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EXISTENCE AND UNIQUENESS OF A CLASSICAL SOLUTION OF FOURIER'S FIRST PROBLEM FOR NONLINEAR PARABOLIC-ELLIPTIC SYSTEMS

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Abstract. This paper deals with the existence and uniqueness of the classical solution of Fourier's first problem for a wide class of systems of two weakly coupled quasi-linear second order partial differential functional equations. One equation is of the parabolic type (the degenerated parabolic equation) and the other of the elliptic type (the elliptic equation with a parameter). The functional dependence is of the Volterra type. The differential functional problem is considered in the one-dimensional case. A suitable theorem is formulated and proved. The proof is based on some monotone iterative method with use of Green's function and basic theorems of integral calculus. It is a new technique of solving of the specific mixed systems considered. Examples of physical applications are given.

1. Introduction. The aim of the paper is to give a theorem on the existence and uniqueness of the classical solution of Fourier's first problem for systems of two weakly coupled quasi-linear second order partial differential functional equations. One equation of each system is of the parabolic type (the degenerated parabolic equation) and the other of the elliptic type (the equation with a parameter) in $D := [0, T] \times [0, \delta] \subset \mathbf{R}^2$. The paper is motivated by the question whether classical existence and uniqueness results can

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be transferred from single-type systems to systems of mixed types. The author was inspired by the works by G. Sweers [**20**] and M.A. Abdrachmanow [**123123123**].

There are a lot of well-known mathematical models describing physical phenomena by weakly or strongly coupled parabolic-elliptic systems with different initial-boundary conditions. Weakly coupled system (1) realizes the process of incompressible fluid flow in a porous medium [**123123123**]. P. Segall [**18**] used them for computing poroelastic stress changes due to fluid extraction. System (1) can describe the process of heat exchange with flow of a substance when temperature changes are small – modifications of the very important Navier–Stokes system. Parabolic-elliptic systems similar to (1) are also used in medicine, in the theory of chemotaxis (the Keller–Segal model) [**19**]. The mentioned systems occur in certain problems of astrophysics (the evolutional version of Chandrasekhar's model), hydromechanics (a statistics of whirls in Euler's equations) and statistical mechanics (the equation of Vlasov–Poisson–Boltzmann) [**511155111551115**]. R.C. MacCamy and M. Suri [**13**] use them to describe rotary currents in electrodynamics. Moreover, they arise in a groundwater flow problem [**10**], a model of evolution of water waves (the systems of Davey–Stewartson) [**21**] and in the theory of magnetism (the Myrzakulov equations) [**14**]. Another example is the Poisson–Schrödinger non-stationary system in the theory of semiconductors.

Such systems also have many various applications. But unfortunately they have been less examined than systems of the parabolic, elliptic or hyperbolic types. It is mainly caused by their specific mixed structure.

The statements of principal Theorem 1 on the existence and uniqueness of the classical solution are argued with use of the monotone iterative method for parabolic differential functional equations proposed by S. Brzychczy [**6969**] (the method of upper and lower solutions), Green's function and basic theorems of integral calculus. The fundamental assumptions about the Hölder continuity, the Lipschitz condition, monotonicity, the Volterra type functional dependence and the existence of uppper and lower solutions are typical of similar single-type problems (cf. [**6969**]). There is also new assumption Z_9 which deals especially with a sign constancy of Green's function. It is true, among others, for the important differential operator $\mathbf{L} := \frac{d^2}{dx^2} + c$ in G and the boundary operator $\mathbf{B} := 1$ on ∂G, where $c = \mathrm{const} \leq 0$.

The author does not know theorems on the existence of the classical solutions for such a general class of differential functional systems as in (1), even in a case of linear equations.

A certain linear differential system of type (1) with conditions (2) was considered by G. Sweers [**20**], who in 1994 proved a theorem on the existence of the weak and classical solutions. G. Sweers used the semi-group theory in his proof.

Numerous interesting results from a domain of study of linear parabolic-elliptic differential systems concerning the existence of the weak solutions and their estimate were obtained in 1990s by M.A. Abdrachmanow (cf. [**123123123**]).

Theorems on the existence and uniqueness of the classical solutions of quasi-linear parabolic-elliptic systems of a special form without a functional term were obtained by M.S. Mock [**15**] in 1974, A. Krzywicki and T. Nadzieja [**11**] in 1992 (for the one-dimensional case) and P. Biler [**5**] in 1992. However, a technique of proving these theorems is different from the one proposed in this paper. These authors used some difficult differential inequalities or Hopf's generalized transformation.

