UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XLV
2007

IMPLICIT DIFFERENCE METHODS FOR QUASILINEAR PARABOLIC FUNCTIONAL DIFFERENTIAL SYSTEMS

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Abstract. Nonlinear parabolic functional differential equations with initial boundary conditions of the Neumann type are considered. A general class of difference methods for the problem is constructed. Theorems on the convergence of difference schemes and error estimates of approximate solutions are presented. The proof of the stability of the difference functional problem is based on a comparison technique. Nonlinear estimates of the Perron type with respect to the functional variable for given functions are used. Numerical examples are given.

1. Introduction. For any two metric spaces X and Y we write $\mathbf{C}(X,Y)$ to denote the class of all continuous functions defined on X and taking values in Y. Let $M[n]$ denote the set of all $n\times n$ real matrices. We will use vectorial inequalities, with the understanding that the same inequality hold between their corresponding components. Let $E=[0,a]\times[-b,b]$, where $a>0$, $b=(b_1,\dots,b_n)$, $b_i>0$ for $1\le i\le n$, and

$$\partial_0 E=[0,a]\times([-b,b]\setminus(-b,b)).$$

Write $\Sigma=E\times\mathbf{C}(E,\mathbf{R}^k)$ and

$$\partial_0 E_j=\{(t,x)\in\partial_0 E:\ x_j=b_j\}\cup\{(t,x)\in\partial_0 E:\ x_j=-b_j\},\ 1\le j\le n,$$

and suppose that

$$f_s:\Sigma\to M[n],\ g_s:\Sigma\to\mathbf{R}^n,$$
$$f_s=[f_{s.ij}]_{i,j=1,\dots,n},\ g_s=(g_{s.1},\dots,g_{s.n}),\ 1\le s\le k,$$
$$G:\Sigma\to\mathbf{R}^k,\ G=(G_1,\dots,G_k)$$

2000 *Mathematics Subject Classification.* 65M12, 35R10.

Key words and phrases. Functional differential equations, difference methods, stability and convergence.

and

$$\varphi : [-b, b] \to \mathbf{R}^k,\ \varphi = (\varphi_1, \ldots, \varphi_k),$$

$$\psi_j : \partial_0 E_j \to \mathbf{R}^k,\ \psi_j = (\psi_{j.1}, \ldots, \psi_{j.k}),\ 1 \le j \le n$$

are given functions. We consider the problem consisting of the quasilinear system of differential functional equations

$$\begin{aligned} \partial_t z_s(t,x) = &\sum_{i,j=1}^{n} f_{s.ij}(t,x,z)\partial_{x_i x_j} z_s(t,x) \\ &+ \sum_{i=1}^{n} g_{s.i}(t,x,z)\partial_{x_i} z_s(t,x) + G_s(t,x,z), \quad s = 1, \ldots, k, \end{aligned} \tag{1}$$

and the initial boundary condition of the Neumann type

$$z(0,x) = \varphi(x) \text{ for } x \in [-b, b], \tag{2}$$

$$\partial_{x_j} z(t,x) = \varphi_j(t,x) \text{ for } (t,x) \in \partial_0 E_j,\ 1 \le j \le n, \tag{3}$$

where $z = (z_1, \ldots, z_k)$.

For $t \in [0,a]$, we write $E_t = [0,t] \times [-b,b]$. The function f is said to satisfy the Volterra condition if for each $(t,x) \in E$ and $z, \bar{z} \in C(E, \mathbf{R}^k)$ such that $z(\bar{t}, y) = \bar{z}(\bar{t}, y)$ for $(\bar{t}, y) \in E_t$ there is $f(t,x,z) = f(t,x,\bar{z})$. Note that the Volterra condition means that the value of f at a point (t,x,z) in the space Σ depends on (t,x) and on the restriction of z to the set E_t.

In a similar way, we define the Volterra condition for functions g and G. We assume that f, g and G satisfy the Volterra condition and we consider classical solutions of (1)–(3). We approximate these solutions with solutions of associated implicit difference functional equations.

We are interested in establishing a method of numerical approximation of classical solutions to (1)–(3) by means of solutions of an associated implicit difference schemes and error estimates.

The classical difference methods for nonlinear partial differential or functional equations consist in replacing partial derivatives by suitable difference operators. Then, under suitable assumptions on given functions and on the mesh, solutions of difference or functional difference equations approximate solutions of the original problem. The method of difference inequalities and theorems on recurrent inequalities are used in the investigation of the stability of nonlinear difference schemes, The proofs of the convergence are based on a general theorem on error estimate of approximate solutions for functional difference equations of the Volterra type with unknown function in several independent variables.

Finite difference approximations of the initial boundary value problems of the Neumann type for parabolic differential or functional differential equations are considered in [**1, 4**].

Difference methods for nonlinear parabolic equations with nonlinear boundary conditions are investigated in [**10, 16, 17**]. Numerical treatment of initial boundary value problem of the Dirichlet type can be found in [**5, 9, 14**]. Finite difference approximations of initial problems are presented in [**8**].

Error estimates implying the convergence of explicit difference schemes are obtained in those papers by using a comparison technique.

Papers [**11–13**] initiated the theory of implicit difference methods for nonlinear parabolic differential equations. Classical solutions of initial boundary value problems of the Dirichlet type for nonlinear equations without mixed derivatives are approximated in [**11, 12**] by solutions of difference schemes which are implicit with respect to time variable. Paper [**13**] deals with the initial boundary value problem of the Neumann type for nonlinear equations with mixed derivatives. Implicit difference methods for nonlinear parabolic differential functional equations with initial boundary conditions of the Dirichlet type are investigated in [**2, 6, 7**]. The proofs of the convergence of implicit difference schemes are based on the method of difference inequalities.

In the paper, we start the investigation of implicit difference methods for quasilinear parabolic functional differential systems with initial boundary conditions of the Neumann type. We prove that under natural assumptions on given functions and on the mesh, there is a class of implicit difference schemes for (1)–(3) and those schemes are convergent. The stability of the methods is investigated by using a comparison method. It is important in our considerations that we assume the nonlinear estimates of the Perron type for given functions with respect to the functional variable. As a consequence we obtain nonlinear differential equations with a retarded variable as comparison problems for (1)–(3). We show in examples that a connection with a functional differential problem is essential for the convergence analysis of the difference schemes.

The paper is organized as follows. In Section 2 we construct a class of implicit difference schemes for (1)–(3). The existence and uniqueness of approximate solutions, which are not so obvious as in the case of the explicit methods, are proved in Section 3. In Section 4, which is the main part of the paper, we give sufficient conditions for the convergence of implicit difference schemes. Finally, numerical examples are presented in the last part of the paper.

Natural specification of given operators enables the results of this paper to be applied to differential systems with deviated variables and to differential functional problems.

