots \\ J_t^{(2)} &= f_t(J_1^{(1)}, J_2^{(1)}, \dots, J_t^{(1)}), \end{aligned}$$

where $f_1, f_2, \dots, f_t$ are arbitrary smooth functions.
The conditions

$$\widetilde{X}_1^{(2)} J_1^{(1)} = 0, \quad \widetilde{X}_1^{(2)} J_2^{(1)} = 0, \quad \dots, \quad \widetilde{X}_{r_2}^{(2)} J_t^{(1)} = 0$$

imply that the functions $J_1^{(1)}, J_2^{(1)}, \dots, J_t^{(1)}$ are first-order differential invariants for the subalgebra L^2 and, therefore, can be written in the following form:

$$\begin{aligned} J_1^{(1)} &= g_1(J_1^{(2)}, J_2^{(2)}, \dots, J_t^{(2)}) \\ J_2^{(1)} &= g_2(J_1^{(2)}, J_2^{(2)}, \dots, J_t^{(2)}) \\ &\dots\dots\dots\dots\dots\dots \\ J_t^{(1)} &= g_t(J_1^{(2)}, J_2^{(2)}, \dots, J_t^{(2)}), \end{aligned}$$

where $g_1, g_2, \dots, g_t$ are arbitrary smooth functions. Thus, we have obtained the relations $(**)$. The sufficiency is proved. □

Proposition. *There exist 243 non-equivalent functional bases of the first-order differential invariants for the splitting subgroups of the group $P(1,4)$.*

Proof. The list of the splitting subalgebras of the Lie algebra of the group $P(1,4)$ contains 281 non-conjugate ones [6].

Taking into account the general ranks of matrices which contain coordinates of the one-prolonged basis elements of the subalgebras of the Lie algebra considered, and the theorem on number of invariants of the Lie group of the point transformations (see, for example, [14, 15]), we can make sure that all of the splitting subalgebras of the Lie algebra of the group $P(1,4)$ have the functional bases of the first-order differential invariants. Therefore, there are 281 functional bases of the first-order differential invariants. Among them, there are equivalent ones. Equivalent functional bases can only be among those which have the same dimensions. Let L^1 be a splitting subalgebra of the Lie algebra of the group $P(1,4)$ which has the t-dimensional functional basis of the first-order differential invariants $\{J_1^{(1)}, J_2^{(1)}, \dots, J_t^{(1)}\}$. To find the bases which are equivalent to $\{J_1^{(1)}, J_2^{(1)}, \dots, J_t^{(1)}\}$, we use the Lemma. Let $\{J_1^{(2)}, J_2^{(2)}, \dots, J_t^{(2)}\}$ be t-dimensional functional basis of the first-order differential invariants of the other splitting subalgebra L^2. Following to the Lemma, if these functional bases satisfy the conditions $(*)$, then the considered bases are equivalent. Otherwise, the considered bases are not equivalent. In the analogous manner, we check whether other t-dimensional functional bases of the first-order differential invariants are equivalent to the $\{J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}\}$ or not. In this way, we obtain all t-dimensional functional bases which are equivalent to $\{J_1^{(1)}, J_2^{(1)}, \ldots, J_t^{(1)}\}$.

In the analogous manner, we construct classes of the equivalent functional bases of other dimensions.

The direct check provides 243 non-equivalent functional bases of the first-order differential invariants for the splitting subgroups of the group $P(1,4)$. The Proposition is proved. □

It is impossible to present all non-equivalent functional bases here. Some of them can be found in [3]. Therefore, below we only give a short review of the results obtained.

Below, for some of the splitting subalgebras of the Lie algebra of the group $P(1,4)$, we write their basis elements and corresponding functional bases of differential invariants.

1. There exists one two-dimensional functional basis

$$\begin{aligned}
&\langle G,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&\langle L_3+eG,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4,\ e>0\rangle,\\
&\langle G,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&\langle G,\ L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&\langle G,\ C_1,\ C_2,\ C_3,\ L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&J_1=u,\quad J_2=u_0^2-u_1^2-u_2^2-u_3^2-u_4^2\ ;\\
&u_\mu\equiv\frac{\partial u}{\partial x_\mu},\ \mu=0,1,2,3,4.
\end{aligned}$$

2. There exist 7 three-dimensional non-equivalent functional bases. Let us give some examples.

   1. $$\begin{aligned}
&\langle G,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&\langle L_3+eG,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4,\ e>0\rangle,\\
&\langle G,\ L_3,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&\langle G,\ L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle,\\
&J_1=u,\quad J_2=u_0^2-u_1^2-u_2^2-u_3^2-u_4^2,\quad J_3=\frac{x_0+x_4}{u_0-u_4}\ ;
\end{aligned}$$

2. $\langle P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $\langle L_3 - P_3,\ P_1,\ P_2,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $\langle L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $\langle L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $J_1 = u, \quad J_2 = u_0 - u_4, \quad J_3 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2$.