A finite difference method of approximate solving differential problem (1) with conditions (2) is given in [**17**].

2. Notation and definitions. Denote by $\mathbf{R}$ the Euclidean space and define the following sets

$$G := (0, \delta) \subset \mathbf{R}, \quad D := [0, T] \times G,$$

$$\sigma := \{(t, x) : \ t \in [0, T], \ x = 0 \text{ or } x = \delta\},$$

$$\Sigma := \sigma \cup \{(t, x) : \ t = 0, \ x \in G\},$$

where $0 < \delta < +\infty$, $0 < T < +\infty$.

Let, moreover,

$$\overline{G} := [0, \delta], \quad \overline{D} := [0, T] \times \overline{G}.$$

For $U \subset \mathbf{R}^n$ compact, denote by $C(U) := C(U, \mathbf{R})$ *the Banach space* of continuous functions $z : U \to \mathbf{R}$ with *the maximum norm*

$$\|z\| := \max_{x \in U} |z(x)| .$$

For a fixed $t \in [0, T]$,

$$z(t, \cdot) : \overline{G} \ni x \to z(t, x) \in \mathbf{R}$$

stands for the restriction of a function $z \in C(\overline{D})$ to the time intersection $\{(t, x) : \ x \in \overline{G}\}$. Observe that $z(t, \cdot) \in C(\overline{G})$ for any $t \in [0, T]$.

We consider a weakly coupled system of two nonlinear second order partial differential functional equations, from which the first is of the parabolic type and the second of the elliptic type in the set D, of the following form

$$(1) \qquad \begin{cases} u_t(t, x) = a(t, x) u_{xx}(t, x) + f(t, x, u(t, x), v(t, x), u(t, \cdot)), \\ v_{xx}(t, x) + b(x) v_x(t, x) + c(x) v(t, x) = g(t, x, u(t, x)), \\ \qquad \text{for } (t, x) \in D \end{cases}$$

with the initial condition and the boundary conditions of the Dirichlet type

$$\left\{ \begin{array}{lll} u(t,x) = h_1(t,x) & \text{for} & (t,x) \in \Sigma, \\ v(t,x) = h_2(t,x) & \text{for} & (t,x) \in \sigma, \end{array} \right. \tag{2}$$

where

$$f : \overline{D} \times \mathbf{R}^2 \times C\left(\overline{G}\right) \ni (t,x,p,s,z) \to f(t,x,p,s,z) \in \mathbf{R},$$

$$g : \overline{D} \times \mathbf{R} \ni (t,x,p) \to g(t,x,p) \in \mathbf{R},$$

$$h_1 : \Sigma \ni (t,x) \to h_1(t,x) \in \mathbf{R},$$

$$h_2 : \sigma \ni (t,x) \to h_2(t,x) \in \mathbf{R}$$

and the coefficients $a(t,x), b(x), c(x)$ are given.

The Hölder space $H^{l,\frac{l}{2}}\left(\overline{D}\right) := H^{k+\alpha,\frac{k+\alpha}{2}}\left(\overline{D},\mathbf{R}\right)$ $(k = 0,1,2,\ 0 < \alpha < 1,\ l = k+\alpha)$ in Ladyženskaja's sense is the Banach space of continuous functions $z : \overline{D} \to \mathbf{R}$ whose all derivatives $D_t^r D_x^s z(t,x)$ $(0 \le 2r+s \le k)$ exist and are Hölder continuous with exponent α in $\overline{D}$, with a finite norm

$$|z|^{(k+\alpha)} := \langle z\rangle^{(k+\alpha)} + \sum_{j=0}^{k} \langle z\rangle^{(j)}.$$

The components $\langle z\rangle^{(k+\alpha)}$ and $\langle z\rangle^{(j)}$ are defined as

$$|z|^{(0)} := \sup_{(t,x)\in\overline{D}} |z(t,x)|, \quad \langle z\rangle^{(j)} := \sum_{2r+s=j} \left|D_t^r D_x^s z\right|^{(0)},$$

$$\langle z\rangle^{(k+\alpha)} := \langle z\rangle_x^{(k+\alpha)} + \langle z\rangle_t^{((k+\alpha)/2)},$$

$$\begin{aligned} \langle z\rangle_x^{(k+\alpha)} &:= \sum_{2r+s=k} \langle D_t^r D_x^s z\rangle_x^{(\alpha)}, \langle z\rangle_t^{((k+\alpha)/2)} \\ &:= \sum_{0<k+\alpha-2r-s<2} \langle D_t^r D_x^s z\rangle_t^{((k+\alpha-2r-s)/2)}, \end{aligned}$$