2. Discretization of mixed problems. We will write $\mathbf{F}(X,Y)$ to denote the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let $\mathbf{N}$ and $\mathbf{Z}$ denote the set of natural numbers and the set of integers, respectively. For $x,y\in\mathbf{R}^n$, $U\in M[n]$, where $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$, $U=[u_{ij}]_{i,j=1,\dots,n}$ we write

$$||x||=\sum_{i=1}^{n}|x_i|,\quad \|p\|_\infty=\max\{|p_i|:\ 1\le i\le k\},$$
$$\|U\|=\max\Big\{\sum_{j=1}^{n}|u_{ij}|:\ 1\le i\le n\Big\},$$

and $x*y=(x_1y_1,\dots,x_ny_n)$. For a function $z\in C(E,R^k)$, we put

$$\|z\|_t=\max\{\|z(\tilde{\tau},x)\|_\infty:\ (\tilde{\tau},x)\in E_t\},\quad 0\le t\le a.$$

We now formulate a difference problem corresponding to (1)–(3). We define a mesh on E in the following way. Let (h_0,h'), where $h'=(h_1,\dots,h_n)$, stand for steps of the mesh. For $h=(h_0,h')$ and $(r,m)\in\mathbf{Z}^{1+n}$, where $m=(m_1,\dots,m_n)$, we define nodal points as follows

$$t^{(r)}=rh_0,\quad x^{(m)}=m*h',\quad x^{(m)}=(x_1^{(m_1)},\dots,x_n^{(m_n)}).$$

Let us denote by H the set of all $h=(h_0,h')$ such that there exist $(N_1,\dots,N_n)=N\in\mathbf{N}^n$ satisfying the condition $N*h'=b$. We write $\|h\|=h_0+h_1+\ldots+h_n$. Let $N_0\in\mathbf{N}$ be defined by the relation $N_0h_0\le a<(N_0+1)h_0$. For $h\in H$, we put

$$\mathbf{R}_h^{1+n}=\{(t^{(r)},x^{(m)}):\ (r,m)\in\mathbf{Z}^{1+n}\}$$

and

$$E_h=E\cap\mathbf{R}_h^{1+n},\quad \partial_0E_h=\partial_0E\cap\mathbf{R}_h^{1+n},$$
$$\partial_0E_{h.j}=\partial_0E_j\cap\mathbf{R}_h^{1+n},\quad j=1,\dots,n,$$
$$E'_h=\{(t^{(r)},x^{(m)})\in E_h:\ 0\le r\le N_0-1\},$$
$$\Sigma_h=E'_h\times C(E,\mathbf{R}^k).$$

Put $E_{h.r}=E_h\cap\big([0,t^{(r)}]\times\mathbf{R}^n\big)$, where $0\le r\le N_0$. For a function $z:E_h\to\mathbf{R}^k$, we write $z^{(r,m)}=z(t^{(r)},x^{(m)})$ and

$$\|z\|_{h.r}=\max\{\|z^{(i,m)}\|_\infty:\ (t^{(i)},x^{(m)})\in E_{h.r}\},\ 0\le r\le N_0.$$

For each $m\in\mathbf{Z}^n$ such that $x^{(m)}\in[-b,b]\setminus(-b,b)$, we consider the class of $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbf{Z}^n$ satisfying the conditions:

i) $\|\alpha\|=1$ or $\|\alpha\|=2$,

ii) if $m=(m_1,\dots,m_n)$ and there is j, $1\le j\le n$, such that $m_j=N_j$, then $\alpha_j\in\{0,1\}$,

iii) if $m = (m_1, \ldots, m_n)$ and there is j, $1 \le j \le n$, such that $m_j = -N_j$, then $\alpha_j \in \{-1, 0\}$.

iv) if $m = (m_1, \ldots, m_n)$ and there is j, $1 \le j \le n$, such that $-N_j < m_j < N_j$, then $\alpha_j = 0$.

The set of all $\alpha \in \mathbf{Z}^n$ satisfying the above conditions will be denoted by $A^{(m)}$. Let us define the following sets.

$$\partial E_h^+ = \{(t^{(r)}, x^{(m+\alpha)}) : \ (t^{(r)}, x^{(m)}) \in \partial_0 E_h \text{ and } \alpha \in A^{(m)}\},$$

$$E_h^+ = \partial E_h^+ \cup E_h.$$

Equation (1) contains the functional variable z which is an element of space $C(E, \mathbf{R}^k)$. Then we need an interpolating operator T_h: $\mathbf{F}(E_h, \mathbf{R}^k) \to \mathbf{C}(E, \mathbf{R}^k)$. We give the following example of such operator. Put

$$\Im = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \ \lambda_i \in \{0, 1\} \text{ for } 0 \le i \le n\}.$$

Let $w \in F(E_h, \mathbf{R})$ and $(t, x) \in E$. There exists $(r, m) \in \mathbf{Z}^{1+n}$ such that $t^{(r)} \le t \le t^{(r+1)}$, $x^{(m)} \le x \le x^{(m+1)}$ and $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in E_h$, where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$. We define

$$T_h[w](t,x) = \frac{t - t^{(r)}}{h_0} \sum_{\lambda \in \Im} w^{(r+1, m+\lambda)} \left(\frac{x - x^{(m)}}{h}\right)^{\lambda} \left(1 - \frac{x - x^{(m)}}{h}\right)^{1-\lambda}$$
$$+ \left(1 - \frac{t - t^{(r)}}{h_0}\right) \sum_{\lambda \in \Im} w^{(r, m+\lambda)} \left(\frac{x - x^{(m)}}{h}\right)^{\lambda} \left(1 - \frac{x - x^{(m)}}{h}\right)^{1-\lambda},$$

where

$$\left(\frac{x - x^{(m)}}{h}\right)^{\lambda} = \prod_{i=1}^{n} \left(\frac{x_i - x_i^{(m_i)}}{h_i}\right)^{\lambda_i},$$

$$\left(1 - \frac{x - x^{(m)}}{h}\right)^{1-\lambda} = \prod_{i=1}^{n} \left(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\right)^{1-\lambda_i}$$

and we assume $0^0 = 1$ in the above formulas. Then we have defined $T_h[w]$ on E. For a function $z \in \mathbf{F}(E_h, \mathbf{R}^k)$, $z = (z_1, \ldots, z_k)$, we put $T_h[z] = (T_h[z_1], \ldots, T_h[z_k])$. It easily follows that $T_h[z] \in C(E, \mathbf{R}^k)$.

The above interpolating operator has been first proposed in [**3**]. Put

$$I = \{(i, j) : \ 1 \le i, j \le n, i \ne j\}.$$

Let $\zeta : E_h^+ \to \mathbf{R}$ and $-N \le m \le N$. We define

$$\delta_i^+ \zeta^{(r,m)} = \frac{1}{h_i} \left(\zeta^{(r, m+e_i)} - \zeta^{(r,m)}\right), \quad \delta_i^- \zeta^{(r,m)} = \frac{1}{h_i} \left(\zeta^{(r,m)} - \zeta^{(r, m-e_i)}\right),$$

where $1 \le i \le n$. Suppose that functions

$$\varphi_h : [-b, b] \to \mathbf{R}^k, \text{ and } \varphi_{h.j} : \partial_0 E_{h.j} \to \mathbf{R}^k, \quad 1 \le j \le n,$$

are given. We approximate solutions of (1)–(3) with solutions of the difference equations

$$\begin{aligned} \delta_0 z_s^{(r,m)} &= \sum_{i,j=1}^{n} f_{s.ij}(P^{(r,m)}[z])\delta_{ij} z_s^{(r+1,m)} \\ &+ \sum_{i=1}^{n} g_{s.i}(P^{(r,m)}[z])\delta_i z_s^{(r+1,m)} + G_s(P^{(r,m)}[z]),\ -N \le m \le N,\ 1 \le s \le k, \end{aligned} \tag{4}$$

$$z^{(r,m+\alpha)} = z^{(r,m-\alpha)} + 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{h.j}^{(r,m)} \text{ on } \partial_0 E_h, \quad \alpha \in A^{(m)}, \tag{5}$$

with the initial condition

$$z^{(0,m)} = \varphi_h^{(m)} \text{ for } x^{(m)} \in [-b, b], \tag{6}$$

where $P^{(r,m)}[z] = (t^{(r)}, x^{(m)}, T_h[z])$. Difference operators

$$\delta_0,\ \delta = (\delta_1, \ldots, \delta_n),\ \delta^{(2)} = [\delta_{ij}]_{i,j=1,\ldots,n}$$

