ON THE ORDER OF HOLOMORPHIC AND C-HOLOMORPHIC FUNCTIONS

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Abstract. In the first part of this paper we prove that the Łojasiewicz exponent of a non-constant holomorphic germ $f\colon (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ is a good exponent for f coinciding with the order of vanishing of f at zero and the degree at zero of its cycle of zeroes Z_f. As an application of this result we show that for any holomorphic curve germ $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^m, 0)$ one has $\mathrm{ord}_0(f \circ \gamma) = \mathrm{ord}_0 f \cdot \mathrm{ord}_0 \gamma$ if and only if γ is transversal to $f^{-1}(0)$ at zero. In a recent paper we have introduced an order of flatness for c-holomorphic functions which allowed us to give some bounds on the Łojasiewicz exponent of c-holomorphic mappings. Answering a question of A. Płoski we show that both notions (the order of flatness and the Łojasiewicz exponent) are intrinsic to the analytic set given (this allows to carry these notions over to analytic spaces). We turn then to considerations about possible Łojasiewicz exponents of c-holomorphic mappings. The last part deals with quotients of c-holomorphic functions. We investigate relations between this newly introduced order of flatness and the possibility of dividing one c-holomorphic function by another.

1. Introduction. For the convenience of the reader we recall some basic notions.

DEFINITION 1.1. We say that a continuous mapping $f\colon \Omega \to \mathbb{C}^n$, where $\Omega \subset \mathbb{C}^m$ is non-empty, satisfies the *Łojasiewicz inequality at the point* $a \in f^{-1}(0)$, if there exist positive constants $\alpha, C > 0$ such that the inequality

$$(\#) \qquad |f(z)| \geq C \mathrm{dist}(z, f^{-1}(0))^\alpha,$$

where the distance is computed in one of the usual norms in $\mathbb{C}^m$, holds in a neighbourhood of a. Note that it is clearly a condition on germs.

By [**7**], every holomorphic mapping f satisfies the Łojasiewicz inequality at each point of its zero set (it still holds true for c-holomorphic mappings by [**3**], see section 4 for the definition). It is natural to consider the *Łojasiewicz exponent* of f at $a \in f^{-1}(0)$ defined to be

$$\mathcal{L}(f; 0) = \inf\{\alpha > 0 \mid (\#) \text{ is satisfied in a neighbourhood of } a\}.$$

By the generalized Mean Value Theorem, it is easy to see that $\mathcal{L}(f; 0) \geq 1$ in the holomorphic case (see e.g. [**3**]).

In the first part of this paper we shall consider the following situation: $\Omega \subset \mathbb{C}^m$ is an open neighbourhood of 0 and $f \colon \Omega \to \mathbb{C}$ is a non constant holomorphic function such that $f(0) = 0$. Let Γ_f be the graph of f. The intersection $\Gamma_f \cap (\Omega \times \{0\})$ is proper, i.e., has the minimal possible (pure) dimension, namely $m - 1$. Therefore, using Draper's results from [**6**], we can define the *proper intersection cycle*

$$Z_f := \Gamma_f \cdot (\Omega \times \{0\}) = \sum_\iota \alpha_\iota S_\iota,$$

where $S_\iota \subset f^{-1}(0)$ are irreducible components, and $\alpha_\iota = i(\Gamma_f \cdot (\Omega \times \{0\}); S_\iota)$ are the intersection multiplicities along S_ι ([**6**]).

We may assume that $0 \in S_\iota$ for each ι and $\iota \in \{1, \ldots, r\}$.

DEFINITION 1.2. We call degree of the cycle Z_f at a point $a \in f^{-1}(0)$, the number

$$\deg_a Z_f := \sum_{j=1}^r \alpha_j \deg_a S_j,$$

where $\deg_a S_j$ stands for the classical degree of the analytic set S_j at the point a. (If $a \notin S_j$, then $\deg_a S_j = 0$ by definition.)

Finally, three more notations: let $\operatorname{ord}_z f$ denote the *order of vanishing* of f at $z \in \Omega$ i.e. $\operatorname{ord}_z f = \min\{\alpha_1 + \ldots + \alpha_m \mid \frac{\partial^{\alpha_1 + \ldots + \alpha_m} f}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m}(z) \neq 0\}$; if $F \colon \Omega \to \mathbb{C}^m$ is a holomorphic proper mapping, then $m_z(F)$ denotes its *(geometric) multiplicity* at $z \in \Omega$ (i.e. the generic number of points in the fibre $F^{-1}(w)$ for w close to $F(z)$, see e.g. [**2**]); if $Z \subset \Omega$ is analytic and $a \in Z$, then $C_a(Z)$ is its *tangent cone* at a.

2. A remark on the degree of the cycle of zeroes.

PROPOSITION 2.1. *In the introduced setting,* $\deg_0 Z_f = \operatorname{ord}_0 f$.