$$\langle z\rangle_x^{(\alpha)} := \sup_{(t,x),\ (t,x')\in\overline{D}} \frac{\left|u(t,x) - u\left(t,x'\right)\right|}{|x-x'|^{\alpha}},$$

$$\langle z\rangle_t^{(\alpha)} := \sup_{(t,x),\ (t',x)\in\overline{D}} \frac{\left|u(t,x) - u\left(t',x\right)\right|}{|t-t'|^{\alpha}}$$

(cf. [**12**], pp. 7–8).

We now define the operators $\mathbf{P}(t) : C\left(\overline{G}\right) \to C\left(\overline{G}\right)$ as follows

$$\begin{aligned} \text{(3)} \qquad \mathbf{P}(t)[z](x) : &= \int_0^\delta G(x,y)\,p(y)\,g(t,y,z(y))\,dy \\ &+ \left(1-\delta^{-1}x\right)h_2(t,0) + \delta^{-1}xh_2(t,\delta), \end{aligned}$$

where $t \in [0,T]$ is any fixed parameter, $x \in \overline{G}$, $z \in C\left(\overline{G}\right)$, $p(x) := \exp\left(\int_0^x b(s)\,ds\right)$ and $G(x,y)$ is Green's function for the differential operator $\mathbf{L} := \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + c(x)\,p(x)$ in G and the boundary operator $\mathbf{B} := 1$ on ∂G.

We also define the function $F : \overline{D} \times \mathbf{R} \times C\left(\overline{G}\right) \ni (t,x,p,z) \to F(t,x,p,z) \in \mathbf{R}$ by the formula

$$\text{(4)} \qquad F(t,x,p,z) := f(t,x,p,\mathbf{P}(t)[z](x),z).$$

Note that Green's function $G(x,y)$ exists, and that the operators $\mathbf{P}(t)$ and function F are, by Lemmas 1 and 2, well defined if suitable assumptions are satisfied (see Assumption Z).

A function $z : \overline{D} \to \mathbf{R}$ will be called *regular* in $\overline{D}$ if $z, z_t, z_x, z_{xx} \in C\left(\overline{D}\right)$ (cf. [**9**], p. 76). We briefly write $z \in \overline{C}_{reg}\left(\overline{D}\right)$.

A pair of mappings $u, v \in \overline{C}_{reg}\left(\overline{D}\right)$ is *a classical solution* of differential functional problem (1), (2) if u, v satisfy system of equations (1) and initial-boundary conditions (2).

Functions $\widehat{u}, \widetilde{u} \in \overline{C}_{reg}\left(\overline{D}\right)$ satisfying the systems of inequalities

$$\begin{cases} \widehat{u}_t(t,x) \le a(t,x)\,\widehat{u}_{xx}(t,x) + F(t,x,\widehat{u}(t,x),\widehat{u}(t,\cdot)) & \text{for} \quad (t,x) \in D, \\ \widehat{u}(t,x) \le h_1(t,x) & \text{for} \quad (t,x) \in \Sigma, \end{cases}$$

$$\begin{cases} \widetilde{u}_t(t,x) \ge a(t,x)\,\widetilde{u}_{xx}(t,x) + F(t,x,\widetilde{u}(t,x),\widetilde{u}(t,\cdot)) & \text{for} \quad (t,x) \in D, \\ \widetilde{u}(t,x) \ge h_1(t,x) & \text{for} \quad (t,x) \in \Sigma \end{cases}$$

are called, respectively, a *lower* and an *upper solution* of the differential functional problem

(5)

$$\begin{cases} u_t(t,x) = a(t,x)\,u_{xx}(t,x) + F(t,x,u(t,x),u(t,\cdot)) & \text{for} \quad (t,x) \in D, \\ u(t,x) = h_1(t,x) & \text{for} \quad (t,x) \in \Sigma \end{cases}$$

in $\overline{D}$, where the function F is defined by (4).

3. Assumptions. We need the following assumptions on f, g, h_1, h_2 and a, b, c.

Assumption Z

Z_1. $a(\cdot,\cdot) \in H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right)$, where $\alpha = \text{const} \in (0,1)$.

Z_2. $a(t,x) > 0$ for $(t,x) \in D$.

Z_3. $b(\cdot), c(\cdot) \in C(\overline{G})$.