are defined in the following way

$$\delta_0 z_s^{(r,m)} = \frac{1}{h_0}\left(z_s^{(r+1,m)} - z_s^{(r,m)}\right), \tag{7}$$

$$\delta_i z_s^{(r+1,m)} = \frac{1}{2}\left(\delta_i^+ z_s^{(r+1,m)} + \delta_i^- z_s^{(r+1,m)}\right), \quad 1 \le i \le n, \tag{8}$$

and

$$\delta_{ii} z_s^{(r+1,m)} = \delta_i^+ \delta_i^- z_s^{(r+1,m)} \text{ for } 1 \le i \le n, \tag{9}$$

where $1 \le s \le k$. The difference expressions $\delta_{ij} z_s^{(r+1,m)}$ for $(i,j) \in J$ are defined in the following way:

$$\text{if } f_{s.ij}(P^{(r,m)}[z]) \le 0, \text{ then } \delta_{ij} z_s^{(r+1,m)} = \frac{1}{2}\left(\delta_i^+ \delta_j^- z_s^{(r+1,m)} + \delta_i^- \delta_j^+ z_s^{(r+1,m)}\right), \tag{10}$$

$$\text{if } f_{s.ij}(P^{(r,m)}[z]) > 0, \text{ then } \delta_{ij} z_s^{(r+1,m)} = \frac{1}{2}\left(\delta_i^+ \delta_j^+ z_s^{(r+1,m)} + \delta_i^- \delta_j^- z_s^{(r+1,m)}\right). \tag{11}$$

Difference functional problem (4)–(6) with $\delta_0,\ \delta,\ \delta^{(2)}$ defined by (7)–(11) is considered as an implicit difference method for (1)–(3). It is important in our considerations that the difference expressions $\delta_i z,\ \delta_{ij} z,\ 1 \le i, j \le n$, appear in (4) at the point $(t^{(r+1)}, x^{(m)})$. The corresponding explicit difference scheme consists of (5), (6) and the system of equations

$$\begin{aligned}\delta_0 z_s^{(r,m)} &= \sum_{i,j=1}^{n} f_{s.ij}(P^{(r,m)}[z])\delta_{ij} z_s^{(r,m)} \\ &+ \sum_{i=1}^{n} g_{s.i}(P^{(r,m)}[z])\delta_i z_s^{(r,m)} + G_s(P^{(r,m)}[z]),\ -N \le m \le N,\ 1 \le s \le k.\end{aligned} \tag{12}$$

It is clear that there exists exactly one solution of problem (5), (6), (12). We prove that under natural assumptions on given functions there exists exactly one solution $u_h : E_h^+ \to \mathbf{R}^k$ of implicit difference problem (4)–(6).

Lemma 1. *Suppose that $\tilde{z} : E \to \mathbf{R}^k$ and*

1) $\tilde{z}(t, \cdot) : [-b, b] \to \mathbf{R}^k$ *is of class* C^2 *for* $t \in [0, a]$ *and* $\tilde{z}_h = \tilde{z}|_{E_h}$,
2) $\tilde{d} \in \mathbf{R}_+$ *is such a constant that*

$$\|\partial_{x_j x_k} \tilde{z}(t, x)\|_\infty \le \tilde{d},\ (t, x) \in E, \quad j, k = 1, \ldots, n, \tag{13}$$

3) *there is* $L \in \mathbf{R}_+$ *such that*

$$\|\tilde{z}(t, x) - \tilde{z}(\bar{t}, x)\|_\infty \le L|t - \bar{t}|. \tag{14}$$

Then

$$\|T_h[\tilde{z}_h] - z\|_t \le Lh_0 + \tilde{d}\|h'\|^2, \quad 0 \le t \le k. \tag{15}$$

A proof of the above lemma can be found in [**7**].

Estimate (15) states that the function $\tilde{z}$ is approximated by $T_h[\tilde{z}_h]$ and the error of this approximation is estimated by $Lh_0 + \tilde{d}\|h'\|^2$.

It is easy to prove by induction with respect to n that

$$\sum_{\lambda \in \Im} \left(\frac{x - x^{(m)}}{h'}\right)^{\lambda} \left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\lambda} = 1 \text{ for } x^{(m)} \le x \le x^{(m+1)}.$$

This gives $\|T_h[z]\|_{t^{(r)}} = \|z\|_{h.r}$, $0 \le r \le N_0$, where $z \in \mathbf{F}(E_h, \mathbf{R}^k)$.

3. Solutions of difference functional problems.

For a function $z \colon E_h^* \to \mathbf{R}^k$ and a point $t^{(r)}, x^{(m)}) \in E_h$, we put

$$J_{s.-}^{(r,m)}[z] = \{(i, j) \in J :\ f_{s.ij}(P^{(r,m)}[z]) \le 0\}, \quad J_{s.+}^{(r,m)}[z] = J \setminus J_{s.-}^{(r,m)}[z]. \tag{16}$$

Assumption H$[f, g]$. Suppose that functions $f_s : \Sigma \to M[n]$ and $g_s : \Sigma \to \mathbf{R}^n$, $1 \le s \le k$, are bounded on Σ_h and

$$-\frac{1}{2}|g_{s.i}(P)| + \frac{1}{h_i} f_{s.ii}(P) - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{h_j} |f_{s.ij}(P)| \ge 0, \quad i = 1, \dots, n, \tag{17}$$

for $1 \le s \le k$, $P = (t, x, z) \in \Sigma$.

Remark 1. Suppose that for each s, $1 \le s \le k$, there is

$$f_{s.ii}(P) - \sum_{\substack{j=1 \\ j \ne j}}^{n} f_{s.ij}(P) \ge \varepsilon, \quad P \in \Sigma, \tag{18}$$

where $\varepsilon > 0$, and $h_1 = h_2 = \ldots = h_n$ are sufficiently small. Then condition (17) is satisfied. Note that condition (18) implies that

$$\sum_{i,j=1}^{n} f_{s.ij}(P) \xi_i \xi_j \ge 0, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n,$$

which means that problem (1), (2) is parabolic, as defined in [**15**].

Lemma 2. *If Assumption H[f, g] is satisfied and* $\varphi_h : [-b, b] \to \mathbf{R}^k$, *and* $\varphi_{h.j} : \partial_0 E_{h.j} \to \mathbf{R}^k$, $\varphi_{h.j} = (\varphi_{h.1.j}, \dots, \varphi_{h.k.j})$, $j = 1, \dots, n$, *then there is exactly one solution* $u_h : E_h^+ \to \mathbf{R}^k$, $u_h = (u_{h.1}, \dots, u_{h.1})$, *of problem (4)–(6).*

Proof. Suppose that $0 \le r \le N_0 - 1$ is fixed and the solution u_h of (4)–(6) is defined on $E_{h.r}$. We prove that the vectors $u_h^{(r+1,m)}$, where $(t^{(r+1)}, x^{(m)}) \in E_h^+$, exist and are unique. There is $Q_h > 0$ such that

$$Q_h \ge 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} f_{s.ii}(P^{(r,m)}[u_h]) + h_0 \sum_{(i,j) \in J} \frac{1}{h_i h_j} |f_{s.ij}(P^{(r,m)}[u_h])|, \ 1 \le s \le k. \tag{19}$$