PROOF. Let us choose coordinates in $\mathbb{C}^m$ so that the degree is realized in the following manner: $\deg_0 Z_f = i(Z_f \cdot \{z_1 = \ldots = z_{m-1} = 0\}; 0)$. Then there must be $C_0(f^{-1}(0)) \cap (\{0\}^m \times \mathbb{C}) = \{0\}^{m+1}$ (transversality). But the intersection multiplicity above coincides, by [**2**], p. 140 (the intersection multiplicity being associative in the proper intersection case), with the multiplicity $m_0(f, z_1, \ldots, z_{m-1})$ and so by a Tsikh–Yuzhakov result (see [**2**], p. 112), in view of the transversality of the intersection, we obtain $m_0(f, z_1, \ldots, z_{m-1}) = \operatorname{ord}_0 f \prod_{j=1}^{m-1} \operatorname{ord}_0 z_j = \operatorname{ord}_0 f$, which is the result sought for. $\qquad\square$

Let $k_j := \min\{\operatorname{ord}_z f \mid z \in \operatorname{Reg} S_j\}$ for $j = 1, \ldots, r$. Obviously $k_j \geq 1$.

PROPOSITION 2.2. *For each* $j \in \{1, \ldots, r\}$, $\operatorname{ord}_z f = k_j$ *for* $z \in S_j$ *apart from a nowheredense analytic subset of* S_j *(i.e. for the generic* $z \in S_j$*). Moreover,* $k_j = \alpha_j$.

PROOF. The set $Z_j := \{z \in \operatorname{Reg} S_j \mid D^\alpha f(z) = 0, |\alpha| \leq k_j\}$ is clearly analytic and nowheredense in $\operatorname{Reg} S_j$. This gives the first part of the assertion.

So as to compute the intersection multiplicity $\alpha_j = i(\Gamma_f \cdot (\Omega \times \{0\}); S_j)$ along S_j, we take a generic point $a \in \operatorname{Reg} S_j$ and any affine complex line $L \subset \mathbb{C}^m$ through a, transversal to S_j. Then we obtain $i(Z_f \cdot L; a) = \alpha_j i(S_j \cdot L; a)$ as the isolated proper intersection multiplicity. Transversality means that $i(S_j \cdot L; a) = \deg_a S_j$ and the latter is equal to 1 by the choice of a. Hence $\alpha_j = i(Z_f \cdot L; a)$ and the latter coincides with $\deg_a Z_f$ by the choice of L. We may as well assume that $a \notin Z_j$.

Take linear forms $l_1, \ldots, l_{m-1}$ on $\mathbb{C}^m$ such that $L - a = \bigcap_j \operatorname{Ker} l_j$. Then the mapping $\varphi(x) = (f(x), l_1(x-a), \ldots, l_{m-1}(x-a)) = (\varphi_1(x), \ldots, \varphi_m(x))$ has an isolated zero at a. After a linear change of coordinates $y = x - a$ we may apply the preceding proof. By assumptions, we have transversality $\bigcap_{\iota=1}^m C_a(\varphi_\iota^{-1}(0)) = \{0\}^m$. Thus the result of Tsikh–Yuzhakov leads to $m_a(\varphi) = \prod_{\iota=1}^m \operatorname{ord}_a \varphi_\iota = \operatorname{ord}_a f$, but $m_a(\varphi) = \deg_a Z_f$. Finally, the choice of a ensures that $\operatorname{ord}_a f = k_j$. $\qquad\square$

Now using the Weierstrass Preparation Theorem, we link the degree $\deg_0 Z_f$ with the Łojasiewicz exponent of f.

THEOREM 2.3. *If* $f \colon (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *is a non-constant holomorphic germ, then* $\mathcal{L}(f; 0) = \operatorname{ord}_0 f = \deg_0 Z_f$ *and* (#) *is satisfied with this exponent.*

PROOF. We may assume that $f \colon \Omega \to \mathbb{C}$ is holomorphic and coordinates in $\mathbb{C}^m = \mathbb{C}^{m-1} \times \mathbb{C}$ are chosen in such a way that

$$C_0(f^{-1}(0)) \cap (\{0\}^m \times \mathbb{C}) = \{0\}^{m+1}.$$

Then applying the Weierstrass Preparation Theorem to f in a neighbourhood $U \times V$ of $0 \in \mathbb{C}^{m-1} \times \mathbb{C}$, we obtain a distinguished Weierstrass polynomial $P \in \mathcal{O}(U)[t]$ and a holomorphic function $h \in \mathcal{O}(U \times V)$ with empty zero set and such that $f = hP$ in $U \times V$. Then $\mathrm{ord}_0 f = \mathrm{ord}_0 P$ and the latter is equal to $\deg P(x, \cdot)$ by the choice of the coordinates.

