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EXISTENCE OF A SOLUTION FOR THE INITIAL BOUNDARY VALUE PROBLEM FOR A HIGH ORDER PARABOLIC EQUATION IN AN UNBOUNDED DOMAIN

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Abstract. In this paper we consider the initial boundary value problem for the equation $u_{tt}+A_1u+A_2u_t+g(u_t) = f(x,t)$ in an unbounded domain, where A_1 is a linear elliptic operator of the fourth order and A_2 is a linear elliptic operator of the second order. We obtain the conditions of the existence of a weak solution for this problem.

Many authors have considered initial boundary value problems in unbounded domains for parabolic equations of a high order with the first derivative with respect to time. There are much fewer papers concerning the problems for parabolic equations with the second derivative with respect to time. In particular, in [**123123123**] and [**5656**], the authors have considered the Cauchy problem and initial boundary value problem for a parabolic equation of a high order and the properties of its solutions. In [**7**], the authors obtained some conditions for the uniqueness of the solution of the initial boundary value problem for the general linear parabolic systems in unbounded domains. By introducing a parameter, they have shown the uniqueness of a weak solution in the class of functions which do not grow faster than the function $e^{a|x|^\alpha}$ for $|x| \to \infty$. In [**11**], the author proved with this method that the solution of problem (1)–(3) is unique. The main goal of this paper is to obtain some conditions for the existence of a weak solution for a parabolic equation of the fourth order with the second derivative with respect to time and with Dirichlet boundary conditions on some weighted Sobolev space.

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain and $\partial\Omega \in C^1$, $\Omega \cap B_R = \Omega^R$ be a regular domain for all $R > 0$, where $B_R = \{x \in R^n, |x| < R\}$ and $Q_T = \Omega \times (0,T)$, $Q_T^R = \Omega^R \times (0,T)$, $\Omega_\tau = Q_\tau \cap \{t = \tau\}$, $Q_{\tau_0,\tau_1} = \Omega \times (\tau_0, \tau_1)$.

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We shall consider the equation of the form

$$
\begin{aligned}
A(u) \equiv u_{tt}(x,t) + \sum_{i,j,k,l=1}^{n} (a_{ij}^{kl}(x,t)u_{x_i x_j}(x,t))_{x_k x_l} - \sum_{i,j=1}^{n} (a_{ij}(x,t)u_{x_i}(x,t))_{x_j} \\
- \sum_{i,j=1}^{n} (b_{ij}(x,t)u_{tx_i}(x,t))_{x_j} + a(x,t)u(x) + g(x,u_t) = f(x,t)
\end{aligned} \tag{1}
$$

in the domain Q_T.
For this equation, we put the following boundary and initial conditions

$$
u|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}|_{S_T} = 0, \tag{2}
$$

$$
u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \tag{3}
$$

where $S_T = \partial\Omega \times (0,T)$ and ν is the normal vector for S_T.

We define the function $\Psi \in C^2(\mathbb{R})$, $\Psi : [0,+\infty) \to (0,+\infty)$ such that $\Psi > 0$ in $[0,+\infty)$, $\Psi'(\xi) \leq 0$ or $\Psi'(\xi) \geq 0$ in $[0,+\infty)$ and

$$
\left| \frac{\Psi_{x_i}(|x|)}{\Psi(|x|)} \right| \leq \Psi_0, \quad \left| \frac{\Psi_{x_i x_j}(|x|)}{\Psi(|x|)} \right| \leq \Psi_0, \quad i,j = 1,\dots,n, \ \Psi_0 < \infty. \tag{4}
$$

Let us introduce the weighted spaces

$$
L_\Psi^p(\Omega) = \left\{ u \ : \ \int_\Omega |u(x)|^p \Psi(|x|)dx < \infty \right\}, \ p \in (1,\infty),
$$

$$
H_\Psi^{0,k}(\Omega) = \left\{ u \ : \ \int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u(x)|^2 \Psi(|x|)dx < \infty, \ u|_{\partial\Omega} = 0, \ \frac{\partial^{k-1} u}{\partial \nu^{k-1}}|_{\partial\Omega} = 0 \right\},
$$

where $k = 1,2$; $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and ν is the normal vector for $\partial\Omega$.

For equation (1), we adopt the following system of assumptions:

(A) $a_{ij}^{kl}, (a_{ij}^{kl})_{x_k x_l}, (a_{ij}^{kl})_{tt} \in L^\infty(Q_T)$; $a_{ij}^{kl}(x,t) = a_{kl}^{ij}(x,t)$, $i,j,k,l = 1,\dots,n$ for almost all $(x,t) \in Q_T$;
$\sum\limits_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t)\xi_{ij}\xi_{kl} \geq a_2 \sum\limits_{i,j=1}^{n} \xi_{ij}^2$, $\quad i,j,k,l = 1,\dots,n$,
$\sum\limits_{i,j,k,l=1}^{n} (a_{ij}^{kl}(x,t))_t \xi_{ij}\xi_{kl} \leq a_2^1 \sum\limits_{i,j=1}^{n} \xi_{ij}^2$ $\quad$ for almost all $(x,t) \in Q_T$ and for all $\xi \in R^{n(n-1)/2}$, where $a_2 > 0$ is a constant;
$a_{ij}, (a_{ij})_{x_j}, (a_{ij})_t \in L^\infty(Q_T)$, $\quad i,j = 1,\dots,n$; $\quad a, a_t \in L^\infty(Q_T)$,

(B) $b_{ij}, (b_{ij})_{x_j}, (b_{ij})_t \in L^\infty(Q_T), \quad i,j = 1,\dots,n;$
$\sum\limits_{ij=1}^{n} b_{ij}(x,t)\xi_i\xi_j \geq b_0 \sum\limits_{i=1}^{n} \xi_i^2$ for almost all $(x,t) \in Q_T$ and for all $\xi \in \mathbb{R}^n$, where $b_0 > 0$ is a constant;

(G) The functions $x \to g(x,\xi)$, $x \to g_\xi(x,\xi)$ are continuous for every $\xi \in \mathbb{R}$ and the functions $\xi \to g(x,\xi)$, $\xi \to g_\xi(x,\xi)$ are measurable for almost all $x \in \Omega$ and satisfy the following inequalities:

$$(g(x,\xi) - g(x,\mu))(\xi - \mu) \geq g_0|\xi - \mu|^q$$

for almost all $x \in \Omega$ and for all $\xi, \mu \in \mathbb{R}$, $g_0 = const > 0$;
$|g(x,\xi)| \leq g_1|\xi|^{q-1}$ for almost all $x \in \Omega$ and for all $\xi \in \mathbb{R}$, $q \in (2, +\infty)$.

Under these assumptions, we will obtain the existence of a weak solution of problem (1)–(3).