Z_4. $f(\cdot,\cdot,p,s,z) \in H^{\alpha,\frac{\alpha}{2}}(\overline{D})$.

Z_5. The function $f = f(t,x,p,s,z)$ satisfies the Lipschitz condition with respect to $p, s \in \mathbf{R}$ and $z \in C(\overline{G})$:

$$|f(t,x,p,s,z) - f(t,x,\overline{p},\overline{s},\overline{z})| \le L_1 |p-\overline{p}| + L_2 |s-\overline{s}| + L_3 \|z-\overline{z}\|$$

for $(t,x) \in \overline{D}$.

Z_6. The function f is non-decreasing with respect to p and z.

Z_7. $g(\cdot,\cdot,\cdot) \in C(\overline{D} \times \mathbf{R})$.

Z_8. The function $g = g(t,x,p)$ satisfies the Lipschitz condition with respect to $p \in \mathbf{R}$:

$$|g(t,x,p) - g(t,x,\overline{p})| \le L |p-\overline{p}| \quad \text{for} \quad (t,x) \in \overline{D}.$$

Z_9. One of the following cases holds:

a) $G(x,y) \ge 0$ for $(x,y) \in \overline{G} \times \overline{G}$ and the both functions f (with respect to s) and g (with respect to p) are non-decreasing or they are both non-increasing,

b) $G(x,y) \le 0$ for $(x,y) \in \overline{G} \times \overline{G}$ and the function f is non-decreasing with respect to s while the function g is non-increasing with respect to p, or vice versa (i.e. f is non-increasing while g is non-decreasing).

Z_{10}. $g_t(\cdot,\cdot,\cdot), g_p(\cdot,\cdot,\cdot) \in C(\overline{D} \times \mathbf{R})$.

Z_{11}. There exists a function $\overline{h}_1(\cdot,\cdot) \in H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D})$ such that $h_1(t,x) = \overline{h}_1(t,x)$ for $(t,x) \in \Sigma$.

Z_{12}. $h_2(\cdot,0), h_2(\cdot,\delta) \in C^1([0,T])$.

We will also adopt the fundamental following assumption.

Assumption A. There exists at least one pair $\widehat{u}_0$, $\widetilde{u}_0$ of a lower and an upper solution, respectively, of differential functional problem (5) in $\overline{D}$ and $\widehat{u}_0, \widetilde{u}_0 \in H^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D})$.

4. Main results. Before formulating and proving the main result of the paper, we give three lemmas and construct some two successive approximation sequences.

Let us consider the boundary value problem

$$y'' + b(x)\,y' + c(x)\,y = d(x), \quad x \in G, \tag{6}$$

$$y(0) = A, \quad y(\delta) = B. \tag{7}$$

Lemma 1. *Suppose that Z_3 holds and*

(1) $d(\cdot) \in C(\overline{G})$,

(2) *the function* $y \equiv 0$ *is the unique solution of problem* (6), (7) *for* $d \equiv 0,\ A = B = 0$.

Then

(*i*)

$$G(x,s) = \begin{cases} \frac{1}{\Delta_0}\, y_1(x)\, y_2(s) & \text{for} \quad 0 \le x \le s, \\ \frac{1}{\Delta_0}\, y_1(s)\, y_2(x) & \text{for} \quad s \le x \le \delta, \end{cases}$$

where $\Delta_0 := y_1(0)\, y_2'(0) - y_1'(0)\, y_2(0)$ *and* $y_1, y_2 \in C^2\left(\overline{G}\right)$ *are any non-zero solutions of homogeneous equation* (6), *satisfying* $y_1(0) = 0$ *and* $y_2(\delta) = 0$,

(*ii*) *the function*

$$y(x) := \int_0^{\delta} G(x,s)\, p(s)\, d(s)\, ds + \left(1 - \delta^{-1}x\right) A + \delta^{-1} B x \quad for \quad x \in \overline{G}$$

is the unique solution of boundary value problem (6), (7).

Proof. It is clear that equation (6) is equivalent to a self-adjoint equation of the form

$$\left(p(x)\, y'\right)' + c(x)\, p(x)\, y = d(x)\, p(x), \quad x \in G.$$

Moreover, $p(0) = 1$. Then the statements of Lemma 1 follow from [**16**], pp. 174–176. $\square$

Further, the following lemma holds.