Indeed, $C_0(f^{-1}(0)) = C_0(P^{-1}(0)) = \mathrm{in}\, P^{-1}(0)$, where $\mathrm{in}\, P$ denotes the initial form in the expansion of P into homogenous forms near zero. If $P(x,t) = t^d + a_1(x)t^{d-1} + \ldots + a_d(x)$ with a_j holomorphic in U, then clearly $\mathrm{ord}_0 P = \min\{d, \mathrm{ord}_0 a_1 + d - 1, \ldots, \mathrm{ord}_0 a_d\}$. The condition on the tangent cone is equivalent to $\mathrm{ord}_0 a_j \geq j$ for $j = 1, \ldots, d$ and so in this case $\mathrm{ord}_0 P = d = \deg P(x, \cdot)$.

Shrinking U if necessary, we may assume that $c := \inf_U |h| > 0$. Take a point $x \in U$ for which there are exactly d distinct roots $t_1(x), \ldots, t_d(x)$ of P. Then $|f(x,t)| \geq c \prod_{j=1}^d |t - t_j(x)|$ and it is obvious that for each j, $|t - t_j(x)| \geq \mathrm{dist}((x,t), f^{-1}(0))$. Therefore, $|f(x,t)| \geq c\,\mathrm{dist}((x,t), f^{-1}(0))^d$ and by continuity this holds for all $(x,t) \in U \times V$. Hence $\mathcal{L}(f;0) \leq d = \mathrm{ord}_0 f$.

On the other hand, by [3], Lemma (4.8), we have $\mathcal{L}(f;0) \geq \mathrm{ord}_0 f$. This, together with Proposition 2.1, gives the result. $\qquad\square$

NOTE. To prove that $\mathcal{L}(f;0) \leq \mathrm{ord}_0 f$ one can also recall theorem (4.9) from [3] making use of the main result of [1] by which there is $\mathcal{L}(f;0) \leq \deg_0 Z_f$, since $\mathcal{L}(f;0)$ coincides in the holomorphic case with the Łojasiewicz regular separation exponent of Γ_f and $\Omega \times \{0\}$ (see e.g. [3] (2.5)). Proposition 2.1 then yields the result.

3. Application. As an example of application of our preceding result we shall give here a useful formula for the order of vanishing of a holomorphic function restricted to an analytic curve. It seems that such a theorem was not written anywhere till now. Recall that for any non-constant holomorphic map germ $h\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ one has $\mathrm{ord}_0 h = \min_{j=1}^n \mathrm{ord}_0 h_j$.

THEOREM 3.1. *Let* $f\colon (\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ *be a holomorphic germ and let* $\Gamma \subset \mathbb{C}^m$ *be an analytic irreducible curve germ at zero. Then for any local parametrization* $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^m, 0)$ *of* Γ *the following three statements are equivalent:*

  (i) $\mathrm{ord}_0(f \circ \gamma) = \mathrm{ord}_0 f \cdot \mathrm{ord}_0 \gamma$;

  (ii) *The tangent cones at zero of* $f^{-1}(0)$ *and* Γ *intersect only at zero:*

$$C_0(f^{-1}(0)) \cap C_0(\Gamma) = \{0\};$$

  (iii) *There exist positive constants* $\varepsilon, C > 0$ *such that*

$$C|\gamma(t)| \leq \mathrm{dist}(\gamma(t), f^{-1}(0)), \quad |t| < \varepsilon.$$

PROOF. It is easy to see (see e.g. [**3**]) that for any non-constant holomorphic map germ $h\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ one has

$$\operatorname{ord}_0 h = \max\{\eta > 0 \mid |h(x)| \le \operatorname{const}.|x|^\eta \text{ holds in a neighbourhood of } 0\}.$$

It follows thence that $\operatorname{ord}_0(f \circ \gamma) \ge \operatorname{ord}_0 f \cdot \operatorname{ord}_0 \gamma$ always holds.

The equivalence (ii)$\Leftrightarrow$(iii) follows directly from [**10**]. Let us start with (ii)$\Rightarrow$(i): by [**10**], we know that the assumption on the tangent cones leads to

$$\operatorname{const}.\operatorname{dist}(x, f^{-1}(0) \cap \Gamma) \le \operatorname{dist}(x, f^{-1}(0)) + \operatorname{dist}(x, \Gamma)$$

in a neighbourhood of zero. On the other hand, by the preceding result we have

$$|f(x)| \ge \operatorname{const}.\operatorname{dist}(x, f^{-1}(0))^{\operatorname{ord}_0 f}$$

in a neighbourhood of zero. Now, since $\operatorname{dist}(x, f^{-1}(0) \cap \Gamma) = |x|$, we obtain taking $x \in \Gamma$,

$$\operatorname{const}.|\gamma(t)| \le |f(\gamma(t))|^{1/\operatorname{ord}_0 f}, \quad |t| < \varepsilon,$$

with some suitable $\varepsilon > 0$. It is known that $\mathcal{L}(\gamma, 0) = \operatorname{ord}_0 \gamma$ since γ is a curve (see e.g. [**3**] and Section 4) and so for some constants $C, C' > 0$,

$$C|t|^{\operatorname{ord}_0 \gamma} \le |f(\gamma(t))|^{1/\operatorname{ord}_0 f} \le C'|t|^{\operatorname{ord}_0(f \circ \gamma)/\operatorname{ord}_0 f},$$

for $|t| < \varepsilon' \le \varepsilon$, whence

$$0 < \frac{C}{C'} \le |t|^{(\operatorname{ord}_0(f \circ \gamma)/\operatorname{ord}_0 f) - \operatorname{ord}_0 \gamma}, \quad |t| \le \varepsilon'.$$

Therefore, $\operatorname{ord}_0(f \circ \gamma) \le \operatorname{ord}_0 f \cdot \operatorname{ord}_0 \gamma$.