Definition 1. *We call a function u a weak solution of problem* (1)–(3) *if*

$$u \in L^\infty\left((0,T); H_\Psi^{0,2}(\Omega)\right) \cap C\left([0,T]; H_\Psi^{0,1}(\Omega)\right),$$

$$u_t \in L^2\left((0,T); H_\Psi^{0,1}(\Omega)\right) \cap L^q\left((0,T); L_\Psi^q(\Omega)\right) \cap C\left([0,T]; L_\Psi^2(\Omega)\right)$$

and u satisfies the following integral equality

$$\int\limits_{\Omega_T} u_t(x,T)w(x,T)dx + \int\limits_{Q_T} \Bigg[-u_t w_t + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t)u_{x_i x_j} w_{x_k x_l}$$
$$(5) \qquad + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}w_{x_j} + a(x,t)uw + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{x_i t}w_{x_j}$$
$$+ g(x,u_t)w\Bigg] dxdt = \int\limits_{Q_T} f(x,t)wdxdt + \int\limits_{\Omega} u_1(x)w(x,0)dx$$

$\forall w \in C^1([0,T]; C_0^\infty(\Omega))$ *and* $u(x,0) = u_0(x)$.

We consider the following equation

$$A(u) = f^R(x,t) \tag{6}$$

in the domain $Q_T^R = \Omega^R \times (0,T)$, $R > 1$, where

$$f^R(x,t) = \begin{cases} f(x,t), & \text{for } (x,t) \in Q_T^R, \\ 0, & \text{for } (x,t) \in Q_T \setminus Q_T^R, \quad R = 2,3,4,\dots \end{cases}$$

For this equation, we put the following boundary and initial conditions

$$u|_{t=0} = u_0^R(x), \quad u_t|_{t=0} = u_1^R(x), \tag{7}$$

$$u|_{\partial\Omega^R} = \frac{\partial u}{\partial \nu}|_{\partial\Omega^R} = 0, \tag{8}$$

where $u_0^R(x) = u_0(x)\cdot\zeta^R(x)$, $u_1^R(x) = u_1(x)\zeta^R(x)$ for $0 \le \zeta^R(x) \le 1$, $\zeta \in C^2(\mathbb{R}^n)$ and $\zeta^R(x) = \begin{cases} 1, & \text{for } |x| \le R-1, \\ 0, & \text{for } |x| \ge R. \end{cases}$

Let $H^{0,2}(\Omega^R) = \left\{u : u \in H^2(\Omega^R); u|_{\partial\Omega^R} = 0, \frac{\partial u}{\partial\nu}|_{\partial\Omega^R} = 0\right\}$.

Definition 2. *We call a function u^R a weak solution of problem (6)–(8) if*

$$u^R \in L^\infty\left((0,T); H^{0,2}(\Omega^R)\right), \quad u_{tt}^R \in L^2\left((0,T); L^2(\Omega^R)\right),$$
$$u_t^R \in L^2\left((0,T); H^{0,1}(\Omega^R)\right) \cap L^q(Q_T^R)$$

and u^R satisfies the following integral equality

$$\int\limits_{Q_\tau^R}\Bigg[u_{tt}^R w + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t) u_{x_i x_j}^R w_{x_k x_l} + \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^R w_{x_j} + a(x,t) u^R w$$
$$+ \sum_{i,j=1}^n b_{ij}(x,t) u_{x_i t}^R w_{x_j} + g(x,u^R) w\Bigg] dxdt = \int\limits_{Q_\tau^R} f^R(x,t) w dxdt \tag{9}$$

$\forall\tau \in (0,T]$, $\forall w \in L^2\left((0,T); H^{0,2}(\Omega^R)\right) \cap L^q(Q_T^R)$ *and the initial conditions (7).*

Theorem 1. *Suppose that conditions* **(A), (B), (G)** *hold and $u_0 \in H^{0,2}(\Omega^R) \cap H^4(\Omega^R)$, $u_1 \in H^{0,1}(\Omega^R) \cap L^{2q-2}(\Omega^R)$, $f^R, f_t^R \in L^2(Q_T^R)$. Then problem* (4)–(6) *has a weak solution.*

Proof. Consider the space $H^{0,2}(\Omega^R) \cap H^4(\Omega^R) \cap L^q(\Omega^R)$ and the basis of this space $\{\Phi_k(x)\}$.

Next, consider the sequence of functions of the form

$$u^{R,N}(x,t) = \sum_{s=1}^N C_s^N(t)\Phi_s(x)$$

for $N = 1, 2, \dots$, where the functions $C_1^N, \dots, C_N^N$ are the solution of the following Cauchy problem

$$\int\limits_{\Omega^R}\Bigg[u_{tt}^{R,N}(x,t)\Phi_s(x) + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t) u_{x_i x_j}^{R,N}(x,t)\Phi_{s x_k x_l}(x)$$
$$+ \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^{R,N}(x,t)\Phi_{s x_j}(x) + \sum_{i,j=1}^n b_{ij}(x,t) u_{t x_i}^{R,N}(x,t)\Phi_{s x_i}(x) \tag{10}$$
$$+ a(x,t) u^{R,N}(x,t)\Phi_s(x) + g(x, u_t^{R,N})\Phi_s(x) - f^R(x,t)\Phi_s(x)\Bigg] dx = 0,$$
$$t \in [0,T]$$

with the conditions

$$C_s^N(0) = u_{0,s}^{R,N},\ C_{st}^N(0) = u_{1,s}^{R,N},\quad s = 1,\dots,N, \tag{11}$$

and $\|u_0^{R,N} - u_0^R\|_{H^{0,2}(\Omega^R)\cap H^4(\Omega^R)} \to 0$, $\|u_1^{R,N} - u_1^R\|_{H^{0,1}(\Omega^R)\cap L^{2q-2}(\Omega^R)} \to 0$, where $u_0^{R,N}(x) = \sum\limits_{s=1}^{N} u_{0,s}^{R,N}\Phi_s(x)$ and $u_1^{R,N}(x) = \sum\limits_{s=1}^{N} u_{1,s}^{R,N}\Phi_s(x)$.

From the Carathéodory theorem, there exists the solution of problem (10), (11), where C_s^N are continuous and C_{st}^N, $s = 1,\dots,N$ are absolutely continuous on the interval $[0, h_0]$. Taking into account the estimations obtained below, we can state that $h_0 = T$.

