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JACOBIAN PROBLEM FOR FACTORIAL VARIETIES

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Abstract. In this paper we give the solution to the Jacobian Problem for non-singular factorial varieties under the additional assumption that the counterimage of any hypersurface is a hypersurface.

1. Introduction. The aim of this paper is to give an answer to a question: Does an injective homomorphism $f : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, X_n]$ which maps irreducible polynomials into irreducible ones have to be an automorphism? We answer this question in the affirmative for a special class of mappings – étale endomorphisms. In the rest of this paper we use the following conventions and notations.

We work over the field $\mathbb{C}$ of complex numbers. All rings concerned in this paper are assumed to be Noetherian. For a ring R, by $U(R)$ we denote the group of units of R and by $\operatorname{ht} I$ we denote the height of an ideal I.

By a variety we always mean a variety defined over $\mathbb{C}$ and all varieties are assumed to be irreducible.

2. Étale morphisms. In this paragraph we recall briefly the notion of étale morphisms. References for all facts and definitions from this section are, for example, [**4, 3, 5**] and [**2**]. Let us start with the following

Definition 2.1. Let X, Y be algebraic varieties. A morphism $f : X \to Y$ is called étale if

1. f is flat.
2. For all $x \in X$ such that $y = f(x)$, there is $m_{y,Y}\mathcal{O}_{x,X} = m_{x,X}$, where by $\mathcal{O}_{p,V}$ we denote the local ring of a point p on variety V and by $m_{p,V}$ its maximal ideal.

Next definition is just a reformulation of the above, geometric one, in the algebraic setting.

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Definition 2.2. Let A,B be finitely generated $\mathbb{C}$-algebras and let $\varphi : A \to B$ be a morphism. We say that φ is étale if the induced morphism $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is étale, i.e.

1. φ is flat – in this setting this is equivalent to saying that B is flat as an A-module.
2. For all $P \in \operatorname{Spec} B$ such that $Q = \varphi^{-1}(P)$, there is $m_Q B_P = m_P$, where B_P denotes the localization of B at P and m_P, m_Q are maximal ideals of B_P and A_Q,respectively, and the extension is done by the standard homomorphism $\varphi_Q : A_Q \to B_P$.

In the case of non-singular varieties we can give another characterization of étale morphisms. Namely, if X, Y are non-singular varieties then $f : X \to Y$ is an étale morphism if and only if the induced mapping $T_f : T_{x,X} \to T_{f(x),Y}$ on tangent spaces at closed points is an isomorphism (cf. [**2**], p. 270). In particular one can show that étale endomorphisms $\varphi : \mathbb{C}[X_1, \dots, X_n] \to \mathbb{C}[X_1, \dots, X_n]$ are precisely those for which the mapping $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ with $F_i = \varphi(X_i)$ has nowhere vanishing jacobian, i.e. polynomial mapping such that $\operatorname{Jac}(F) \equiv const \neq 0$ (see for example [**3**]). Also, if $f : X \to Y$ is an étale morphism, then it is dominating (cf. [**5**]).

3. Main results. We begin this section with recalling the theorem known as the Going Down for flat extensions, which is very important in this paper. It is taken from [**1**].

Lemma 3.1 (Going Down for flat extensions). *Suppose that $\varphi : R \to S$ is a flat ring homomorphism. If $Q' \subset Q$ are primes of R and P is a prime of S with $\varphi^{-1}(P) = Q$, then there exists a prime P' in S such that $\varphi^{-1}(P') = Q'$ and $P' \subset P$.*

From this lemma one in a standard way obtains the next result. For the convenience of the reader, we present it here with a proof.

Proposition 3.2. *Let $\varphi : R \to S$ be a flat ring homomorphism. Then for all $P \in \operatorname{Spec} S$ there is* $\operatorname{ht} \varphi^{-1}(P) \leq \operatorname{ht} P$.

Proof. Denote by Q the counterimage $\varphi^{-1}(P)$ and let $n = \operatorname{ht} Q$. Then there exist primes $Q_0, \dots, Q_n$ of R such that $Q_0 \subset Q_1 \subset \dots \subset Q_n = Q$ and the inclusions are sharp. From Lemma 3.1 we can construct a chain of prime ideals $P_0 \subset P_1 \subset \dots \subset P_n = P$. This means that $\operatorname{ht} P \geq \operatorname{ht} Q$. □

Definition 3.3. We will say that a ring endomorphism $\varphi : R \to R$ satisfies condition (H_1) if for all prime ideals $P \in \operatorname{Spec} R$ of height 1 there is $\operatorname{ht} \varphi^{-1}(P) = 1$.

Next proposition describes three cases in which the endomorphism satisfies the above condition.

Proposition 3.4. *Let* $\varphi : A \to A$ *be a monomorphism of a finitely generated and integral* $\mathbb{C}$*-algebra. Then each of the following assumptions is sufficient for* φ *to fulfill condition* (H_1)*:*

1. *A is normal and $\varphi : A \to A$ induces an integral extensions of rings.*
2. *$\varphi : A \to A$ is étale.*
3. *$A = \mathbb{C}[X_1, \ldots, X_n]$ and there exists a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\mathrm{Jac}(F) \equiv const \neq 0$ such that $\varphi(X_i) = F_i$.*

Proof. 1. Put $B = \varphi(A)$. By the assumptions, the extension $B \subset A$ is integral with A normal. It is well known that in such setting there is $\mathrm{ht}\, P = \mathrm{ht}(P \cap B)$ for all $P \in \mathrm{Spec}\, A$.