Lemma 2. *If Assumption Z holds and homogeneous boundary value problem* (6), (7) *has the trivial (null) solution only, then*

(*i*) *the operators* $\mathbf{P}(t)$ *in* (3) *are well defined for an arbitrary value of* $t \in [0,T]$,

(*ii*) $F(\cdot,\cdot,p,z) \in H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right)$, *where* F *is defined by* (4),

(*iii*) *the function* $F = F(t,x,p,z)$ *satisfies the Lipschitz condition with respect to* p *and* z,

(*iv*) *the function* F *is non-decreasing with respect to* p *and* z.

Proof. We first prove (*i*). Fix a parameter $t \in [0,T]$ and a function $z \in C\left(\overline{G}\right)$. It follows from Lemma 1 that Green's function $G(x,y)$ in definition (3) of $\mathbf{P}(t)$ exists and is continuous. By continuity of p and g (see assumption Z_7) and the theorem on the continuity of an integral with respect to a parameter, we conclude that $\mathbf{P}(t)[z] \in C\left(\overline{G}\right)$.

We now demonstrate (ii). Let $t, \overline{t} \in [0,T]$ and $x, \overline{x} \in \overline{G}$ be fixed. Assumptions Z_4 and Z_5 imply the estimate

$$\begin{aligned}
& \left|F(t,x,p,z) - F\left(\overline{t},\overline{x},p,z\right)\right| \\
= \; & \left|f(t,x,p,\mathbf{P}(t)[z](x),z) - f\left(\overline{t},\overline{x},p,\mathbf{P}\left(\overline{t}\right)[z](\overline{x}),z\right)\right| \\
\leq \; & \left|f(t,x,p,\mathbf{P}(t)[z](x),z) - f\left(t,x,p,\mathbf{P}\left(\overline{t}\right)[z](\overline{x}),z\right)\right| \\
& + \left|f\left(t,x,p,\mathbf{P}\left(\overline{t}\right)[z](\overline{x}),z\right) - f\left(\overline{t},\overline{x},p,\mathbf{P}\left(\overline{t}\right)[z](\overline{x}),z\right)\right| \\
\leq \; & L_2\left|\mathbf{P}(t)[z](x) - \mathbf{P}\left(\overline{t}\right)[z](\overline{x})\right| + H\left(\left|t-\overline{t}\right|^{\frac{\alpha}{2}} + |x-\overline{x}|^{\alpha}\right)
\end{aligned}$$

for all $p \in \mathbf{R}$ and $z \in C\left(\overline{G}\right)$, where H is the Hölder coefficient for the function $f(\cdot,\cdot,p,s,z)$.

Observe that $\mathbf{P}(\cdot)[z](\cdot)$ can be treated as a function in t,x. We now prove that $\mathbf{P}(\cdot)[z](\cdot) \in C^1\left(\overline{D}\right)$.

We first show that $\mathbf{P}(\cdot)[z](\cdot) \in C\left(\overline{D}\right)$. Indeed, this function is uniformly continuous with respect to $x \in \overline{G}$ for fixed $t \in [0,T]$. Using continuity of Green's function and assumptions Z_7, Z_{12} we get, from the theorem on the continuity of an integral with respect to a parameter, that it is a uniformly continuous function with respect to $t \in [0,T]$ for fixed $x \in \overline{G}$. Thus $\mathbf{P}(\cdot)[z](\cdot)$ is uniformly continuous in $\overline{D}$.

Next, we prove that the derivative $(\mathbf{P}(\cdot)[z](\cdot))_t \in C\left(\overline{D}\right)$. From the regularity of Green's function, assumptions Z_7, Z_{10}, Z_{12} and the theorem on the differentiation of an integral with respect to a parameter, we can write

$$\begin{aligned}
(\mathbf{P}(t)[z](x))_t \; = \; & \int_0^{\delta} G(x,y)\,p(y)\,g_t(t,y,z(y))\,dy + \\
& + \left(1-\delta^{-1}x\right)(h_2)_t(t,0) + \delta^{-1}x\,(h_2)_t(t,\delta) \quad \text{for} \quad (t,x) \in \overline{D}.
\end{aligned}$$

Continuity of $(\mathbf{P}(\cdot)[z](\cdot))_t$ follows from the same reasoning as in the proof of the continuity of $\mathbf{P}(\cdot)[z](\cdot)$.