Multiplying (10) by the functions $C_{st}^N(t)e^{-\eta t}$, $\eta > 0$, respectively, then summing over s from 1 to N and integrating with respect to t from 0 to τ, $\tau \in (0, T]$, we obtain

$$\begin{aligned}\int\limits_{Q_\tau^R} \Bigg[& u_{tt}^{R,N}u_t^{R,N} + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t)u_{x_ix_j}^{R,N}u_{tx_kx_l}^{R,N} \\ & + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i}^{R,N}u_{tx_j}^{R,N} + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{tx_i}^{R,N}u_{tx_j}^{R,N} \\ & + a(x,t)u^{R,N}u_t^{R,N} + g(x,u_t^{R,N})u_t^{R,N} - f^R(x,t)u_t^{R,N}\Bigg] e^{-\eta t}dxdt = 0.\end{aligned} \tag{12}$$

If we consider the respective components of the last equality we will obtain

$$\begin{aligned}\int\limits_{Q_\tau^R} u_{tt}^{R,N}u_t^{R,N}e^{-\eta t}dxdt = & \frac{1}{2}\int\limits_{\Omega_\tau^R} |u_t^{R,N}|^2 e^{-\eta\tau}dx \\ & - \frac{1}{2}\int\limits_{\Omega_0^R} |u_1^{R,N}|^2dx + \frac{\eta}{2}\int\limits_{Q_\tau^R} |u_t^{R,N}|^2e^{-\eta t}dxdt,\end{aligned}$$

where $\Omega_\tau^R = Q_T^R \cap \{t = \tau\}$. Next from **(A)** there follows:

$$\begin{aligned}&\int\limits_{Q_\tau^R} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t)u_{x_ix_j}^{R,N}u_{tx_kx_l}^{R,N}e^{-\eta t}dxdt \\ &\qquad \geq \frac{a_2}{2}\int\limits_{\Omega_\tau^R} \sum_{i,j=1}^{n} |u_{x_ix_j}^{R,N}|^2e^{-\eta\tau}dx - \frac{1}{2}\int\limits_{\Omega_0^R} \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,0)u_{0x_ix_j}^{R,N}u_{0x_kx_l}^{R,N}dx \\ &\qquad\quad + \frac{1}{2}(\eta a_2 - a_2^1)\int\limits_{Q_\tau^R} \sum_{i,j=1}^{n} |u_{x_ix_j}^{R,N}|^2e^{-\eta t}dxdt.\end{aligned}$$

It is easy to prove that

$$(13)\quad \int\limits_{Q_\tau^R}(u^{R,N}(x,t))^2dxdt \le 2T\int\limits_{\Omega^R}(u^{R,N}(x,0))^2dx + 2T^2\int\limits_{Q_\tau^R}(u_t^{R,N}(x,t))^2dxdt,$$

$$(14)\quad \begin{aligned}&\int\limits_{Q_\tau^R}(u_{x_i}^{R,N}(x,t))^2e^{-\eta t}dxdt\\ &\qquad\le 2T\int\limits_{\Omega^R}(u_{x_i}^{R,N}(x,0))^2dx + 2T^2\int\limits_{Q_\tau^R}(u_{x_it}^{R,N}(x,t))^2e^{-\eta t}dxdt,\end{aligned}$$

and

$$\int\limits_{Q_\tau^R}\sum_{i=1}^n(u_{x_i}^{R,N})^2e^{-\eta t}dxdt \le \frac{1}{2}\int\limits_{Q_\tau^R}\sum_{i,j=1}^n(u_{x_ix_j}^{R,N})^2e^{-\eta t}dxdt + \frac{M}{2}\int\limits_{Q_\tau^R}(u^{R,N})^2e^{-\eta t}dxdt,$$

where M does not depend on N.

Hence from assumption **(A)**, (13) and the initial condition, we obtain

$$\begin{aligned}&\int\limits_{Q_\tau^R}\sum_{i,j=1}^n a_{ij}(x,t)u_{x_i}^{R,N}u_{tx_j}^{R,N}e^{-\eta t}dxdt\\ &\qquad\le \frac{\delta}{2}\int\limits_{Q_\tau^R}\sum_{i=1}^n(u_{tx_i}^{R,N})^2e^{-\eta t}dxdt + \frac{A_1}{4\delta}\int\limits_{Q_\tau^R}\sum_{i,j=1}^n(u_{x_ix_j}^{R,N})^2e^{-\eta t}dxdt\\ &\qquad+\frac{A_1nT^2}{2\delta}\int\limits_{Q_\tau^R}(u_t^{R,N})^2e^{-\eta t}dxdt + \frac{A_1nT}{2\delta}\int\limits_{Q_\tau^R}(u_0^{R,N})^2dx,\end{aligned}$$

and

$$\begin{aligned}&\int\limits_{Q_\tau^R}a(x,t)u^{R,N}u_t^{R,N}e^{-\eta t}dxdt \le \frac{A_0}{2}\int\limits_{Q_\tau^R}\left[(u^{R,N})^2+(u_t^{R,N})^2\right]e^{-\eta t}dxdt\\ &\qquad\le A_0T\int\limits_{\Omega^R}(u_0^{R,N})^2dx + (T^2+1)A_0\int\limits_{Q_\tau^R}(u_t^{R,N})^2e^{-\eta t}dxdt,\end{aligned}$$

where $A_1 = \operatorname{ess\,sup}\limits_{Q_T}\sum\limits_{i,j=1}^n a_{ij}^2(x,t)$, $A_0 = \operatorname{ess\,sup}\limits_{Q_T}|a(x,t)|$ and $\delta > 0$.

Next, from the assumption **(B)** there follows:

$$\int\limits_{Q_\tau^R}\sum_{i,j=1}^n b_{ij}(x,t)u_{tx_j}^{R,N}u_{tx_i}^{R,N}e^{-\eta t}dxdt \ge b_0\int\limits_{Q_\tau^R}\sum_{i,j=1}^n|u_{tx_i}^{R,N}|^2e^{-\eta t}dxdt.$$

Moreover, by virtue of condition **(G)** we get

$$\int\limits_{Q_\tau^R} (g(x, u_t^{R,N}), u_t^{R,N}) e^{-\eta t} dxdt \geq g_0 \int\limits_{Q_\tau^R} |u_t^{R,N}|^q e^{-\eta t} dxdt$$

and it is evident that

$$\int\limits_{Q_\tau^R} (f^R, u_t^{R,N}) e^{-\eta t} dxdt \leq \frac{1}{2} \int\limits_{Q_\tau^R} |f^R|^2 e^{-\eta t} dxdt + \frac{1}{2} \int\limits_{Q_\tau^R} |u_t^{R,N}|^2 e^{-\eta t} dxdt.$$

Taking into account the above estimates, from (9) we obtain the following inequality