We turn now to proving (i)$\Rightarrow$(ii). Let $d := \operatorname{ord}_0 f$ and let $f = \sum_{\nu \ge d} f_\nu$ be the expansion of f into a series of homogenous forms near zero (f_ν being a form of degree ν). Let $\operatorname{ord}_0 \gamma = \operatorname{ord}_0 \gamma_1$ where $\gamma = (\gamma_1, \zeta)$, $\zeta = (\gamma_2, \ldots, \gamma_m)$. Obviously, $\operatorname{ord}_0 \zeta \ge \operatorname{ord}_0 \gamma_1$.

In a neighbourhood of zero we can write $\gamma_1(t) = t^{\operatorname{ord}_0 \gamma} \widetilde{\gamma_1}(t)$ with $\widetilde{\gamma_1}$ holomorphic and such that $\widetilde{\gamma_1}(0) \ne 0$, as well as $\zeta(t) = t^{\operatorname{ord}_0 \gamma} \widetilde{\zeta}(t)$ with $\widetilde{\zeta}$ holomorphic (possibly vanishing at zero). Thus for each $\nu \ge d$ we have $f_\nu(\gamma_1(t), \zeta(t)) = t^{d \operatorname{ord}_0 \gamma} t^{(\nu - d)\operatorname{ord}_0 \gamma} f_\nu(\widetilde{\gamma_1}(t), \widetilde{\zeta}(t))$ and so we may write in a neighbourhood of zero

$$f(\gamma_1(t), \zeta(t)) = t^{d \operatorname{ord}_0 \gamma}(f_d(\widetilde{\gamma_1}(t), \widetilde{\zeta}(t)) + R(t)),$$

where R is holomorphic and such that $R(0) = 0$. Now, by assumption,

$$(\star) \qquad 0 \ne \lim_{t \to 0} \frac{f(\gamma_1(t), \zeta(t))}{t^{d \operatorname{ord}_0 \gamma}} = f_d(\widetilde{\gamma_1}(0), \widetilde{\zeta}(0)).$$

Note that $C_0(\Gamma)$ is a complex line, since Γ is irreducible at zero. On the other hand, for any sequence $t_\nu \to 0$, if we put $\lambda_\nu := \frac{1}{t_\nu^{\operatorname{ord}_0 \gamma}}$, then

$$\lambda_\nu \gamma(t_\nu) \to (\widetilde{\gamma_1}(0), \widetilde{\zeta}(0)) \in C_0(\Gamma) \setminus \{0\}.$$

Therefore, $C_0(\Gamma) = \mathbb{C} \cdot (\widetilde{\gamma_1}(0), \widetilde{\zeta}(0))$ and since $C_0(f^{-1}(0)) = f_d^{-1}(0)$, we obtain the result sought thanks to $(\star)$. $\hfill\square$

NOTE. As observed by A. Płoski, the equivalence (i)$\Leftrightarrow$(ii) may also be proved in the following, somewhat more direct, way:

Set $\nu_0 := \mathrm{ord}_0\gamma = \min_j \mathrm{ord}_0\gamma_j$. We may expand each $\gamma_j(t)$ into a power series starting with $a_{j,\nu_0} t^{\nu_0}$. Therefore, $\gamma(t) = at^{\nu_0} + \eta(t)$, where $a = (a_{1,\nu_0}, \ldots, a_{m,\nu_0})$ is non-zero and η is holomorphic such that $\mathrm{ord}_0\eta > \nu_0$. In particular, $\eta(t)/t^{\nu_0}$ has a removable singularity at zero. Then $C_0(\Gamma) = \mathbb{C}a$, since for $0 \neq t_\nu \to 0$ and $\lambda_\nu := 1/(t_\nu)^{\nu_0}$, $\lambda_\nu\gamma(t_\nu) = a + \eta(t_\nu)/(t_\nu)^{\nu_0}$ converges to a. Clearly, (ii) is equivalent to $\mathrm{in}f(a) \neq 0$ (here $\mathrm{in}f$ denotes the initial form of f). Now, if $n_0 := \mathrm{ord}_0 f$ and $f = \sum_{n \geq n_0} f_n$ is the expansion of f into homogenous forms, then $f_n(\gamma(t)) = t^{\nu_0 n} f_n(a + \eta(t)/t^{\nu_0})$. Hence

$$\frac{f(\gamma(t))}{t^{\nu_0 n_0}} = \sum_{n \geq n_0} t^{\nu_0(n-n_0)} f_n\left(a + \frac{\eta(t)}{t^{\nu_0}}\right).$$

Finally, letting $t \to 0$ we obtain $\lim_{t\to 0} f(\gamma(t))/t^{\nu_0 n_0} = f_{n_0}(a)$ and the equivalence follows.