In the same way as above, we prove that $(\mathbf{P}(\cdot)[z](\cdot))_x \in C\left(\overline{C}\right)$. After elementary calculations we obtain

$$\begin{aligned}
(\mathbf{P}(t)[z](x))_x \; = \; & \frac{1}{\Delta_0}\,G_2'(x)\int_0^{x} G_1(y)\,p(y)\,g(t,y,z(y))\,dy \\
& + \frac{1}{\Delta_0}\,G_1'(x)\int_x^{\delta} G_2(y)\,p(y)\,g(t,y,z(y))\,dy \\
& + \delta^{-1}\left[h_2(t,\delta) - h_2(t,0)\right] \quad \text{for} \quad (t,x) \in \overline{D}.
\end{aligned}$$

Continuity of this derivative in $\overline{D}$ is a consequence of the regularity of the functions G_1, G_2, p, g, h_2 and the theorems on the continuity of an integral with respect to a parameter and the regularity of an integral as a function of a limit of integration.

Since $C^1\left(\overline{D}\right) \subset H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right)$, then $\mathbf{P}(\cdot)[z](\cdot) \in H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right)$. Thus, $F(\cdot,\cdot,p,z) \in H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right)$.

To prove (iii) we fix $p,\overline{p} \in \mathbf{R}$ and $z,\overline{z} \in C\left(\overline{G}\right)$. Addition of assumptions Z_5 and Z_8, the properties of an integral and the definition of the norm in $C\left(\overline{G}\right)$ yield the inequalities

$$
\begin{aligned}
& |F(t,x,p,z) - F(t,x,\overline{p},\overline{z})| \\
= \ & |f(t,x,p,\mathbf{P}(t)[z](x),z) - f(t,x,\overline{p},\mathbf{P}(t)[\overline{z}](x),\overline{z})| \\
\leq \ & L_1|p-\overline{p}| + L_2|\mathbf{P}(t)[z](x) - \mathbf{P}(t)[\overline{z}](x)| + L_3\|z-\overline{z}\| \\
\leq \ & L_1|p-\overline{p}| + L_2\int_0^\delta |G(x,y)||p(y)||g(t,y,z(y)) - g(t,y,\overline{z}(y))|dy \\
& \qquad + L_3\|z-\overline{z}\| \\
\leq \ & L_1|p-\overline{p}| + LL_2\int_0^\delta |G(x,y)||p(y)||z(y)-\overline{z}(y)|dy + L_3\|z-\overline{z}\| \\
\leq \ & L_1|p-\overline{p}| + \left(LL_2\int_0^\delta |G(x,y)||p(y)|dy + L_3\right)\|z-\overline{z}\|
\end{aligned}
$$

for all $(t,x) \in \overline{D}$. Hence (iii) is proved.

Statement (iv) follows immediately from the monotonicity of an integral and assumptions Z_6, Z_9.

This completes the proof of Lemma 2. □

We now define by recurrence two successive approximation sequences of functions $\{\widehat{u}_n\}_{n\in\mathbf{N}}$, $\{\widetilde{u}_n\}_{n\in\mathbf{N}}$, such that $\widehat{u}_n, \widetilde{u}_n$ $(n \in \mathbf{N})$ are the unique solutions in $H^{2+\alpha,1+\frac{\alpha}{2}}\left(\overline{D}\right)$ of the following linear differential functional problems

$$
(8) \qquad \begin{cases}
(\widehat{u}_n)_t(t,x) = a(t,x)(\widehat{u}_n)_{xx}(t,x) & \\
\qquad\qquad +F(t,x,\widehat{u}_{n-1}(t,x),\widehat{u}_{n-1}(t,\cdot)) & \text{for } (t,x) \in D, \\
\widehat{u}_n(t,x) = h_1(t,x) & \text{for } (t,x) \in \Sigma
\end{cases}
$$

and

$$
(9) \qquad \begin{cases}
(\widetilde{u}_n)_t(t,x) = a(t,x)(\widetilde{u}_n)_{xx}(t,x) & \\
\qquad\qquad +F(t,x,\widetilde{u}_{n-1}(t,x),\widetilde{u}_{n-1}(t,\cdot)) & \text{for } (t,x) \in D, \\
\widetilde{u}_n(t,x) = h_1(t,x) & \text{for } (t,x) \in \Sigma,
\end{cases}
$$

($n \in \mathbf{N}$), where $\widehat{u}_0, \widetilde{u}_0$ are a lower and an upper solution of differential functional problem (5).

Note that if Assumption A and the assumptions of Lemma 2 hold, then, by next Lemma 3, the above definition is correct.