(15)

$$\begin{aligned}
&\int\limits_{\Omega_\tau^R} \left[(u_t^{R,N})^2 + a_2 \sum_{i,j=1}^n (u_{x_i x_j}^{R,N})^2 \right] e^{-\eta \tau} dx \\
&\quad + \int\limits_{Q_\tau^R} \Bigg[\left(\eta - 2A_0(T^2+1) - 1 - \frac{A_1 n T^2}{\delta} \right) (u_t^{R,N})^2 + g_0 |u_t^{R,N}|^q \\
&\quad + \left(\eta a_2 - \frac{A_1}{2\delta} - a_2^1 \right) \sum_{i,j=1}^n (u_{x_i x_j}^{R,N})^2 + (b_0 - \delta) \sum_{i=1}^n (u_{tx_i}^{R,N})^2 \Bigg] e^{-\eta t} dxdt \\
&\leq \int\limits_{\Omega_0^R} \left[(u_1^{R,N})^2 + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{0x_i x_j}^{R,N} u_{0x_k x_l}^{R,N} + \left(2A_0 T + \frac{A_1 n T}{\delta} \right) (u_0^{R,N})^2 \right] dx \\
&\quad + \int\limits_{Q_\tau^R} (f^R(x,t))^2 e^{-\eta t} dxdt \leq \mu_1.
\end{aligned}$$

Then from (15), choosing

$$\delta = \frac{b_0}{2}, \quad \eta \geq \max \left\{ \frac{A_1 + b_0 a_2^1}{a_2 b_0} \, ; \, 2 + 2A_0(T^2+1) + \frac{2A_1 n T^2}{b_0} \right\}$$

we get the estimate

$$\begin{aligned}
&\int\limits_{\Omega_\tau^R} \left[(u_t^{R,N})^2 + \sum_{ij=1}^n (u_{x_i x_j}^{R,N})^2 \right] e^{-\eta \tau} dx \\
&\quad + \int\limits_{Q_T^R} \left[(u_t^{R,N})^q + \sum_{i=1}^n (u_{tx_i}^{R,N})^2 \right] e^{-\eta t} dxdt \leq \mu_2,
\end{aligned} \tag{16}$$

$\tau \in [0,T]$ and the constant μ_2 does not depend on N. Moreover, assumption **(G)** implies the estimate

(17) $$\int\limits_{Q_T^R} |g(x,t,u^{R,N})|^{q'} dxdt \leq \mu_3, \quad q' = \frac{q}{q-1}.$$

Due to the assumptions of Theorem 1, we can differentiate system (7) with respect to t :

(18)
$$\int\limits_{\Omega^R} \Bigg[u_{ttt}^{R,N}(x,t)\phi_s(x) + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{tx_ix_j}^{R,N}(x,t)\phi_{sx_kx_l}(x)$$
$$+ \sum_{i,j=1}^{n} a_{ij}(x,t)u_{tx_i}^{R,N}(x,t)\phi_{sx_j}(x) + \sum_{i,j=1}^{n} b_{ij}(x,t)u_{ttx_i}^{R,N}(x,t)\phi_{sx_j}(x)$$
$$+ a(x,t)u_t^{R,N}(x,t)\phi_s(x) + g_\xi(x,u_t^{R,N})u_{tt}^{R,N}(x,t)\phi_s(x) - f_t(x,t)\phi_s(x)$$
$$+ \sum_{i,j,k,l=1}^{n} a_{ijt}^{kl}(x,t)u_{x_ix_j}^{R,N}(x,t)\phi_{sx_kx_l}(x) + \sum_{i,j=1}^{n} a_{ijt}(x,t)u_{x_i}^{R,N}(x,t)\phi_{sx_j}(x)$$
$$+ a_t(x,t)u^{R,N}(x,t)\phi_s(x) + \sum_{i,j=1}^{n} b_{ijt}(x,t)u_{tx_i}^{R,N}(x,t)\phi_{sx_i}(x)\Bigg] dx = 0.$$

Hence, multiplying (18) by the functions $C_{stt}^N(t)e^{-\alpha t}$, $\alpha > 0$, respectively, then summing over s from 1 to N and integrating with respect to t from 0 to τ, $\tau \in (0,T]$, we obtain

$$\int\limits_{Q_\tau^R} \Bigg[u_{ttt}^{R,N}u_{tt}^N + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t)u_{tx_ix_j}^{R,N}u_{ttx_kx_l}^{R,N} + \sum_{i,j=1}^{n} a_{ij}(x,t)u_{tx_i}^{R,N}u_{ttx_j}^{R,N}$$
$$+ \sum_{i,j=1}^{n} b_{ij}(x,t)u_{ttx_i}^{R,N}u_{ttx_i}^{R,N} + a(x,t)u_t^{R,N}u_{tt}^{R,N}$$
(19) $$+ g_\xi(x,u_t^{R,N})(u_{tt}^{R,N})^2 - f_t(x,t)u_{tt}^{R,N}\Bigg] e^{-\alpha t} dxdt$$
$$+ \int\limits_{Q_\tau^R} \Bigg[\sum_{i,j,k,l=1}^{n} a_{ijt}^{kl}(x,t)u_{x_ix_j}^{R,N}u_{ttx_kx_l}^{R,N} + \sum_{i,j=1}^{n} a_{ijt}(x,t)u_{x_i}^{R,N}u_{ttx_j}^{R,N}$$
$$+ \sum_{i,j=1}^{n} b_{ijt}(x,t)u_{tx_i}^{R,N}u_{ttx_i}^{R,N} + a_t(x,t)u^{R,N}u_{tt}^{R,N}\Bigg] e^{-\alpha t} dxdt = 0.$$

where $\xi :=: u_t^{R,N}$. By analogy with (12), the following estimate for the first integral can be obtained

$$\int\limits_{Q_\tau^R}\Big[u_{ttt}^{R,N}u_{tt}^N+\sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t)u_{tx_ix_j}^{R,N}u_{ttx_kx_l}^{R,N}+\sum_{i,j=1}^n a_{ij}(x,t)u_{tx_i}^{R,N}u_{ttx_j}^{R,N}$$
$$+\sum_{i,j=1}^n b_{ij}(x,t)u_{ttx_i}^{R,N}u_{ttx_i}^{R,N}+a(x,t)u_t^{R,N}u_{tt}^{R,N}+g_\xi(x,u_t^{R,N})(u_{tt}^{R,N})^2$$
$$-f_t(x,t)u_{tt}^{R,N}\Big]e^{-\alpha t}dxdt$$

(20)
$$\geq\frac12\int\limits_{\Omega_\tau^R}\Big[(u_{tt}^{R,N})^2+a_2\sum_{i,j=1}^n(u_{tx_ix_j}^{R,N})^2\Big]e^{-\alpha\tau}dx+\frac12\int\limits_{Q_\tau^R}\Big[(\alpha-A_0-1)(u_{tt}^{R,N})^2$$
$$+(\alpha a_2-a_2^1)\sum_{i,j=1}^n(u_{tx_ix_j}^{R,N})^2+(2b_0-\delta_1)\sum_{i=1}^n(u_{ttx_i}^{R,N})^2\Big]e^{-\alpha t}dxdt$$
$$-\frac12\int\limits_{Q_\tau^R}\Big[(u_t^{R,N})^2+\frac{A_1}{\delta_1}\sum_{i=1}^n(u_{tx_i}^{R,N})^2+(2b_0-\delta_1)\sum_{i=1}^n(u_{ttx_i}^{R,N})^2$$
$$+(f_t^R(x,t)))^2\Big]e^{-\alpha t}dxdt-\frac12\int\limits_{\Omega_0^R}\Big[(u_{tt}^{R,N})^2+\sqrt{A_3}\sum_{i,j=1}^n(u_{1,x_ix_j}^{R,N})^2\Big]dx,$$

where $A_3=\operatorname{ess\,sup}\limits_{\Omega}\sum\limits_{i,j,k,l=1}^n(a_{ij}^{kl}(x,0))^2$, because $g_\xi(x,\xi)\geq 0$.