Problem (4)–(6) is equivalent to the system of equations

$$\begin{aligned} z_s^{(r+1,m)} = \frac{1}{Q_h + 1} \Big[& Q_h z_s^{(r+1,m)} + u_{h.s}^{(r,m)} \\ & + h_0 \sum_{i,j=1}^{n} f_{s.ij}(P^{(r,m)}[u_h]) \delta_{ij} z_s^{(r+1,m)} \\ & + h_0 \sum_{i=1}^{n} g_{s.i}(P^{(r,m)}[u_h]) \delta_i z_s^{(r+1,m)} + h_0 G_s(P^{(r,m)}[u_h]) \Big], \end{aligned} \tag{20}$$

where $-N \le m \le N$, $1 \le s \le k$, and

$$z(t^{(r+1)}, x^{(m+\alpha)}) = z(t^{(r+1)}, x^{(m-\alpha)}) + 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{h.j}(t^{(r+1)}, x^{(m)}), \tag{21}$$

where $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_h$, $\alpha \in A^{(m)}$, and $z^{(r+1,m)}$ are unknown. Write

$$S_h = \{x^{(m)} :\ (t^{(r+1)}, x^{(m)}) \in E_h^+\}.$$

We consider the space $\mathbf{F}(S_h, \mathbf{R}^k)$. Elements of $\mathbf{F}(S_h, \mathbf{R}^k)$ are denoted by ξ, $\bar{\xi}$. For $\xi \in \mathbf{F}(S_h, \mathbf{R}^k)$, $\xi = (\xi_1, \ldots, \xi_k)$, we write $\xi^{(m)} = \xi(x^{(m)})$ and

$$\delta\xi_s^{(m)} = (\delta_1\xi_s^{(m)}, \ldots, \delta_n\xi_s^{(m)}),\ \delta^{(2)}\xi_s^{(m)} = \big[\delta_{ij}\xi_s^{(m)}\big]_{i,j=1,\ldots,n},\ 1 \le s \le k,$$

where δ_i and δ_{ij}, $1 \le i, j \le n$, are defined by (8)–(11). The norm in the space $\mathbf{F}(S_h, \mathbf{R}^k)$ is defined by

$$\|\xi\|_* = \max\{\|\xi^{(m)}\|_\infty :\ x^{(m)} \in S_h\}.$$

We consider the linear operator

$$U_{h.r} : F(S_h, \mathbf{R}^k) \to F(S_h, \mathbf{R}^k), \quad U_{h.r}[\xi] = (U_{h.r.1}[\xi], \ldots, U_{h.r.k}[\xi]),$$

defined by

$$\begin{aligned} U_{h.r.s}[\xi]^{(m)} = \frac{1}{Q_h + 1}\Big[&Q_h\xi_s^{(m)} + h_0 \sum_{i,j=1}^{n} f_{s.ij}(P^{(r,m)}[u_h])\delta_{ij}\xi_s^{(r+1,m)} \\ &+ h_0 \sum_{i=1}^{n} g_{s.i}(P^{(r,m)}[u_h])\delta_i\xi_s^{(r+1,m)}\Big], \end{aligned} \tag{22}$$

where $-N \le m \le N$ and

$$U_{h.r.s}[\xi]^{(m+\alpha)} = U_{h.r.s}[\xi]^{(m-\alpha)} \text{ on } \partial_0 E_h,\ \alpha \in A^{(m)}. \tag{23}$$

We prove that for $\xi \in \mathbf{F}(S_h, \mathbf{R}^k)$ there holds

$$||U_{h.r}[\xi]||_* \le \frac{Q_h}{1 + Q_h}||\xi||_*. \tag{24}$$

Write

$$A_{i.s.+}^{(r,m)}[z] = \frac{h_0}{2h_i} g_{s.i}(P^{(r,m)}[z]) + \frac{h_0}{h_i^2} f_{s.ii}(P_s^{(r,m)}[z]) - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{h_0}{h_i h_j} |f_{s.ij}(P_s^{(r,m)}[z])|,$$

$$A_{i.s.-}^{(r,m)}[z] = -\frac{h_0}{2h_i} g_{s.i}(P^{(r,m)}[z]) + \frac{h_0}{h_i^2} f_{s.ii}(P_s^{(r,m)}[z]) - \sum_{\substack{j=1 \\ j \ne i}}^{n} \frac{h_0}{h_i h_j} |f_{s.ij}(P_s^{(r,m)}[z])|,$$

$$A_s^{(r,m)}[z] = -2\sum_{i=1}^{n} \frac{h_0}{h_i^2} f_{s.ii}(P_s^{(r,m)}[z]) + \sum_{(i,j)\in J}^{n} \frac{h_0}{h_i h_j} |f_{s.ii}(P_s^{(r,m)}[z])|,$$

where $1 \le i \le n$, $1 \le s \le k$.

From Assumption H$[f, g]$ there follows that for each m, $-N \le m \le N$, $1 \le s \le k$, there is

$$\begin{aligned}
|U_{h.r.s}[\xi]^{(m)}|(Q_h+1) &\le |(Q_h + A_s^{(r,m)})[u_h]^{(m)}| \\
&+ \left|\sum_{i=1}^{n} A_{+.i.s}^{(r,m)}[u_h]^{(m+e_i)}\right| + \left|\sum_{i=1}^{n} A_{-.i.s}^{(r,m)}[u_h]^{(m-e_i)}\right| \\
+h_0 \sum_{(i,j)\in J_{s.+}^{(r,m)}[u_h]}^{n} &\frac{1}{2h_i h_j} f_{s.ii}(P^{(r,m)}[u_h]) \Big[|\xi_s^{(m+e_i+e_j)}| + |\xi_s^{(m-e_i-e_j)}| \Big] \\
-h_0 \sum_{(i,j)\in J_{s.-}^{(r,m)}[u_h]}^{n} &\frac{1}{2h_i h_j} f_{s.ij}(P^{(r,m)}[u_h]) \Big[|\xi_s^{(m+e_i-e_j)}| + |\xi_s^{(m-e_i+e_j)}| \Big].
\end{aligned}$$

From Assumption H$[f_h]$ and (19), there follows:

$$Q_h + A_s^{(r,m)}[u_h] \ge 0,\ A_{i.s.+}^{(r,m)}[u_h] \ge 0, \quad A_{i.s.-}[u_h]^{(r,m)} \ge 0,$$
$$1 \le i \le n, \quad 1 \le s \le k,$$

and

$$\begin{aligned}
A^{(r,m)}[u_h] + \sum_{i=1}^{n} A_{i.s.+}^{(r,m)}[u_h] &+ \sum_{i=1}^{n} A_{i.s.-}^{(r,m)}[u_h] \\
&+ h_0 \sum_{(i,j)\in J_{s.+}^{(r,m)}[u_h]}^{n} \frac{1}{2h_i h_j} f_{s.ij}(P^{(r,m)}[u_h]) \\
&- h_0 \sum_{(i,j)\in J_{s.-}^{(r,m)}[u_h]}^{n} \frac{1}{2h_i h_j} f_{s.ij}(P^{(r,m)}[u_h]) = 0.
\end{aligned}$$

Thus we get

$$\|U_{r.h}[\xi]^{(m)}\|_* \le \frac{Q_h}{Q_h+1} ||\xi||_*, \quad -N \le m \le N.$$

From (23), we conclude that the above inequality is satisfied for $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_h$. This completes the proof of (24). It follows that the norm of the operator $U_{h.r}$ is less that 1. Then there exists exactly one solution of (20)–(21). Since u_h is given on the initial set $\{0\} \times [-b, b]$, the proof of the lemma is completed by induction with respect to r, $0 \le r \le N_0$. $\square$

Now we formulate assumptions on the regularity of G, f_s, g_s, $a \le s \le k$, with respect to the functional variables.