Remark 1. If the assumptions of Lemma 2 hold, then the nonlinear generalized Nemytskij operator $\mathbf{F} : H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right) \to H^{\alpha,\frac{\alpha}{2}}\left(\overline{D}\right)$ given by

$$\mathbf{F}\left[z\right]\left(t,x\right) := F\left(t,x,z\left(t,x\right),z\left(t,\cdot\right)\right)$$

(see [**69874698746987469874698746987469874**]) is well defined. Note that operators of this type play an important role in the theory of nonlinear equations.

Proof. Remark 1 is a consequence of Lemma 2. For more details we refer the reader to [**6**] and [**9**]. $\square$

Lemma 3. *If Assumption A, the assumptions of Lemma* 2 *and the compatibility conditions*

$$\left(h_1\right)_t\left(0,0\right) = a\left(0,0\right)\left(h_1\right)_{xx}\left(0,0\right) + f\left(0,0,h_1\left(0,0\right),\mathbf{P}\left(0\right)\left[h_1\left(0,\cdot\right)\right]\left(0\right),h_1\left(0,\cdot\right)\right),$$

$$\left(h_1\right)_t\left(0,\delta\right) = a\left(0,\delta\right)\left(h_1\right)_{xx}\left(0,\delta\right) + f\left(0,\delta,h_1\left(0,\delta\right),\mathbf{P}\left(0\right)\left[h_1\left(0,\cdot\right)\right]\left(\delta\right),h_1\left(0,\cdot\right)\right)$$

are true, then the successive approximation sequences $\{\widehat{u}_n\}_{n\in\mathbf{N}}$, $\{\widetilde{u}_n\}_{n\in\mathbf{N}}$ *given by* (8), (9) *are uniquely defined in* $H^{2+\alpha,1+\frac{\alpha}{2}}\left(\overline{D}\right)$ *and are convergent.*

Proof. Let $\widehat{u}_0$ and $\widetilde{u}_0$ be a lower and an upper solution of problem (5), respectively. Applying the regularity of the functions $\widehat{u}_0$ and $\widetilde{u}_0$, Remark 1, assumptions Z_1, Z_2, Z_{11}, the compatibility conditions and Lemma 3.2 in [**9**], we get that $\widehat{u}_1, \widetilde{u}_1 \in H^{2+\alpha,1+\frac{\alpha}{2}}\left(\overline{D}\right)$ and are unique. Using the same method, we conclude that $\widehat{u}_n, \widetilde{u}_n \in H^{2+\alpha,1+\frac{\alpha}{2}}\left(\overline{D}\right)$ are unique for every $n \in \mathbf{N}$.

The convergence of the sequences follows directly from Theorems 3.1 in [**6969**]. $\square$

As a consequence of the above lemmas we obtain the following conclusion.

Theorem 1. *If the assumptions of Lemma* 3 *are satisfied, then the pair of functions*

$$u\left(t,x\right) := \lim_{n\to\infty} \widehat{u}_n\left(t,x\right), \tag{10}$$

$$v\left(t,x\right) := \mathbf{P}\left(t\right)\left[u\left(t,\cdot\right)\right]\left(x\right) \tag{11}$$

is the unique classical solution of differential functional problem (1), (2) *in* $\overline{D}$*; moreover,* $u \in H^{2+\alpha,1+\frac{\alpha}{2}}\left(\overline{D}\right)$.

Proof. First observe that differential functional system (1) with conditions (2) is equivalent to differential functional initial-boundary value problem (5) in the class $\overline{C}_{reg}\left(\overline{D}\right)$ if we put $v\left(t,x\right) := \mathbf{P}\left(t\right)\left[u\left(t,\cdot\right)\right]\left(x\right)$ for $\left(t,x\right) \in \overline{D}$ (see (11)). This follows directly from the theory of ordinary differential equations.

Theorem 3.1 in [**9**] (compare also Theorem 3.1 in [**6**]), by virtue of the assumptions adopted and the lemmas given, implies that (5) has a solution $u \in H^{2+\alpha,1+\frac{\alpha}{2}}\left(\overline{D}\right) \subset \overline{C}_{reg}\left(\overline{D}\right)$ defined by (10) and it is the unique solution in $\overline{C}_{reg}\left(\overline{D}\right)$.