Taking into account **(A)**, we get

$$\int\limits_{Q_\tau^R}\sum_{i,j,k,l=1}^n a_{ijt}^{kl}(x,t)u_{x_ix_j}^{R,N}u_{ttx_kx_l}^{R,N}e^{-\alpha t}dxdt$$
$$\geq-\frac12\int\limits_{\Omega_\tau^R}\Big[\frac{A_2}{\delta_1}\sum_{i,j=1}^n(u_{x_ix_j}^{R,N})^2+\delta_1\sum_{i,j=1}^n(u_{tx_ix_j}^{R,N})^2\Big]e^{-\alpha\tau}dx$$
$$-\frac12\int\limits_{\Omega_0^R}\Big[\sqrt{A_2}\sum_{i,j=1}^n(u_{0,x_ix_j}^{R,N})^2+\sqrt{A_2}\sum_{i,j=1}^n(u_{1x_ix_j}^{R,N})^2\Big]dx$$
$$-\frac12\int\limits_{Q_\tau^R}\Big[\frac{\alpha^2A_2+A_4}{\delta_1}\sum_{i,j=1}^n(u_{x_ix_j}^{R,N})^2+(\delta_1+2a_2^1)\sum_{i,j=1}^n(u_{tx_ix_j}^{R,N})^2\Big]e^{-\alpha t}dxdt,$$

$$\int\limits_{Q_\tau^R}\Bigg[\sum_{i,j=1}^n a_{ijt}(x,t)u_{x_i}^{R,N}u_{ttx_j}^{R,N}+\sum_{i,j=1}^n b_{ijt}(x,t)u_{tx_i}^{R,N}u_{ttx_j}^{R,N}$$
$$+a_t(x,t)u^{R,N}u_{tt}^{R,N}\Bigg]e^{-\alpha t}dxdt$$
$$\geq -\frac{1}{2}\int\limits_{Q_\tau^R}\Bigg[\frac{A_5}{\delta_1}\sum_{i=1}^n(u_{x_i}^{R,N})^2+2\delta_1\sum_{i=1}^n(u_{ttx_i}^{R,N})^2+\frac{B_1}{\delta_1}\sum_{i=1}^n(u_{tx_i}^{R,N})^2$$
$$+(u_{tt}^{R,N})^2+A_0^2(u^{R,N})^2\Bigg]e^{-\alpha t}dxdt,$$

where $A_2=\operatorname{ess\,sup}\limits_{Q_T}\sum\limits_{i,j,k,l=1}^n(a_{ijt}^{kl}(x,t))^2$, $A_4=\operatorname{ess\,sup}\limits_{Q_T}\sum\limits_{i,j,k,l=1}^n(a_{ijtt}^{kl}(x,t))^2$, $A_5=\operatorname{ess\,sup}\limits_{Q_T}\sum\limits_{i,j=1}^n(a_{ijt}(x,t))^2$, $B_1=\operatorname{ess\,sup}\limits_{Q_T}\sum\limits_{i,j=1}^n(b_{ijt}(x,t))^2$.

Hence

$$\int\limits_{\Omega_\tau^R}\Bigg[(u_{tt}^{R,N})^2+(a_2-\delta_1)\sum_{i,j=1}^n(u_{tx_ix_j}^{R,N})^2\Bigg]e^{-\alpha\tau}dx$$
$$+\frac{1}{2}\int\limits_{Q_\tau^R}\Bigg[(\alpha-A_0-2)(u_{tt}^{R,N})^2+(\alpha a_2-3a_2^1-\delta_1)\sum_{i,j=1}^n(u_{tx_ix_j}^{R,N})^2$$
$$+(2b_0-3\delta_1)\sum_{i=1}^n(u_{ttx_i}^{R,N})^2\Bigg]e^{-\alpha t}dxdt$$
$$\leq\int\limits_{Q_\tau^R}\Bigg[(u_t^{R,N})^2+\frac{A_1+B_1}{\delta_1}\sum_{i=1}^n(u_{tx_i}^{R,N})^2+\frac{\alpha^2A_2+A_4}{\delta_1}\sum_{i,j=1}^n(u_{x_ix_j}^{R,N})^2 \tag{21}$$
$$+\frac{A_5}{\delta_1}\sum_{i=1}^n(u_{x_i}^{R,N})^2+A_0^2(u^{R,N})^2+(f_t^R(x,t)))^2\Bigg]e^{-\alpha t}dxdt$$
$$+\int\limits_{\Omega_0^R}\Bigg[(u_{tt}^{R,N})^2+\sqrt{A_2}\sum_{i,j=1}^n(u_{0,x_ix_j}^{R,N})^2+(\sqrt{A_2}+\sqrt{A_3})\sum_{i,j=1}^n(u_{1,x_ix_j}^{R,N})^2\Bigg]dx$$
$$+\frac{A_2}{\delta_1}\int\limits_{\Omega_\tau^R}\sum_{i,j=1}^n(u_{x_ix_j}^{R,N})^2e^{-\alpha\tau}dx.$$

Observe that for $t = 0$, multiplying (18) by $C_{stt}(0)$, by analogy with (19), we obtain

$$\begin{aligned}
\int\limits_{\Omega_0^R} \Big[(u_{tt}^{R,N}(x,0))^2 &+ \sum_{i,j,k,l=1}^{n} \Big(a_{ij}^{kl}(x,0) u_{x_i x_j}^{R,N}(x,0) \Big)_{x_k x_l} u_{tt}^{R,N}(x,0) \\
&- \sum_{i,j=1}^{n} \big(a_{ij}(x,0) u_{x_i}^{R,N}(x,0) \big)_{x_j} u_{tt}^{R,N}(x,0) \\
&- \sum_{i,j=1}^{n} \Big(b_{i,j}(x,0) u_{tx_i}^{R,N} \Big)_{x_i} u_{tt}^{R,N}(x,0) + a(x,0) u^{R,N}(x,0) u_{tt}^{R,N}(x,0) \\
&+ g(x, u_t^{R,N}(x,0)) u_{tt}^{R,N}(x,0) u_{tt}^{R,N}(x,0) - f(x,0) u_{tt}^{R,N}(x,0) \Big] dx = 0.
\end{aligned} \tag{22}$$