Assumption H$[\sigma, f, g, G]$. Suppose that Assumption H$[f]$ is satisfied and

1) there are $\sigma : [0,a] \times [0, \mathbf{R}_+] \to \mathbf{R}_+$ and $\mu : [0,a] \to \mathbf{R}_+$ such that
   i) σ is continuous, nondecreasing with respect to the both variables and $\sigma(t,0) = 0$ for $t \in [0,a]$,
   ii) μ is continuous, nondecreasing and $\mu(t) \le t$ for $t \in [0,a]$,
   iii) for each $c \ge 1$, the maximal solution of the Cauchy problem

$$\zeta'(t) = c\sigma(t, \zeta(\mu(t))),\ \zeta(0) = 0, \tag{25}$$

   is $\tilde{\zeta}(t) = 0$ for $t \in [0,a]$,
2) the estimates

$$\|f_s(t,x,z) - f_s(t,x,\bar{z})\| \le \sigma(t, ||z - \bar{z}||_{\mu(t)}),$$
$$\|g_s(t,x,z) - g_s(t,x,\bar{z})\| \le \sigma(t, ||z - \bar{z}||_{\mu(t)}),$$
$$\|G_s(t,x,z) - G_s(t,x,\bar{z})\| \le \sigma(t, ||z - \bar{z}||_{\mu(t)}),$$

   where $1 \le s \le k$, are satisfied on Σ.

For a function $\eta : I_h \to \mathbf{R}$, we write $\eta^{(r)} = \eta(t^{(r)})$.

Now we can prove a theorem on the convergence of method (4)–(6).

Theorem 3.1. *Suppose that Assumption H$[\sigma, f, g, G]$ is satisfied, $\Omega \in \mathbf{R}^{1+n}$ is an open, bounded set such that $E \subset \Omega$ and*

1) *the function $v : \Omega \to \mathbf{R}^k$, $v = (v_1, \dots, v_k)$, is the solution of (1)–(3) and v is of class C^2 on Ω, and $\tilde{c} \in \mathbf{R}_+$ is defined by the relations*

$$\|\partial_{x_i} v(t,x)\|_\infty,\ \left\|\partial_{x_i x_j} v(t,x)\right\|_\infty \le \tilde{c}, \quad i,j = 1,\dots,n,\ (t,x) \in E,$$

2) *there is $c_0 > 0$ such that $h_i h_j^{-1} \le c_0$, $i,j = 1,\dots,n$,*
3) *there exists $\bar{c} \in \mathbf{R}$ such that $\|h'\|^2 \le \bar{c} h_0$,*
4) *the function $u_h : E_h \to \mathbf{R}^k$, $u = (u_1, \dots, u_k)$, is a solution of (4)–(6) and there are $\gamma_0, \gamma_1 : H \to \mathbf{R}_+$ such that*

$$\|v^{(r,m)} - u_h^{(r,m)}\|_\infty \le \gamma_0(h) \text{ on } E_{0.h}, \tag{26}$$

$$\|v_h^{(r,m+\alpha)} - v_h^{(r,m-\alpha)} - 2\sum_{j=1}^n \alpha_j h_j \varphi_{j.h}^{(r,m)}\|_\infty \le \gamma_1(h)\|h'\|^2 \quad \text{on } \partial_0 E_h,\ \alpha \in A^{(m)}, \tag{27}$$

and

$$\lim_{h\to 0} \gamma_0(h) = 0, \quad \lim_{h \to 0} \gamma_1(h) = 0.$$

Then there exists a function $\alpha : H \to \mathbf{R}_+$ *such that*

$$\|u_h^{(r,m)} - v_h^{(r,m)}\|_\infty \le \alpha(h) \quad \text{on} \quad E'_h \quad \text{and} \quad \lim_{h\to 0}\alpha(h) = 0, \tag{28}$$

where v_h *is the restriction of* v *to the set* E_h*.*

Proof. Let $\Gamma_h : E'_h \to \mathbf{R}^k,\ \Gamma_{0.h} : E_{0.h} \to \mathbf{R}^k,\ \Gamma_{\partial.h} : \partial_0 E_h \to \mathbf{R}^k$ be defined by the relations

$$\delta_0 v_h^{(r,m)} = F_h[v_h]^{(r,m)} + \Gamma_h^{(r,m)} \text{ on } E'_h,$$

$$v_h^{(r,m+\alpha)} - v_h^{(r,m-\alpha)} = 2\sum_{j=1}^n \alpha_j h_j \varphi_{j.h}^{(r,m)} + \Gamma_{\partial.h}^{(r,m)} \text{ on } \partial_0 E_h \text{ and } \alpha \in A^{(m)},$$

$$v_h^{(0,m)} = \varphi_h^{(m)} + \Gamma_{0.h}^{(m)}, \text{ for } x^{(m)} \in [-b,b].$$

From Lemma 1, condition 1 of the Theorem and (26), (27) there follows, that there are $\gamma, \gamma_1, \gamma_0 : H \to \mathbf{R}_+$ such that

$$\left\|\Gamma_h^{(r,m)}\right\|_\infty \le \gamma(h) \text{ on } E'_h, \ \left\|\Gamma_{\partial.h}^{(r,m)}\right\|_\infty \le \gamma_1(h)\|h'\|^2 \text{ on } \partial_0 E_h,$$

$$\left\|\Gamma_{0.h}^{(m)}\right\|_\infty \le \gamma_0(h) \text{ for } x^{(m)} \in [-b,b]$$

and

$$\lim_{h\to 0}\gamma(h) = 0, \ \lim_{h\to 0}\gamma_0(h) = 0, \ \lim_{h\to 0}\gamma_1(h) = 0.$$

Write $z_h = v_h - u_h$. Then

$$\begin{aligned}\delta_0 z_{h.s}^{(r,m)} = &\sum_{i,j=1}^n f_{s.ij}(P^{(r,m)}[u_h])\delta_{ij} z_h^{(r+1,m)} \\ &+ \sum_{i=1}^n g_{s.i}(P^{(r,m)}[u_h])\delta_i z_h^{(r+1,m)} + \Lambda_{h.s}^{(r,m)} + \Gamma_{h.s}^{(r,m)},\end{aligned}$$

where

$$\begin{aligned}\Lambda_{h.s}^{(r,m)} = &\sum_{i,j=1}^n [f_{s.ij}(P^{(r,m)}[v_h]) - f_{s.ij}(P^{(r,m)}[u_h])]\delta_{ij} v_{h.s}^{(r+1,m)} \\ &+ \sum_{i=1}^n [g_{s.i}(P^{(r,m)}[v_h]) - g_{s.i}(P^{(r,m)}[u_h])]\delta_i v_{h.s}^{(r+1,m)} \\ &+ G_s(P^{(r,m)}[v_h]) - G_s(P^{(r,m)}[u_h]),\end{aligned}$$

and $1 \le s \le k$. The above relations and (7)–(11) imply

(29)

$$
\begin{aligned}
&z_{h.s}^{(r+1,m)}\Big[1 - A_s^{(r,m)}[u_h]\Big] \\
&= h_0 z_{h.s}^{(r,m)} + h_0 \sum_{i=1}^{n} A_{i.s.+}^{(r,m)}[u_h] z_{h.s}^{(r+1,m+e_i)} + h_0 \sum_{i=1}^{n} A_{i.s.-}^{(r,m)}[u_h] z_{h.s}^{(r+1,m-e_i)} \\
&\quad - h_0 \sum_{(i,j)\in J_{s.-}^{(r,m)}[u_h]} \frac{1}{2h_ih_j} f_{h.s.ij}(P^{(r,m)}[u_h])\Big[z_{h.s}^{(r+1,m+e_i-e_j)} + z_{h.s}^{(r+1,m-e_i+e_j)}\Big] \\
&\quad + h_0 \sum_{(i,j)\in J_{s.+}^{(r,m)}[u_h]} \frac{1}{2h_ih_j} f_{h.s.ij}(P^{(r,m)}[u_h])\Big[z_{h.s}^{(r+1,m+e_i+e_j)} + z_{h.s}^{(r+1,m-e_i-e_j)}\Big] \\
&\quad + h_0 \Lambda_{h.s}^{(r,m)} + h_0 \Gamma_{h.s}^{(r,m)}.
\end{aligned}
$$