3. There exist 18 four-dimensional non-equivalent functional bases. Let us give some examples.

   1. $\langle G,\ P_3,\ L_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4}, \quad J_3 = u_1^2 + u_2^2, \quad J_4 = u_0^2 - u_3^2 - u_4^2;$$

   2. $\langle L_3,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2, \quad J_3 = u_0 - u_4,$
   $J_4 = u_1^2 + u_2^2$.

4. There exist 37 five-dimensional non-equivalent functional bases. Let us give some examples.

   1. $\langle G,\ P_3,\ L_3,\ X_0,\ X_3,\ X_4\rangle, \quad \langle G,\ P_3,\ C_3,\ L_3,\ X_0,\ X_3,\ X_4\rangle$,
   $J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = u, \quad J_3 = x_1 u_2 - x_2 u_1,$
   $J_4 = u_1^2 + u_2^2, \quad J_5 = u_0^2 - u_3^2 - u_4^2$;

   2. $\langle L_1,\ L_2,\ L_3,\ X_1,\ X_2,\ X_3,\ X_4\rangle$,
   $J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_0, \quad J_4 = u_4, \quad J_5 = u_1^2 + u_2^2 + u_3^2$.

5. There exist 51 six-dimensional non-equivalent functional bases. Let us give some examples.

   1. $\langle L_3 + eG,\ P_1,\ P_2,\ X_3,\ X_4,\ e > 0\rangle$,
   $$J_1 = u, \quad J_2 = \frac{x_0 + x_4}{u_0 - u_4},$$
   $$J_3 = \left(x_1 + \frac{x_0 + x_4}{u_0 - u_4} u_1\right)^2 + \left(x_2 + \frac{x_0 + x_4}{u_0 - u_4} u_2\right)^2,$$
   $$J_4 = e \arctan\left(\frac{u_1(x_0 + x_4) + x_1(u_0 - u_4)}{u_2(x_0 + x_4) + x_2(u_0 - u_4)}\right) + \ln(x_0 + x_4),$$
   $J_5 = u_3, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_4^2$;

2. $\langle L_3,\ P_1,\ P_2,\ X_3,\ X_4\rangle$,

$J_1 = x_0 + x_4, \quad J_2 = u,$

$$J_3 = \left(\frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4}\right)^2 + \left(\frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4}\right)^2,$$

$J_4 = u_3, \quad J_5 = u_0 - u_4, \quad J_6 = u_0^2 - u_1^2 - u_2^2 - u_4^2$.

6. There exist 58 seven-dimensional non-equivalent functional bases. Let us give some examples.

1. $\langle G,\ L_3,\ X_3,\ X_4\rangle$,

$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = u, \quad J_3 = x_1u_2 - x_2u_1,$

$J_4 = (x_0 + x_4)(u_0 + u_4), \quad J_5 = u_3, \quad J_6 = u_1^2 + u_2^2, \quad J_7 = u_0^2 - u_4^2$;

2. $\langle P_1,\ P_2,\ X_3,\ X_4\rangle$,

$J_1 = x_0 + x_4, \quad J_2 = u, \quad J_3 = u_1(x_0 + x_4) + x_1(u_0 - u_4),$

$J_4 = u_2(x_0 + x_4) + x_2(u_0 - u_4), \quad J_5 = u_3, \quad J_6 = u_0 - u_4,$

$J_7 = u_0^2 - u_1^2 - u_2^2 - u_4^2$.