By conditions of Theorem 1,

$$\begin{aligned}
\int\limits_{\Omega_0^R} \Bigg[\Bigg| \sum_{i,j,k,l=1}^{n} \Big(a_{ij}^{kl} u_{0x_i x_j}^{R,N} \Big)_{x_k x_l} \Bigg|^2 &+ \Bigg| \sum_{i,j=1}^{n} \Big(a_{ij} u_{0x_i}^{R,N} \Big)_{x_j} \Bigg|^2 \\
&+ \Bigg| \sum_{i=1}^{n} \Big(b_i(x) u_{1x_i}^{R,N} \Big)_{x_i} \Bigg|^2 + |a u_0^{R,N}|^2 + |g(x, u_1^N)|^2 \Bigg] dx \le K_1.
\end{aligned}$$

Then from (22) it follows that

$$\int\limits_{\Omega_0^R} |u_{tt}^N|^2 dx \le M_1, \tag{23}$$

where M_1 does not depend on N.

Choosing

$$\delta_1 = \min\left\{ \frac{a_2}{2}\,;\, \frac{b_0}{3} \right\}, \quad \alpha = \min\left\{ A_0 + 2\,;\, \frac{3a_2^1 + \delta_1}{a_2} \right\}$$

and using (16), (17), we obtain

$$\|u^{R,N}\|_{L^2((0,T);H^{0,2}(\Omega^R))} \le M_2, \quad \|u_t^{R,N}\|_{L^2((0,T);H^{0,2}(\Omega^R) \cap L^q(Q_T^R)} \le M_2, \tag{24}$$

$$\|u_{tt}^{R,N}\|_{L^2((0,T);H^{0,1}(\Omega^R))} \le M_2, \quad \|g(\cdot, u_t^{R,N}\|_{L^{q'}(Q_T^R)} \le M_2, \tag{25}$$

where M_2 does not depend on N.

Then due to (24), (25) and [**4**, p. 70] there exists such subsequence $\{u^{R,N_k}\} \subset \{u^{R,N}\}$ that $u^{R,N_k} \to u^R$ weakly in $L^2((0,T); H^{0,2}(\Omega^R))$, $u_t^{R,N_k} \to u_t^R$ weakly in $L^2((0,T); H^{0,2}(\Omega^R)) \cap L^q(Q_T^R)$, $u_{tt}^{R,N_k} \to u_{tt}^R$ weakly in $L^2((0,T); H^{0,1}(\Omega^R))$, $g(\cdot, u_t^{R,N_k}) \to \chi^R$ weakly in $L^{q'}(Q_T^R)$, $u_t^{R,N_k} \to u_t^R$ strongly in $L^2(Q_T^R)$ when $N_k \to \infty$. Hence $\chi^R = g(\cdot, u_t^R)$.

Taking into account (7), (8) it is easy to show that the function u^R is a weak solution of problem (4), (6):

$$\int_{Q_T^R} \Big[u_{tt}^R w + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t) u_{x_i x_j}^R w_{x_k x_l} + \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^R w_{x_j} + a(x,t) u^R v + \sum_{i,j=1}^n b_{ij}(x,t) u_{x_i t}^R w_{x_j} + g(x,u^R) w \Big] dxdt = \int_{Q_T^R} f^R(x,t) w dxdt \tag{26}$$

$\forall w \in L^2\left((0,T); H^{0,2}(\Omega^R)\right) \cap L^q(Q_T^R)$ and initial conditions (7). Moreover, $u_t^R \in C([0,T]; H^{0,1}(\Omega^R))$. □

THEOREM 2. *Suppose that conditions* **(A)**, **(B)** *and* **(G)** *hold and* $u_0 \in H_\Psi^{0,2}(\Omega), u_1 \in L_\Psi^2(\Omega);\ f \in L^2\Big((0,T); L_\Psi^2(\Omega)\Big)$. *Then there exists a weak solution of problem* (1)–(2), *and this solution satisfies the following inequality*

$$\int_{Q_T} \Big[|u|^2 + |u_t|^2 + \sum_{i=1}^n |u_{x_i t}|^2 + \sum_{i,j=1}^n |u_{x_i x_j}|^2 \Big] \Psi(|x|) dxdt < \mu, \tag{27}$$

where μ *is the constant which depends on* f, u_0, u_1 *and coefficients of equation* (1).

PROOF. We consider the sequence of domains Q^R for $R = 2, 3, \ldots$ Observe that the assumptions of Theorem 1 guarantee the existence of a weak solution u^s of problem (1)–(3) in the domain Q^s, where $s = 2, 3, \ldots$, in the sense of Definition 2. Every function u^s satisfies Definition 1 and equality (5). We extend all of these solutions by 0 on Q_T. We obtain the sequence $\{u^s\}$. For this sequence, by analogy to (9), we can obtain the following equality

$$\int_{Q_T} \Big[u_{tt}^s u_t^s \Psi(|x|) + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,t) u_{x_i x_j}^s (u_t^s \Psi(|x|))_{x_k x_l} + \sum_{i,j=1}^n (b_{ij}(x,t) u_{t x_i}^s (u_t^s \Psi(|x|))_{x_j} + \sum_{i,j=1}^n a_{ij}(x,t) u_{x_i}^s (u_t^s \Psi(|x|))_{x_j} + a(x,t) u^s u_t^s \Psi(|x|) + g(x, u_t^s) u_t^s \Psi(|x|) \Big] e^{-\beta t} dxdt = \int_{Q_T} f^s(x,t) u_t^s \Psi(|x|) e^{-\beta t} dxdt,\ \beta > 0. \tag{28}$$

Now we transform and estimate every term of (28). Obviously,

$$I_1 := \int\limits_{Q_T} u^s_{tt} u^s_t \Psi(|x|) e^{-\beta t} dxdt = \frac{\beta}{2} \int\limits_{Q_T} (u^s_t)^2 \Psi(|x|) e^{-\beta t} dxdt$$
$$+ \frac{1}{2} \int\limits_{\Omega_T} (u^s_t)^2 \Psi(|x|) e^{-\beta t} dx - \frac{1}{2} \int\limits_{\Omega_0} (u^s_1(x))^2 \Psi(|x|) dx.$$