From (17) we conclude that

$$
A_{i.s.+}^{(r,m)}[u_h] \geq 0, \quad A_{i.s.-}^{(r,m)}[u_h] \geq 0, \quad 1 - A_s^{(r,m)}[u_h] \geq 0 \quad \text{for } 1 \leq i \leq n \tag{30}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{n} A_{i.s.+}^{(r,m)}[u_h] + \sum_{i=1}^{n} A_{i.s.-}^{(r,m)}[u_h] \\
&\qquad\qquad + \sum_{(i,j)\in J} \frac{1}{h_ih_j} |f_{h.ij}(P^{(r,m)}[u_h])| + A_s^{(r,m)}[u_h] = 0.
\end{aligned} \tag{31}
$$

Write $I_{h_0} = [t^{(0)}, t^{(1)}, \ldots, t^{(N_0)}]$. Let functions $\varepsilon_h^{(r)}$, $\tilde{\varepsilon}_h^{(r)} : I_{h_0} \to \mathbf{R}^+$ be defined by

$$
\varepsilon_h^{(r)} = \max\Big\{\|z_h^{(r,m)}\|_\infty : \ (t^{(r)}, x^{(m)}) \in E_{h.r}\Big)\Big\},
$$

$$
\tilde{\varepsilon}_h^{(r)} = \max\Big\{\|z_h^{(r,m)}\|_\infty : \ (t^{(r)}, x^{(m)}) \in E_h^+ \cap \big([0, t^{(r)}] \times \mathbf{R}^n\big)\Big\},
$$

where $0 \leq r \leq N_0$.

Let us introduce an operator T_{h_0} of linear interpolation on I_{h_0}. If $\zeta : I_{h_0} \to \mathbf{R}^+$ then $T_{h_0} : [0, a] \to \mathbf{R}$ is defined by

$$
T_{h_0}[\zeta](t) = \zeta^{(r+1)}(t - t^{(r)})h_0^{-1} + \zeta^{(r)}[1 - (t - t^{(r)})h_0^{-1}], \quad t^{(r)} \leq t \leq t^{(r+1)},
$$

where $\zeta^{(r)} = \zeta(t^{(r)})$. One can observe that

$$
\|T_h[u_h - v_h]\|_{\mu(t^{(r)})} = T_{h_0}[\varepsilon_h](\mu(t^{(r)})).
$$

From (29)–(31) it follows that the function ε_h satisfies the recurrent inequality

$$
\varepsilon_h^{(r+1)} \leq \tilde{\varepsilon}_h^{(r)} + (2\tilde{c} + 1)h_0\sigma(t^{(r)}, T_{h_0}[\varepsilon_h](\mu(t^{(r)}))) + h_0\gamma(h),
$$

where $0 \le r \le N_0 - 1$. It is easily seen that

$$\tilde{\varepsilon}_h^{(r)} \le \varepsilon_h^{(r)} + h_0\gamma_1(h)\bar{c}, \quad 0 \le r \le N_0 - 1.$$

Thus we see that the function ε_h satisfies the recurrent inequality

$$\varepsilon_h^{(r+1)} \le \varepsilon_h^{(r)} + (2\tilde{c}+1)h_0\sigma_h(t^{(r)}, T_{h_0}[\varepsilon_h](\mu(t^{(r)}))) + h_0\left(\gamma(h) + \bar{c}\gamma_1(h)\right), \tag{32}$$

where $0 \le r \le N_0 - 1$ and $\varepsilon_h^{(0)} \le \gamma_0(h)$.
Consider the Cauchy problem

$$\zeta'(t) = (2\tilde{c}+1)\sigma(t, T_{h_0}[\varepsilon_h](\mu(t^{(r)}))) + (\gamma(h) + \bar{c}\gamma_1(h)), \quad \zeta(0) = \gamma_0(h). \tag{33}$$

It is clear that there exists $\varepsilon_0 > 0$ such that the maximal solution ω_h of (33) is defined on $[0,a]$ for $\|h\| \le \varepsilon_0$. Moreover,

$$\lim_{h\to 0} \omega_h(t) = 0 \text{ uniformly on } [0,a].$$

Then ω_h satisfies the recurrent inequality

$$\omega_h^{(r+1)} \le \omega_h^{(r)} + (2\tilde{c}+1)h_0\sigma_h(t^{(r)}, T_{h_0}[\varepsilon_h](\mu(t^{(r)}))) + h_0\left(\gamma(h) + \bar{c}\gamma_1(h)\right), \tag{34}$$

where $0 \le r \le N_0 - 1$. From (32)–(34) it follows that

$$\varepsilon_h^{(r)} \le \omega_h^{(r)} \text{ for } r = 1, \ldots, N_0,$$

and, consequently,

$$\varepsilon_h^{(r)} \le \omega_h(a) \text{ for } r = 1, \ldots, N_0.$$

Then the assertion of the theorem is satisfied with $\alpha(h) = \omega_h(a)$ and the theorem is proved. □

Remark 2. Suppose that Assumption H$[\sigma, f, g]$ is satisfied with

$$\sigma(t,p) = Lp, \;\; (t,p) \in [0,a] \times \mathbf{R}_+, \quad \text{where } L \in \mathbf{R}_+.$$

Then we have assumed that f and g satisfy the Lipschitz condition with respect to the functional variable. We obtain the following error estimates

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \le \tilde{\alpha}(h)e^{cLa} + \tilde{\gamma}(h)\frac{e^{cLa}-1}{cL} \text{ on } E_h \text{ if } L > 0,$$

and

$$\|u_h^{(i,m)} - v_h^{(i,m)}\| \le \tilde{\alpha}(h) + a\tilde{\gamma}(h) \text{ on } E_h \text{ if } L = 0.$$

The above inequalities follow from (28) with $\alpha(h) = \omega_h(a)$, where $\omega_h : [0,a] \to \mathbf{R}_+$ is a solution of the problem

$$\zeta'(t) = cL\zeta(t) + \tilde{\gamma}(h), \quad \zeta(0) = \alpha_0(h).$$

Remark 3. Theorem (3.1) holds true if Assumption H$[\sigma, f, g, G]$ is replaced by the following conditions.

Assumption H$[\sigma_0, f, g, G]$. Suppose that Assumption H$[f]$ is satisfied and

1) there are $\sigma_0 : [0,a] \times \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, $\mu : [0,a] \to \mathbf{R}_+$ such that
   i) σ_0 is continuous, nondecreasing with respect to all variables and $\sigma_0(t,a,0) = 0$ for $t \in [0,a]$,
   ii) μ is continuous, nondecreasing and $\mu(t) \leq t$ for $t \in [0,a]$,
   iii) for each $c \geq 1$, the maximal solution of the Cauchy problem

$$\zeta'(t) = c\sigma_0(t, \zeta(t), \zeta(\mu(t))), \quad \zeta(0) = 0, \tag{35}$$

   is $\bar{\zeta}(t) = 0$ for $t \in [0,a]$,

2) the expressions

$$\|f_s(t,x,z) - f_s(t,x,\bar{z})\|, \ \|g_s(t,x,z) - g_s(t,x,\bar{z})\|, \ \|G_s(t,x,z) - G_s(t,x,\bar{z})\|,$$

   where $1 \leq s \leq k$, are bounded by $\sigma_0(t, ||z-\bar{z}||_t, ||z-\bar{z}||_{\mu(t)})$.