As a corollary to this we have the following completion of [**3**], part 3:

COROLLARY 3.2. *If $\Gamma \subset \mathbb{C}^m$ is an irreducible curve germ at zero and the function germ $f \colon (\Gamma, 0) \to (\mathbb{C}, 0)$ is non-constant and strongly holomorphic, then for any holomorphic extension $F \supset f$ one has $\mathrm{ord}_0 f \geq \mathrm{ord}_0 F$ (where $\mathrm{ord}_0 f$ is computed as for c-holomorphic functions – see below) and equality holds iff $C_0(\Gamma) \cap C_0(F^{-1}(0)) = \{0\}$. In such a case $\mathcal{L}(f; 0) = \mathrm{ord}_0 F$.*

PROOF. It follows from Theorem 3.1 together with [**3**], Theorem (3.2). $\hfill\square$

4. Basic facts about c-holomorphic functions. For the convenience of the reader we recall the definition of a c-holomorphic mapping. Let $A \subset \Omega$ be an analytic subset of an open set $\Omega \subset \mathbb{C}^m$.

DEFINITION 4.1. ([**7, 11**]) A mapping $f \colon A \to \mathbb{C}^n$ is called *c-holomorphic* if it is continuous and the restriction of f to the subset of regular points $\mathrm{Reg}A$ is holomorphic. We denote by $\mathcal{O}_c(A, \mathbb{C}^n)$ the ring of c-holomorphic mappings, and by $\mathcal{O}_c(A)$ the ring of c-holomorphic functions.

It is a way (due to R. Remmert) of generalizing the notion of holomorphic mapping onto sets having singularities and a more convenient one than the usual notion of *weakly holomorphic functions* (i.e. functions defined and holomorphic on $\mathrm{Reg}A$ and locally bounded on A). Recall also that mappings having locally a holomorphic extension to a neighbourhood of the ambient space are called *(strongly) holomorphic*. The notion is obviously valid also on

analytic spaces. The following theorem is fundamental for all what we shall do (cf. [**11**], 4.5Q):

THEOREM 4.2. *A mapping $f\colon A \to \mathbb{C}^n$ is c-holomorphic iff it is continuous and its graph $\Gamma_f := \{(x, f(x)) \mid x \in A\}$ is an analytic subset of $\Omega \times \mathbb{C}^n$.*

We introduce also the following useful criterion of c-holomorphicity.

THEOREM 4.3. *Let $A \subset \Omega$ be an analytic subset of an open set $\Omega \subset \mathbb{C}^m$ and let $f\colon A \to \mathbb{C}^n$ be a mapping.*
If there exist an analytic set A' in an open set $U \subset \mathbb{C}^r$ and a proper c-holomorphic surjection $\varphi\colon A' \to A$ for which $f \circ \varphi \in \mathcal{O}_c(A', \mathbb{C}^n)$, then f is c-holomorphic.

PROOF. Obviously, we may assume that $n = 1$, since a mapping is c-holomorphic if and only if its components are c-holomorphic.

First let us check that f is continuous. Let $F \subset A$ be closed. Then $(f \circ \varphi)^{-1}(F)$ is closed. Since φ is proper, $\varphi((f \circ \varphi)^{-1}(F))$ is a closed set too. Since φ is a surjection, we have actually shown that $f^{-1}(F)$ is closed. Hence Γ_f is a closed set.

So as to check that the graph Γ_f is analytic we proceed as follows. Since f is c-holomorphic iff its restriction to each irreducible component is c-holomorphic, we may assume that A has pure dimension k as well as that $k \geq 1$. Consider the natural projection

$$\pi\colon U \times \Omega \times \mathbb{C} \ni (t, x, y) \to (x, y) \in \Omega \times \mathbb{C}$$

and the analytic set

$$\Gamma := \{(t, x, y) \mid x = \varphi(t), y = f(\varphi(t)), t \in A'\}.$$

It is easy to see that $\Gamma_f = \pi(\Gamma)$. On the other hand, one can easily check that the fibres of the restriction of π to Γ are of the form

$$\big(\varphi^{-1}(\varphi(t)) \cap \{\tilde{t} \in U \mid f(\varphi(\tilde{t})) = f(\varphi(t))\} \big) \times \{\varphi(t)\} \times \{f(\varphi(t))\},$$

where t is fixed. Since φ is proper, it is obvious that the restriction of π to Γ is proper as well. Thanks to Remmert's Proper Mapping Theorem, we conclude that Γ_f is an analytic set. $\qquad\square$

NOTE. It is clear, by Remmert's Theorem, that under the assumptions of this theorem there is $\dim_a A' = \dim_{\varphi(a)} A$.