From assumption **(A)** there follows

$$I_2 := \int\limits_{Q_T} \sum_{i,j,k,l=1}^{n} a^{kl}_{ij}(x,t) u^s_{x_i x_j} (u^s_t \Psi(|x|))_{x_k x_l} e^{-\beta t} dxdt \geq I^1_2 + I^2_2 + I^3_2,$$

where

$$I^1_2 \geq \frac{a_2}{2} \int\limits_{\Omega_T} \sum_{i,j=1}^{n} |u^s_{x_i x_j}|^2 \Psi(|x|) e^{-\beta t} dx$$
$$- \frac{1}{2} \int\limits_{\Omega_0} \sum_{i,j,k,l=1}^{n} a^{kl}_{ij}(x,0) u^s_{0,x_i x_j} u^s_{0 x_k x_l} \Psi(|x|) dx$$
$$+ \frac{1}{2}(\beta a_2 - a^1_2) \int\limits_{Q_T} \sum_{i,j=1}^{n} |u^s_{x_i x_j}|^2 \Psi(|x|) e^{-\beta t} dxdt,$$

$$I^2_2 \leq \frac{A_3 n \Psi_0}{\delta_3} \int\limits_{Q_T} \sum_{i,j=1}^{n} (u^s_{x_i x_j})^2 \Psi(|x|) e^{-\beta t} dxdt$$
$$+ \delta_3 \Psi_0 \int\limits_{Q_T} \sum_{k=1}^{n} (u^s_{t x_k})^2 \Psi(|x|) e^{-\beta t} dxdt,$$

and

$$I^3_2 \leq \frac{A_3 n^2 \Psi_0}{2} \int\limits_{Q_T} \sum_{i,j=1}^{n} (u^s_{x_i x_j})^2 \Psi(|x|) e^{-\beta t} dxdt + \frac{\Psi_0}{2} \int\limits_{Q_T} (u^s_t)^2 \Psi(|x|) e^{-\beta t} dxdt.$$

Next from assumption **(B)** we obtain:

$$I_3 := \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij}(x,t) u^s_{t x_i} (u^s_t \Psi(|x|))_{x_j} e^{-\beta t} dxdt \geq I^1_3 + I^2_3,$$

where

$$I^1_3 := \int\limits_{Q_T} \sum_{i,j=1}^{n} b_{ij}(x,t) u^s_{t x_i} u^s_{t x_j} \Psi(|x|) e^{-\beta t} dxdt \geq b_0 \int\limits_{Q_T} \sum_{i=1}^{n} (u^s_{t x_i})^2 \Psi(|x|) e^{-\beta t} dxdt,$$

$$I_3^2 \le \frac{B_0 n \Psi_0 \delta_3}{2} \int\limits_{Q_T} \sum_{i=1}^{n} (u^s_{tx_i})^2 \Psi(|x|) e^{-\beta t} dxdt + \frac{\Psi_0}{2\delta_3} \int\limits_{Q_T} (u^s_t)^2 \Psi(|x|) e^{-\beta t} dxdt,$$

where $B_0 = \operatorname{ess\,sup}\limits_{Q_T} \sum\limits_{i,j=1}^{n} b^2_{ij}(x,t)$. Next,

$$I_4 := \int\limits_{Q_T} \left[\sum_{i,j=1}^{n} a_{ij}(x,t) u^s_{x_i} (u^s_t \Psi(|x|))_{x_j} + a(x,t) u^s u^s_t \Psi(|x|) \right] e^{-\beta t} dxdt \ge I_4^1 + I_4^2,$$

where

$$I_4^1 \le \frac{A_1}{2\delta_3} \int\limits_{Q_T} \sum_{i=1}^{n} (u^s_{x_i})^2 \Psi(|x|) e^{-\beta t} dxdt + \frac{\delta_3}{2} \int\limits_{Q_T} \sum_{i=1}^{n} (u^s_{x_i t})^2 \Psi(|x|) e^{-\beta t} dxdt,$$

$$I_4^2 \le \frac{A_1 \Psi_0}{2} \int\limits_{Q_T} \sum_{i=1}^{n} (u^s_{x_i})^2 \Psi(|x|) e^{-\beta t} dxdt$$
$$+ \frac{n + A_0}{2} \int\limits_{Q_T} (u^s_t)^2 \Psi(|x|) e^{-\beta t} dxdt + \frac{A_0}{2} \int\limits_{Q_\tau} (u^s)^2 \Psi(|x|) e^{-\beta t} dxdt.$$

From **(G)**, there follows

$$I_5 := \int\limits_{Q_T} g(x, u^s_t) u^s_t \Psi(|x|) e^{-\beta t} dxdt \ge g_0 \int\limits_{Q_T} |u^s_t|^q \Psi(|x|) e^{-\beta t} dxdt.$$

It is evident that

$$I_6 := \int\limits_{Q_T} f^s(x,t) u^s_t \Psi(|x|) e^{-\beta t} dxdt \le \frac{1}{2} \int\limits_{Q_T} \left[(f^s)^2 + (u^s_t)^2 \right] \Psi(|x|) e^{-\beta t} dxdt.$$

Using the property of Ψ, it is easy to prove the following inequality:

$$\int\limits_{\Omega} \sum_{i=1}^{n} (u^s_{x_i})^2 \Psi(|x|) dx \le \int\limits_{\Omega} \sum_{i,j=1}^{n} (u^s_{x_i x_j})^2 \Psi(|x|) dx + (1 + \Psi_0^2) \int\limits_{\Omega} (u^s)^2 \Psi(|x|) dx. \tag{29}$$

Moreover,

$$\int\limits_{Q_T} (u^s(x,t))^2 \Psi(|x|) e^{-\beta t} dxdt \le 2T \int\limits_{\Omega_0} (u^s(0,t))^2 \Psi(|x|) dx + 2T^2 \int\limits_{Q_T} (u^s_t)^2 \Psi(|x|) e^{-\beta t} dxdt. \tag{30}$$

Using the estimates for the integers I_1–I_6, from (29), (30), we get

$$\begin{aligned}
&\int\limits_{\Omega_T}\Big[(u_t^s)^2+a_2\sum_{i,j=1}^n(u^s_{x_ix_j})^2\Big]\Psi(|x|)e^{-\beta t}dx\\
&+\int\limits_{Q_T}\Big[(2b_0-2\Psi_0\delta_3-\Psi_0nB_0\delta_3-\delta_3)\sum_{i=1}^n(u^s_{tx_i})^2\\
&\qquad+2g_0|u_t^s|^q\Big]\Psi(|x|)e^{-\beta t}dxdt\\
&+\int\limits_{Q_T}\Big[(\beta-\Psi_1-\frac{\Psi_0}{\delta_3}-n-A_0-1-2T^2(A_0+\Psi_0^2+1))|u_t^s|^2\\
&\qquad+\Big(\beta a_2-a_2^1-\frac{2\Psi_0nA_3}{\delta_3}-\Psi_1n^2A_3-\frac{A_1}{\delta_3}-A_1\Psi_0\Big)\sum_{i,j=1}^n|u^s_{x_ix_j}|^2\Big]\\
&\qquad\qquad\cdot\Psi(|x|)e^{-\beta t}dxdt\\
&\le\int\limits_{\Omega_0}\Big[(2TA_0+\Psi_0^2+2T)(u_0^s)^2+|u_1^s|^2\\
&\qquad+\sum_{i,j,k,l=1}^n a_{ij}^{kl}(x,0)u^s_{0,x_ix_j}u^s_{0,x_kx_l}\Big]\Psi(|x|)dx+\int\limits_{Q_T}|f^s|^2e^{-\beta t}\Psi(|x|)dxdt.
\end{aligned}\tag{31}$$