Remark 4. Let us consider explicit difference method (5)–(12). Then we need the following assumption on f and on the steps of the mesh ([**4, 5, 8, 9**]):

$$1 - 2h_0 \sum_{j=1}^{n} \frac{1}{h_j^2} f_{jj}(P) + h_0 \sum_{(i,j)\in J} \frac{1}{h_i h_j} |f_{ij}(P)| \geq 0, \tag{36}$$

where $P \in \Sigma$. If the functions f_{ij}, $i,j = 1,\ldots,1$, are bounded on Σ, then inequality (36) states relations between h_0 and $h' = (h_1,\ldots,h_n)$. It is important in our considerations that condition (36) is omitted in the convergence theorem.

Remark 5. Note that the connection with the functional differential comparison problem is essential in our considerations. The following lemma points out this property.

Lemma 3. *If $\beta \geq \alpha > 1$ and $L \in \mathbf{R}_+$, $c \geq 1$, then the maximal solution of the Cauchy problem*

$$\zeta'(t) = c\left[\zeta(t^\beta)\right]^{\frac{1}{\alpha}} + L\zeta(t), \quad \zeta(0) = 0, \tag{37}$$

is $\bar{\zeta}(t) = 0$ for $t \in [0,a]$, where $a < 1$.

Proof. There are $\tilde{\varepsilon} > 0$ and $\tilde{c} > 0$ such that the maximal solution $\bar{\zeta}$ of (37) satisfies the condition

$$\bar{\zeta}(t) \leq \tilde{C}t \quad \text{for} \quad t \in [0,\tilde{\varepsilon}].$$

Write

$$C = \max\{c, L, \tilde{C}\}.$$

Then $\bar{\zeta}$ satisfies the integral inequality

$$\zeta(t) \leq C\left[\int_0^t \left[\zeta(s^\beta)\right]^{\frac{1}{\alpha}} ds + \int_0^t \zeta(s)ds\right], \quad t \in [0,\tilde{\varepsilon}],$$

and $\bar{\zeta}(t) \leq Ct$ for $t \in [0, \tilde{\varepsilon}]$.

From the above relations it follows that

$$\bar{\zeta}(t) \leq C^k t^k, \quad t \in [0, \tilde{\varepsilon}], \quad k \geq 1.$$

Then there is ε_0 such that $\bar{\zeta} = 0$ for $t \in [0, \varepsilon_0]$ and, consequently, $\bar{\zeta}(t) = 0$ on $[0, a]$. □

Remark 6. Note that the maximal solution of (37) with $\alpha > 1$ and $\beta = 1$ is positive on $(0, a]$.

Now we formulate relations between assumptions on the regularity of given functions in stability theorem presented in [**1, 2**] and our results. For simplicity we assume that $k = 1$. It is assumed in the above papers that there is $\tilde{\sigma}$: $[0, a] \times \mathbf{R}_+ \to \mathbf{R}_+$ such that

1) $\tilde{\sigma}$ is continuous and nondecreasing with respect to both variables and $\tilde{\sigma}(t, a) = 0$ for $t \in [0, a]$,
2) for each $c \geq 1$ the function $\tilde{\zeta}(t) = 0$ is the maximal solution of the Cauchy problem

$$\zeta'(t) = c\sigma(t, \zeta(t)), \quad \zeta(0) = 0, \quad \text{for } t \in [0, a], \tag{38}$$

3) the terms

$$\|f(t, x, z) - f(t, x, \bar{z})\|, \ \|g(t, x, z) - g(t, x, \bar{z})\|, \ |G(t, x, z) - G(t, x, \bar{z})|$$

are bounded from above by $\tilde{\sigma}(t, \|z - \bar{z}\|_t)$.

It is important in our considerations that classical comparison problem (38) is replaced with (35), which is the Cauchy problem for an equation with retarded variable.

From Lemma 3 it follows that there are functional differential comparison problems of the Perron type and the corresponding classical initial problems have positive maximal solutions.

In Section 4, we give examples of equations which satisfy our comparison conditions.

4. Numerical examples.

Example 1. Write

$$E = [0, 0.3] \times [-1, 1] \times [-1, 1],$$
$$\partial_0 E = [0, 0.2] \times \big[\big([-1, 1] \times [-1, 1]\big) \setminus \big((-1, 1) \times (-1, 1)\big)\big].$$

Consider the differential equation with deviated variables

$$\begin{aligned} \partial_t z(t, x, y) &= \partial_{xx} z(t, x, y) + \partial_{yy} z(t, x, y) + xy\partial_{xy} z(t, x, y) \\ &\quad + \sqrt{|z(t^2, x, y)|} + f(t, x, y) z(t, x, y) \end{aligned} \tag{39}$$

and the initial boundary conditions

(40) $$z(0,x,y)=1 \quad \text{for } (x,y)\in[-1,1]\times[-1,1],$$

(41) $$\partial_x z(t,0,y)=1,\ \partial_x z(t,1,y)=e^{ty} \quad \text{for } t\in[0,0.2],\ y\in[-1,1],$$

(42) $$\partial_y z(t,x,0)=1,\ \partial_y z(t,x,1)=e^{tx} \quad \text{for } t\in[0,0.2],\ x\in[-1,1],$$

where

$$f(t,x,y)=xy(1-t)-t^2(x^2+y^2+x^2y^2)-e^{xy}(\frac{t^2}{2}-t).$$

The solution of (39)–(42) is known to be

$$v(t,x,y)=e^{txy}.$$

Remark 7. Write

$$G(t,x,y,z)=\sqrt{z(t^2,x,y)}+f(t,x,y)z(t,x,y).$$

Then

$$|G(t,x,y,z)-G(t,x,y,\bar{z})|\le\sqrt{\|z-\bar{z}\|_{t^2}}+L\|z-\bar{z}\|_t,\ (t,x,y)\in E.$$

It follows that condition 2) of Assumption $[\sigma_0,f,g,G]$ is satisfied with

$$\sigma_0(t,\tau,s)=\sqrt{s}+L\tau,\ \mu(t)=t^2.$$

Write

$$\varepsilon_h^{(r)}=\frac{1}{N_1\cdot N_2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}|u_h^{(r,i,j)}-v_h^{(r,i,j)}|,$$

$$\tilde{\varepsilon}_h^{(r)}=\max_{\substack{1\le i\le N_1\\ 1\le j\le N_2}}|u_h^{(r,i,j)}-v_h^{(r,i,j)}|.$$

We found the approximate solutions of (39)–(42) using both implicit and explicit numerical methods, and taking the following set of steps of the mesh: $h_0=0.001,\ h_1=0.002,\ h_2=0.002$.

Note that the function f and the steps of the mesh do not satisfy condition (36) which is necessary for the explicit method to be convergent. In our numerical example the average errors of the explicit method exceeded 10^{140}, while the average errors ε_h for fixed $t^{(r)}$ of implicit method are given in the following table.