For a more detailed list of basic properties of c-holomorphic mappings see [**11**, **3**]. In the latter we introduce the *order of flatness* for a non-constant c-holomorphic germ $f\colon (A, 0) \to (\mathbb{C}^n, 0)$ as follows.

DEFINITION 4.4. We call order of flatness of f at zero the number

$$\mathrm{ord}_0 f := \max\{\eta > 0 \mid |f(x)| \leq \mathrm{const} \cdot |x|^\eta \text{ in a neighbourhood of } 0\}.$$

We put by definition $\mathrm{ord}_0 0 := +\infty$.

It is proved in [**3**] that the definition is well-posed, $\mathrm{ord}_0 f \in \mathbb{Q}$ and its denominator is not greater than $\deg_0 A$. It is clear that $\mathrm{ord}_0 f = \min_{j=1}^n \mathrm{ord}_0 f_j$ and for functions (i.e. $n = 1$) there is moreover $\mathrm{ord}_0 f^r = r \cdot \mathrm{ord}_0 f$ for integers $r \geq 1$. By [**3**], Theorem (3.2), if A is an irreducible curve germ at zero parametrized by $\gamma \colon (\mathbb{C}, 0) \to (A, 0)$, then $\mathrm{ord}_0 \gamma = \deg_0 A$ and

$$(*) \qquad\qquad \mathcal{L}(f; 0) = \frac{\mathrm{ord}_0(f \circ \gamma)}{\mathrm{ord}_0 \gamma} = \mathrm{ord}_0 f.$$

5. The Łojasiewicz exponent and the order of flatness of a c-holomorphic mapping are intrinsic. In this section we shall answer a question posed by A. Płoski. When defining the Łojasiewicz exponent or the order of flatness of a c-holomorphic mapping (see above and [**3**]) one uses, so to say, the ambient space – the natural question then is: do these notions depend on the imbedding of the analytic set given or are they intrinsic? Below we prove that both notions are intrinsic. This allows us to carry them over to analytic spaces. The main tool used in the proof is the fact that (strongly) holomorphic functions are locally Lipschitz.

THEOREM 5.1. *Let* $f \colon A \to A'$ *be a c-holomorphic mapping of analytic spaces, mapping the point* $a \in A$ *into* $a' \in A'$. *If* $\varphi \colon (G, a) \to (X, 0)$ *and* $\varphi' \colon (G', a') \to (Y, 0)$ *are two analytic maps, then*

$$\mathcal{L}(f; a) := \mathcal{L}(\varphi' \circ f \circ \varphi^{-1}; 0), \quad \mathrm{ord}_a f := \mathrm{ord}_0(\varphi' \circ f \circ \varphi^{-1})$$

are well-defined as, respectively, the Łojasiewicz exponent and the order of flatness of f *at* $a \in A$.

Having *analytic maps* implies (see [**7**]) that G is a neighbourhood of a in A, $X \subset \mathbb{C}^m$ a locally analytic set, and analogously G' is a neighbourhood of a' in A', Y is locally analytic in $\mathbb{C}^r$.

PROOF OF THEOREM 5.1. Take any two other imbedding analytic maps $\psi \colon A \supset (\widetilde{G}, a) \to (Y, 0) \subset \mathbb{C}^n$ and $\psi' \colon A' \supset (\widetilde{G}', a') \to (W, 0) \subset \mathbb{C}^s$. The problem being local (actually we need to solve it for germs), we may suppose that $\widetilde{G} = G$ and $\widetilde{G}' = G'$. Observe that the strongly holomorphic mappings

$$h := \psi \circ \varphi^{-1} \colon \mathbb{C}^m \supset X \to Y \subset \mathbb{C}^n \text{ and } \widetilde{h} := \psi' \circ \varphi'^{-1} \colon \mathbb{C}^r \supset Z \to W \subset \mathbb{C}^s$$

are both biholomorphisms, mapping zero into zero.

Put $f_X := \varphi' \circ f \circ \varphi^{-1}$ and $f_Y := \psi' \circ f \circ \psi^{-1}$. These are c-holomorphic mappings, respectively $(X,0) \to (Z,0)$ and $(Y,0) \to (W,0)$. There is $f_X = \widetilde{h}^{-1} \circ f_Y \circ h$. Therefore, we need only to prove the following fact:

If $f\colon \mathbb{C}^m \supset (X,0) \to (Z,0) \subset \mathbb{C}^r$ is a c-holomorphic mapping and both $h\colon \mathbb{C}^n \supset (Y,0) \to (X,0)$ and $g\colon (Z,0) \to (W,0) \subset \mathbb{C}^s$ are (strong) biholomorphisms of locally analytic sets X, Y, Z, W, then

$$\mathcal{L}(f;0) = \mathcal{L}(g \circ f \circ h;0) \quad \text{and} \quad \mathrm{ord}_0 f = \mathrm{ord}_0(g \circ f \circ h).$$

We shall prove both equalities at the same time. We assign to each of the considered locally analytic sets neighbourhoods in which they are closed.