7. There exist 40 eight-dimensional non-equivalent functional bases. Let us give some examples.

1. $\langle G,\ L_3,\ X_4\rangle$,

$J_1 = x_3, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = u, \quad J_4 = x_1u_2 - x_2u_1,$

$J_5 = (x_0 + x_4)(u_0 + u_4), \quad J_6 = u_3, \quad J_7 = u_0^2 - u_4^2, \quad J_8 = u_1^2 + u_2^2$;

2. $\langle L_3 - P_3,\ X_3,\ X_4\rangle$,

$J_1 = x_0 + x_4, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = u, \quad J_4 = x_1u_2 - x_2u_1,$

$J_5 = u_0 - u_4, \quad J_6 = u_1^2 + u_2^2, \quad J_7 = u_0^2 - u_3^2 - u_4^2,$

$$J_8 = \arctan\frac{u_1}{u_2} - \frac{u_3}{u_0 - u_4} .$$

8. There exist 21 nine-dimensional non-equivalent functional bases. Let us give some examples.

1. $\langle L_3 + eG,\ X_4,\ e > 0\rangle$,

$J_1 = x_3, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = \ln(x_0 + x_4) + e\arctan\dfrac{x_1}{x_2},$

$J_4 = u, \quad J_5 = x_1u_2 - x_2u_1, \quad J_6 = (x_0 + x_4)(u_0 + u_4), \quad J_7 = u_3,$

$J_8 = u_0^2 - u_4^2, \quad J_9 = u_1^2 + u_2^2$;

2. $\langle P_1,\ P_2\rangle$,

$$J_1 = x_3, \quad J_2 = x_0 + x_4, \quad J_3 = (x_0^2 - x_1^2 - x_2^2 - x_4^2)^{1/2}, \quad J_4 = u,$$
$$J_5 = u_1(x_0 + x_4) + x_1(u_0 - u_4), \quad J_6 = u_2(x_0 + x_4) + x_2(u_0 - u_4),$$
$$J_7 = u_3, \quad J_8 = u_0 - u_4, \quad J_9 = u_0^2 - u_1^2 - u_2^2 - u_4^2\ .$$

9. There exist 10 ten-dimensional non-equivalent functional bases. Let us give some examples.

1. $\langle P_3 + C_3 + eL_3,\ e > 2\rangle$,

$$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_3^2 + x_4^2)^{1/2}, \quad J_3 = x_0, \quad J_4 = u,$$
$$J_5 = 2\arctan\frac{x_1}{x_2} - e\arctan\frac{x_3}{x_4}, \quad J_6 = x_1u_2 - x_2u_1,$$
$$J_7 = x_3u_4 - x_4u_3, \quad J_8 = u_0, \quad J_9 = u_1^2 + u_2^2, \quad J_{10} = u_3^2 + u_4^2\ ;$$

2. $\langle L_3 - P_3\rangle$,

$$J_1 = x_0 + x_4, \quad J_2 = (x_0^2 - x_3^2 - x_4^2)^{1/2}, \quad J_3 = (x_1^2 + x_2^2)^{1/2},$$
$$J_4 = u, \quad J_5 = x_1u_2 - x_2u_1, \quad J_6 = \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4},$$
$$J_7 = \arctan\frac{x_1}{x_2} + \frac{x_3}{x_0 + x_4}, \quad J_8 = u_0 - u_4, \quad J_9 = u_0^2 - u_3^2 - u_4^2,$$
$$J_{10} = u_1^2 + u_2^2\ .$$

It should be noted that in Cases from 2 to 9, the second functional basis is invariant under the splitting subalgebra of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$.

3. On some applications of the results obtained.

It is well known (see, for example, [5–7, 10–15, 18]) that differential equations with non-trivial symmetry groups play an important role in theoretical and mathematical physics, mechanics, gas dynamics etc. Therefore, the construction and investigation of equations of this type are important from physical and mathematical points of view. In particular, the results obtained can be used in order to construct the first-order differential equations in the space $M(1,4)\times R(u)$, which are invariant under the splitting subgroups of the group $P(1,4)$. Indeed, (see, for example, [13–15]), in many cases these equations can be written in the following form:

$$F(J_1, J_2, \ldots, J_t) = 0,$$

where F is an arbitrary smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ are functional bases of the first-order differential invariants of the corresponding splitting subgroups of the group $P(1,4)$. In this way, we have constructed 243 classes of the first-order differential equations in the space $M(1,4)\times R(u)$ with non-trivial symmetry property.

Since the Lie algebra of the group $P(1,4)$ contains, as subalgebras, the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ (see also [7]), the obtained differential equations can be used in relativistic and non-relativistic physics.

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