Choosing β, δ_3 from the inequalities

$$2b_0-2\Psi_0\delta_3-\Psi_0nB_0\delta_3-\delta_3\ge 1,$$

$$\beta-\Psi_1-\frac{\Psi_0}{\delta_3}-n-A_0-1-2T^2(A_0+\Psi_0^2+1)\ge 1,$$

$$\beta a_2-a_2^1-\frac{2\Psi_0nA_3}{\delta_3}-\Psi_1n^2A_3-\frac{A_1}{\delta_3}-A_1\Psi_0\ge 1,$$

we get the estimates $\|u_t^s\|_{L^\infty((0,T);L^2_\Psi(\Omega))}\le\mu_1$, $\|u^s\|_{L^2((0,T);H^{0,2}_\Psi(\Omega))}\le\mu_1$, $\|u_t^s\|_{L^2((0,T);H^{0,1}_\Psi(\Omega))\cap L^q((0,T);L^q_\Psi(\Omega))}\le\mu_1$, $\int\limits_{Q_T}|g(x,u_t^s|^{q'}\Psi e^{-\beta t}dxdt\le\mu_1$, where μ_1 does not depend on s.

Hence we can choose a subsequence $\{u^{s_k}\}$ of the sequence $\{u^s\}$ such that $u_t^{s_k}(\cdot,T)\to w$ weakly in $L^2_\Psi(\Omega)$, $u^{s_k}\to u$ weakly in $L^2\Big((0,T);H^{0,2}_\Psi(\Omega)\Big)$, $u_t^{s_k}\to u_t$ weakly in $L^2\Big((0,T);H^{0,1}_\Psi(\Omega)\Big)\cap L^q\Big((0,T);L^q_\Psi(\Omega)\Big)$, $g(\cdot,u_t^{s_k})\to\chi$ weakly in $L^{q'}\Big((0,T);L^{q'}_\Psi(\Omega)\Big)$.

It is easy to prove similarly to [**4**, p. 24] that

$$\int\limits_{\Omega_T} u_t(x,T)w(x,T)dx + \int\limits_{Q_T} \Big[-u_t w_t + \sum_{i,j,k,l=1}^{n} a_{ij}^{kl}(x,t) u_{x_i x_j} w_{x_k x_l} + \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i} w_{x_j} + a(x,t)uw + \sum_{i,j=1}^{n} b_{ij}(x,t) u_{x_i t} w_{x_j} + \chi w \Big] dxdt = \int\limits_{Q_T} f(x,t) w dxdt + \int\limits_{\Omega} u_1(x) w(x,0) dx \tag{32}$$

$\forall w \in C^1([0,T]; C_0^\infty(\Omega))$ and $u(x,0) = u_0(x)$. Moreover, equality (32) is true for $w(x) = u_t e^{-\beta t}\Psi(|x|)$.

For the function g we obtain

(33)

$$\begin{aligned}
0 \le & \int\limits_{Q_T} \Big(g(x,u_t^s) - g(x,v) \Big)(u_t^s - v)\Psi(|x|)e^{-\beta t} dxdt \\
= & \int\limits_{Q_T} g(x,u_t^s) u_t^s \Psi(|x|) e^{-\beta t} dxdt - \int\limits_{Q_T} \Big[g(x,u_t^s)v + g(x,v)(u_t^s - v) \Big] \Psi(|x|) e^{-\beta t} dxdt \\
= & \int\limits_{Q_T} \Big[f^s u_t^s \Psi(|x|) - u_{tt}^s u_t^s \Psi(|x|) - \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_i x_j}^s (u_t^s \Psi(|x|))_{x_k x_l} \\
& - \sum_{i,j=1}^{n} (b_{ij} u_{t x_i}^s (u_t^s \Psi(|x|))_{x_j} - \sum_{i,j=1}^{n} a_{ij} u_{x_i}^s (u_t^s \Psi(|x|))_{x_j} - a u^s u_t^s \Psi(|x|) \Big] e^{-\beta t} dxdt \\
& - \int\limits_{Q_T} \Big[g(x,u_t^s)v + g(x,v)(u_t^s - v) \Big] \Psi(|x|) e^{-\beta t} dxdt \\
\le & - \int\limits_{\Omega_T} u_t(x,T) u_t(x,T) e^{-\beta t} \Psi(|x|) dx \\
& + \int\limits_{Q_T} \Big[f u_t \Psi(|x|) - \beta u_t u_t \Psi(|x|) - \sum_{i,j,k,l=1}^{n} a_{ij}^{kl} u_{x_i x_j} (u_t \Psi(|x|))_{x_k x_l} \\
& - \sum_{i,j=1}^{n} (b_{ij} u_{t x_i} (u_t \Psi(|x|))_{x_j} - \sum_{i,j=1}^{n} a_{ij} u_{x_i} (u_t \Psi(|x|))_{x_j} - a u u_t \Psi(|x|) \Big] e^{-\beta t} dxdt \\
& - \int\limits_{Q_T} \Big[\chi v + g(x,v)(u_t - v) \Big] \Psi(|x|) e^{-\beta t} dxdt + \int\limits_{\Omega} u_1(x) u_1(x) \Psi(|x|) dx,
\end{aligned}$$

when $s \to +\infty$. Hence, summing (32) for $w(x) = u_t e^{-\beta t}\Psi(|x|)$ and (33), we obtain

$$\int\limits_{Q_T} \Big(\chi - g(x,v)\Big)(u_t - v)\Psi(|x|)e^{-\beta t}dxdt \geq 0.$$

Let $v = u_t - \lambda w$,$\lambda > 0$, $w \in L^2\Big((0,T); H^{0,2}_{\Psi}(\Omega)\Big)$. Then

$$\int\limits_{Q_T} (\chi - g(x,u_t))w\Psi(|x|)e^{-\beta t}dxdt = 0$$

for every w, which means that $\chi = g(x,u_t)$.

From (31) there follows (27). By virtue of Definition 1, the function u is a weak solution of problem (1)–(3). □

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