An analysis similar to that in the proof of statement (ii) in Lemma 2 gives

$$\begin{aligned} v_t\left(t,x\right) &= \int_0^\delta G\left(x,y\right)p\left(y\right)\left[g_t\left(t,y,u\left(t,y\right)\right) + g_p\left(t,y,u\left(t,y\right)\right)u_t\left(t,y\right)\right]dy \\ &+ \left(1-\delta^{-1}x\right)\left(h_2\right)_t\left(t,0\right) + \delta^{-1}x\left(h_2\right)_t\left(t,\delta\right), \end{aligned}$$

$$\begin{aligned} v_x\left(t,x\right) &= \frac{1}{\Delta_0}\,G_2'\left(x\right)\int_0^x G_1\left(y\right)p\left(y\right)g\left(t,y,u\left(t,y\right)\right)dy \\ &+ \frac{1}{\Delta_0}\,G_1'\left(x\right)\int_x^\delta G_2\left(y\right)p\left(y\right)g\left(t,y,u\left(t,y\right)\right)dy \\ &+ \delta^{-1}\left[h_2\left(t,\delta\right) - h_2\left(t,0\right)\right], \end{aligned}$$

$$v_{xx}\left(t,x\right) = -b\left(x\right)v_x\left(t,x\right) - c\left(x\right)v\left(t,x\right) + g\left(t,x,u\left(t,x\right)\right)$$

for $\left(t,x\right) \in \overline{D}$, and, moreover, $v \in \overline{C}_{reg}\left(\overline{D}\right)$.

Therefore, the pair of the functions u, v belongs to the class $\overline{C}_{reg}\left(\overline{D}\right)$ and solves problem (1), (2) in $\overline{D}$.

Suppose that a pair of functions $\overline{u}, \overline{v} \in \overline{C}_{reg}\left(\overline{D}\right)$ is also a solution of problem (1), (2) in $\overline{D}$. Note that $\overline{u} = u$ (see (10)), because, as shown at the beginning of the proof of this theorem, problem (5) has the unique solution in $\overline{C}_{reg}\left(\overline{D}\right)$. It follows from the theory of ordinary differential equations that

$$\forall t \in [0,T]: \quad \overline{v}\left(t,x\right) = v\left(t,x\right) \quad \text{for} \quad x \in \overline{G},$$

where v is given by (11).

The proof of Theorem 1 is finished. $\square$

Remark 2. In the proof of Theorem 1 and Lemma 3, we used results by S. Brzychczy [**6969**], concerning parabolic initial-boundary value problems.

Remark 3. Theorem 1 and the method given in this paper can be extended to systems of equations of the form

$$(12)\qquad \begin{cases} u_t^i\,(t,x) = a_i\,(t,x)\,u_{xx}^i\,(t,x) + f_i\,(t,x,u\,(t,x)\,,v\,(t,x)\,,u\,(t,\cdot))\,, \\ v_{xx}^i\,(t,x) + b_i\,(x)\,v_x^i\,(t,x) + c_i\,(x)\,v^i\,(t,x) = g_i\,(t,x,u\,(t,x))\,, \\ \quad \text{for} \quad (t,x) \in D, \end{cases}$$

where $f_i : \overline{D} \times \mathbf{R}^n \times \mathbf{R}^m \times C\left(\overline{G}, \mathbf{R}^n\right) \to \mathbf{R}$ $(i = 1, ..., n)$ and $g_i : \overline{D} \times \mathbf{R}^n \to \mathbf{R}$ $(i = 1, ..., m)$ are given functions, a_i $(i = 1, ..., n)$, b_i, c_i $(i = 1, ..., m)$ are given coefficients, $u = \left(u^1, ..., u^n\right)$, $v = \left(v^1, ..., v^m\right)$. Assumptions are analogous to Assumptions Z and A.

Proof. An idea of a proof is the same as in Theorem 1. After integration of the elliptic part of system (12) and replacing v in the parabolic part by suitable integrals, we obtain a parabolic differential functional system with an unknown function u. By [**6969**], this system has the unique classical solution u given as the limit of a monotone sequence. The regularity of v is proven as in Theorem 1. □

Example 1. To illustrate the class of problems which can be treated with our method, we give the following example

$$(13)\qquad \begin{cases} u_t\,(t,x) = u_{xx}\,(t,x) + \varepsilon v\,(t,x)\,, \\ v_{xx}\,(t,x) = -u\,(t,x)\,, \end{cases}$$

where $(t,x) \in D := [0,T] \times (0,\delta)$, $\varepsilon = \text{const} \geq 0$. Such a system has been considered in [**20**]. It is easily seen that all the assumptions of Theorem 1 are satisfied. If we add, for instance, the integral term $\int_0^\delta u\,(t,y)\,dy$ in the first equation, these assumptions are true as well.

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