Table of errors $\tilde{\varepsilon}_h$, ε_h

t	$\tilde{\varepsilon}_h(t)$	$\varepsilon_h(t)$
0.20	$41 \cdot 10^{-6}$	$80 \cdot 10^{-6}$
0.21	$34 \cdot 10^{-5}$	$10 \cdot 10^{-5}$
0.22	$44 \cdot 10^{-5}$	$13 \cdot 10^{-5}$
0.23	$55 \cdot 10^{-5}$	$17 \cdot 10^{-5}$
0.24	$68 \cdot 10^{-5}$	$21 \cdot 10^{-5}$
0.25	$81 \cdot 10^{-5}$	$26 \cdot 10^{-5}$
0.26	$96 \cdot 10^{-5}$	$31 \cdot 10^{-5}$
0.27	$11 \cdot 10^{-4}$	$36 \cdot 10^{-5}$
0.28	$12 \cdot 10^{-4}$	$42 \cdot 10^{-5}$
0.29	$14 \cdot 10^{-4}$	$48 \cdot 10^{-5}$
0.30	$10 \cdot 10^{-4}$	$54 \cdot 10^{-5}$

Example 2. Write

$$E = [0, 0.2] \times [-1, 1] \times [-1, 1],$$

$$\partial_0 E = [0, 0.2] \times \big[\big([0,1] \times [0,1]\big) \setminus \big((0,1) \times (0,1)\big)\big].$$

Let us consider the integral-differential equation

$$\begin{aligned} \partial_t z(t,x,y) &= \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) - \partial_{xy} z(t,x,y) \\ &\quad + \frac{\pi^4}{16} \int_{-1}^{x} \int_{-1}^{y} z(t,\tau,s)\,ds\,d\tau - \int_0^t z(s,x,y)\,ds \\ &\quad + \frac{\pi^2}{2} z(t,x,y) + \sin\frac{\pi x}{2} \sin\frac{\pi y}{2} \end{aligned} \tag{43}$$

and the initial boundary conditions

$$z(0,x,y) = 0 \quad \text{for } (x,y) \in [0,1] \times [0,1], \tag{44}$$

$$\partial_x z(t,-1,y) = 0,\ \partial_x z(t,1,y) = 0 \quad \text{for } t \in [0, 0.2],\ y \in [-1,1], \tag{45}$$

$$\partial_y z(t,x,-1) = 0,\ \partial_y z(t,x,1) = 0 \quad \text{for } t \in [0, 0.2],\ x \in [-1,1]. \tag{46}$$

The solution of (43)–(46) is known to be

$$v(t,x,y) = \sin t \sin\frac{\pi x}{2} \sin\frac{\pi y}{2}.$$

As in the previous numerical example, we choose the steps of the mesh which do not satisfy condition (36). As we expected, the explicit method is not convergent, and the average errors are larger than 10^{150}, while the implicit method is convergent and gives the following average errors.

Table of errors $\tilde{\varepsilon}_h$, ε_h

t	$\tilde{\varepsilon}_h(t)$	$\varepsilon_h(t)$
0.10	$26 \cdot 10^{-4}$	$10 \cdot 10^{-4}$
0.11	$30 \cdot 10^{-4}$	$12 \cdot 10^{-4}$
0.12	$35 \cdot 10^{-4}$	$14 \cdot 10^{-4}$
0.13	$40 \cdot 10^{-4}$	$16 \cdot 10^{-4}$
0.14	$46 \cdot 10^{-4}$	$19 \cdot 10^{-4}$
0.15	$53 \cdot 10^{-4}$	$22 \cdot 10^{-4}$
0.16	$61 \cdot 10^{-4}$	$26 \cdot 10^{-4}$
0.17	$70 \cdot 10^{-4}$	$31 \cdot 10^{-4}$
0.18	$80 \cdot 10^{-4}$	$38 \cdot 10^{-4}$
0.19	$93 \cdot 10^{-4}$	$46 \cdot 10^{-4}$
0.20	$10 \cdot 10^{-3}$	$57 \cdot 10^{-4}$

Example 3. Write

$$E = [0, 0.2] \times [-1, 1] \times [-1, 1],$$

$$\partial_0 E = [0, 0.2] \times \big[\big([-1, 1] \times [-1, 1]\big) \setminus \big((-1, 1) \times (-1, 1)\big)\big].$$

Let us consider the integral-differential equation

$$\begin{aligned} \partial_t z(t, x, y) &= \partial_{xx} z(t, x, y) + \partial_{yy} z(t, x, y) + \partial_{xy} z(t, x, y) \\ &\quad + \sqrt{\int_0^{t^2} z(\tau, x, y) d\tau} + f(t, x, y) z(t, x, y) \end{aligned} \tag{47}$$

and the initial boundary conditions

$$z(0, x, y) = 0 \text{ for } (x, y) \in [-1, 1] \times [-1, 1], \tag{48}$$

$$\begin{gathered} \partial_x z(t, -1, y) = 2y \sin t e^{-2y}, \quad \partial_x z(t, 1, y) = 2y \sin t e^{2y} \\ \text{for } t \in [0, 0.2],\ y \in [-1, 1], \end{gathered} \tag{49}$$

$$\begin{gathered} \partial_y z(t, x, -1) = 2x \sin t e^{-2x}, \quad \partial_y z(t, x, 1) = 2x \sin t e^{2x} \\ \text{for } t \in [0, 0.2],\ x \in [-1, 1], \end{gathered} \tag{50}$$

where

$$f(t, x, y) = -4 \left[x^2 + y^2 + xy + \frac{\sqrt{1 - \cos t^2}}{4 e^{xy} \sin t} - \frac{1}{4} \tan t \right].$$

The solution of (47)–(50) is known to be

$$v(t, x, y) = \sin t e^{2xy}.$$

REMARK 8. Write

$$G(t,x,y,z) = \sqrt{\int_0^{t^2} z(\tau,x,y)d\tau} + f(t,x,y)z(t,x,y).$$

Then

$$|G(t,x,y,z) - G(t,x,y,\bar{z})| \le \sqrt{\int_0^{t^2} \|z-\bar{z}\|_\tau d\tau} + L\|z-\bar{z}\|_t$$

$$\le \sqrt{\|z-\bar{z}\|_{t^2}} + L\|z-\bar{z}\|_t, \quad (t,x,y) \in E.$$

It follows that condition 2) of Assumption $\mathrm{H}[\sigma_0, f, g, G]$ is satisfied with

$$\sigma_0(t,\tau,s) = \sqrt{s} + L\tau, \ \ \mu(t) = t^2.$$

As in the previous numerical example, we choose the steps of the mesh which do not satisfy condition (36). As expected, the explicit method is not convergent, and the average errors are larger than 10^{230}, while the implicit method is convergent and yields the following average errors.

TABLE OF ERRORS $\tilde{\varepsilon}_h$, ε_h

t	$\tilde{\varepsilon}_h(t)$	$\varepsilon_h(t)$
0.10	$26 \cdot 10^{-4}$	$10 \cdot 10^{-4}$
0.11	$30 \cdot 10^{-4}$	$12 \cdot 10^{-4}$
0.12	$35 \cdot 10^{-4}$	$14 \cdot 10^{-4}$
0.13	$40 \cdot 10^{-4}$	$16 \cdot 10^{-4}$
0.14	$46 \cdot 10^{-4}$	$19 \cdot 10^{-4}$
0.15	$53 \cdot 10^{-4}$	$22 \cdot 10^{-4}$
0.16	$61 \cdot 10^{-4}$	$26 \cdot 10^{-4}$
0.17	$70 \cdot 10^{-4}$	$31 \cdot 10^{-4}$
0.18	$80 \cdot 10^{-4}$	$38 \cdot 10^{-4}$
0.19	$93 \cdot 10^{-4}$	$45 \cdot 10^{-4}$
0.20	$10 \cdot 10^{-3}$	$58 \cdot 10^{-4}$

As in the first example, the functional nature of the comparison problem is necessary here.

The above examples show that there are implicit difference schemes which are convergent, while the corresponding classical methods are not convergent. This is due to the fact that we need relation (36) for steps of the mesh in the classical case. We do not need this condition in our implicit method. Implicit difference methods presented in this paper have the potential for applications in the numerical solving of differential integral equations or equations with deviated variables.

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Received August 16, 2006

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