Take a neighbourhood U of $0 \in X$, exponents $\alpha, \eta > 0$ and constants $c, C > 0$, such that

$$c \, \mathrm{dist}(x, f^{-1}(0))^\alpha \le |f(x)| \le C|x|^\eta, \quad \text{when } x \in U \cap X.$$

We may assume that U is a ball $B(0, 2r)$ (with $r > 0$ arbitrarily small). Then, for $x \in B(0, r)$, the distance $\mathrm{dist}(x, f^{-1}(0))$ is realized by some point belonging to $B(0, 2r) \cap f^{-1}(0)$.

In view of the fact that g is one-to-one we have $f^{-1}(0) = f^{-1}(g^{-1}(0))$, and g^{-1} being strongly holomorphic it satisfies Lipschitz condition (simply by restriction to W of the Lipschitz condition satisfied by any holomorphic extension $H \supset g^{-1}$ in a neighbourhood of zero) in an arbitrarily small neighbourhood of zero:

$$|g^{-1}(w) - g^{-1}(w')| \le \ell|w - w'|, \quad w, w' \in V \cap W.$$

Taking a smaller radius r, we may assume that $g(f(U \cap X)) \subset V$. Then, putting $w = g(f(x))$, $w' = 0$, we obtain $|f(x)| \le \ell|g(f(x))|$. The same argument with g instead of g^{-1} leads (using the Lipschitz condition satisfied by g) to the inequality $|g(f(x))| \le \ell'|f(x)|$. Finally, combining both inequalities obtained we get for $x \in U \cap X$,

$$(c/\ell)\mathrm{dist}(x, f^{-1}(g^{-1}(0)))^\alpha \le |g(f(x))| \le \ell'C|x|^\eta.$$

In order to shorten notation put $\widetilde{g} := g \circ f$. For any $x \in B(0, r) \cap X$ there exists a point $\hat{x} \in U \cap \widetilde{g}^{-1}(0)$ such that $\mathrm{dist}(x, \widetilde{g}^{-1}(0)) = |x - \hat{x}|$. The set $h^{-1}(U)$ is an open neighbourhood of zero containing uniquely determined points $y, \hat{y}$ such that $h(y) = x$, $h(\hat{y}) = \hat{x}$ and $\hat{y} \in h^{-1}(\widetilde{g}^{-1}(0))$. Besides, by the Lipschitz condition (without loss of generality we assume it holds in the whole of U) satisfied by h^{-1}, we obtain $|y - \hat{y}| \le L|h(y) - h(\hat{y})|$. Therefore,

$$\mathrm{dist}(x, \widetilde{g}^{-1}(0)) = |x - \hat{x}| \ge (1/L)|y - \hat{y}| \ge (1/L)\mathrm{dist}(y, h^{-1}(\widetilde{g}^{-1}(0)))$$

for all $x \in B(0, r)$. On the other hand, h being strongly holomorphic, its order of flatness is ≥ 1 (cf. [3]), and since to each x there is exactly one y such that

$h(y) = x$, we obtain

$$|\widetilde{g}(h(y))| \leq \ell' C |h(y)|^\eta \leq \ell' C K |y|^\eta$$

for $y \in h^{-1}(B(0, r)) =: V$ (taking at any event a smaller r). Thus, for any $y \in V$,

$$(c/\ell L^\alpha)\mathrm{dist}(y, h^{-1}(\widetilde{g}^{-1}(0)))^\alpha \leq |\widetilde{g}(h(y))| \leq \ell' C K |y|^\eta,$$

whence $\mathcal{L}(f; 0) \geq \mathcal{L}(g \circ f \circ h; 0)$ and $\mathrm{ord}_0(g \circ f \circ h) \geq \mathrm{ord}_0 f$.

We obtain the converse inequalities thanks to the same argument using this time $\widetilde{f} := g \circ f \circ h$ instead of f and the biholomorphisms g^{-1}, h^{-1}. $\qquad\square$

NOTE. By the theorem above the Łojasiewicz exponent and the order of flatness do not depend on the imbedding and moreover they are both biholomorphic invariant. It is worth noting that they are not *c-biholomorphic* invariant. It is easy to see by considering the c-biholomorphism $\gamma\colon \mathbb{C} \ni t \mapsto (t^2, t^3) \in \{y^2 = x^3\} =: A$ and the c-holomorphic function $f(x, y) = y/x$ on $A \setminus \{(0,0)\}$, $f(0,0) = 0$. Then $\mathcal{L}(f; 0) = 1/2$ but $\mathcal{L}(f \circ \gamma; 0) = 1$.

Note also that the above f does not satisfy Lipschitz condition in any neighbourhood of zero.