2. We give here an elementary proof of this fact. Let $P \in \mathrm{Spec}\, A$ be a prime ideal of height 1 and let $Q = \varphi^{-1}(P)$. By flatness of φ there is $\mathrm{ht}\, Q \leq \mathrm{ht}\, P = 1$ (see Prop. 3.2). If $\mathrm{ht}\, Q = 0$ then $Q = (0)$ and therefore A_Q is a field and obviously $m_Q = (0)$. Now because $m_Q A_P = m_P$, there would be $m_P = (0)$, which would contradict $\dim A_P = 1$.
3. See remarks after the definition of étale morphisms.

□

Proposition 3.5. *Let A be a factorial, finitely generated $\mathbb{C}$-algebra. Assume that A is not a field and $U(A) = \mathbb{C}^*$. Let $\varphi : A \to A$ be a monomorphism. If φ maps irreducible elements into irreducible ones and satisfies condition (H_1), then φ is an automorphism.*

Proof. Since A is factorial and finitely generated, one can find irreducible elements, say $x_1, \ldots, x_n$, such that $A = \mathbb{C}[x_1, \ldots, x_n]$. Obviously, it suffices to prove the surjectivity of φ and the latter is equivalent to the existence of elements $g_1, \ldots, g_n \in A$ such that $\varphi(g_i) = x_i$. We prove here the case $i = 1$; the other are analogous. So, let $P = (x_1)$ (it is a prime ideal of height 1) and put $Q = \varphi^{-1}(P)$. From condition (H_1) we get $\mathrm{ht}\, Q = 1$ and, therefore, there exists an irreducible element $\tilde{g}_1 \in A$ with $Q = (\tilde{g}_1)$. Now, since $\varphi(\tilde{g}_1) \in P = (x_1)$, there exists a unit $u \in A$ such that $\varphi(\tilde{g}_1) = u x_1$ by irreducibility of $\varphi(\tilde{g}_1)$. To get the desired element g_1, just put $g_1 = u^{-1}\tilde{g}_1$. □

Corollary 3.6. *Let $\varphi : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, X_n]$ be a monomorphism sending irreducible polynomials into irreducible ones. If φ satisfies condition (H_1), then φ is an automorphism.*

Theorem 3.7. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with $\mathrm{Jac}(F) \equiv const \neq 0$. Let $\varphi : \mathbb{C}[X_1, \ldots, X_n] \to \mathbb{C}[X_1, \ldots, X_n]$ be a corresponding monomorphism. If φ maps irreducible polynomials into irreducible ones, then φ is an automorphism.*

We would like to end this paper by reformulating Proposition 3.5 in the geometric settings. We will call an affine variety a factorial variety if its coordinate ring is factorial.

THEOREM 3.8. *Let $X \subset \mathbb{C}^n$ be a non-singular, factorial variety with a coordinate ring A. Assume that $U(A) = \mathbb{C}^*$. If $f = (f_1, \ldots, f_n) : X \to X$ is an étale endomorphism such that for each hypersurface $H \subset X$ the counterimage $f^{-1}(H)$ is again a hypersurface, then f is an automorphism.*

PROOF. Denote by $\varphi : A \to A$ the corresponding monomorphism, i.e. the one for which $\varphi(x_i) = f_i$. Of course, it suffices to show that φ maps irreducible elements into irreducible ones, as φ satisfies condition (H_1). So let $h \in A$ be an irreducible element and consider the set $H = \{(x_1, \ldots, x_n) \in X |\ h(x_1, \ldots, x_n) = 0\}$. It is a hypersurface. Now, $f^{-1}(H) = \{(x_1, \ldots, x_n) \in X |\ g(x_1, \ldots, x_n) = 0\}$ with $g(x_1, \ldots, x_n) = h(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$, i.e. $g = \varphi(h)$. If $g = k_1k_2$ with $k_1 \neq k_2$, then we get the contradiction with the irreducibility of the set $f^{-1}(H)$. The other case, i.e. $g = l^k$ for some irreducible l and $k > 1$ is well known to be impossible, since f is étale. For the convenience of the reader, we include here a proof of this fact. Let $P = (l)$, $Q = (h)$. These are prime ideals of height one. Since $\varphi(h) = l^k$, we obtain $\varphi(Q) \subset P$ and thus $Q \subset \varphi^{-1}(P)$. As φ is étale, by Proposition 3.4 there is $\operatorname{ht} \varphi^{-1}(P) = 1$; thus $Q = \varphi^{-1}(P)$. Once again, since φ is étale, we get $m_Q A_P = m_P$. This implies that for $\xi \in A_Q$ there is $v_P(\varphi_Q(\xi)) = v_Q(\xi)$, where v_P, v_Q are valuations associated with the DVR's A_P and A_Q, respectively. Using the last observation we get the contradiction, since $k = v_P(l^k) = v_P(\varphi_Q(h)) = v_Q(h) = 1$. This completes the proof. □

Since $X = \mathbb{C}^n$ satisfies the assumptions of the previous theorem, the following is true.

COROLLARY 3.9. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial mapping with* $\operatorname{Jac}(F) \equiv$ *const* $\neq 0$. *If for any hypersurface $H \subset \mathbb{C}^n$ the counterimage $F^{-1}(H)$ is again a hypersurface, then F is an automorphism.*

Added in the proof. Recently K. Rusek has noticed that Theorem 3.7 is true without assuming that $\operatorname{Jac}(F) \equiv$ *const* $\neq 0$.

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