6. Possible Łojasiewicz exponents. In the holomorphic case, if we confine ourselves to non-constant holomorphic germs $f\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$, with m fixed, not all rational numbers can be Łojasiewicz exponents of such maps – see [**8**]. The gaps are filled by c-holomorphic mappings on analytic sets of pure dimension m. To be precise let us introduce the set

$$\mathcal{L}_m := \{q > 0 \mid \exists A \text{ a pure } m\text{-dimensional analytic germ at zero}$$
$$\exists f\colon (A, 0) \to (\mathbb{C}^m, 0) \text{ c-holomorphic, such that}$$
$$f^{-1}(0) = \{0\} \text{ and } \mathcal{L}(f; 0) = q\}.$$

Let $\mathbb{Q}_{>0} := \{q \in \mathbb{Q} \mid q > 0\}$. We already know that $\mathcal{L}_m \subset \mathbb{Q}_{>0}$ ([**3**], Theorem (2.6)). Actually, by [**9**], Corollary 3.1 and Theorem 1.5 this remains true if we drop the assumption $f^{-1}(0) = \{0\}$ in the definition of $\mathcal{L}_m$. To show the converse inclusion we consider the following family of sets and functions: for $r, s \in \mathbb{N}$ such that their greatest common denominator $GCD(r, s) = 1$ and $r < s$ we put $\Gamma_{r,s} := \{(x, y) \in \mathbb{C}^2 \mid y^r = x^s\}$, $f_{r,s}(x, y) = y/x$ for $(x, y) \in \Gamma_{r,s} \setminus \{0\}$, $f_{r,s}(0) = 0$, and $g_{r,s}(x, y) = y$ for $(x, y) \in \Gamma_{r,s}$.

Then $f_{r,s}^{-1}(0) = g_{r,s}^{-1}(0) = \{0\}$ and $\mathcal{L}(f_{r,s}; 0) = (s - r)/r$, $\mathcal{L}(g_{r,s}; 0) = s/r$ (cf. (∗)) and it is clear that both fractions are irreducible (indeed, if $s - r = ka$ and $r = kb$, then $s = k(a + b)$, whence $GCD(r, s) \geq k$). Therefore, we obtain

THEOREM 6.1. $\mathcal{L}_1 = \mathbb{Q}_{>0}$.

PROOF. Consider $p/q \in \mathbb{Q}_{>0}$ with $GCD(p,q) = 1$. Then we investigate two cases:

(i) $p/q > 1$. Then obviously $p/q = \mathcal{L}(g_{p,q}; 0)$.

(ii) $p/q \leq 1$. The case $p/q = 1$ being obvious ($h(x) = x$ on $\mathbb{C}$), we may assume that $p/q < 1$. Then we look for integers $1 \leq r < s$ such that $GCD(r,s) = 1$ and $(s-r)/r = p/q$. Put $r := q$ and $s := p + q$. Clearly, $s > r$ and if $p + q = ka$, $q = kb$, then $p = k(a - b)$ which leads to $GCD(p,q) \geq k$. Thus $p/q = \mathcal{L}(f_{q,p+q}; 0)$. $\qquad \square$

COROLLARY 6.2. $\mathcal{L}_m = \mathbb{Q}_{>0}$ for $m = 1, 2, \ldots$

To prove this corollary let us note the following lemma:

LEMMA 6.3. Let X, Y be locally analytic sets in $\mathbb{C}^k, \mathbb{C}^l$, respectively and let $f \in \mathcal{O}_c(X, \mathbb{C}^k)$, $g \in \mathcal{O}_c(Y, \mathbb{C}^l)$ be such that $f^{-1}(0) = \{0\}$ and $g^{-1}(0) = \{0\}$. Then for $h := f \times g \in \mathcal{O}_c(X \times Y, \mathbb{C}^{k+l})$ one has

$$\mathcal{L}(h; 0) = \max\{\mathcal{L}(f; 0), \mathcal{L}(g; 0)\}.$$

PROOF. It is a straightforward computation:

$$|h(x,y)| = |f(x)| + |g(y)| \geq \text{const} \cdot (|x|^{l_f} + |y|^{l_g}) \geq \text{const} \cdot |(x,y)|^{\max\{l_f, l_g\}},$$

in a neighbourhood of zero in $X \times Y$, where $l_f := \mathcal{L}(f; 0)$, $l_g := \mathcal{L}(g; 0)$ (by [3] these are good exponents). Whence $\mathcal{L}(h; 0) \leq \max\{l_f, l_g\}$.

On the other hand, $|h(x,0)| \geq \text{const} \cdot |(x,0)|^{\mathcal{L}(h;0)}$ in a neighbourhood $0 \in U \times V \subset X \times Y$ yields $|f(x)| \geq \text{const} \cdot |x|^{\mathcal{L}(h;0)}$ for $x \in U$, whence $l_f \leq \mathcal{L}(h; 0)$. The same argument for g gives the result. $\qquad \square$

PROOF OF COROLLARY 6.2. Whenever m is fixed it is clear that it suffices to take $A := (\Gamma_{r,s})^m$ and $h = f_{r,s} \times \ldots \times f_{r,s}$ or $g_{r,s} \times \ldots \times g_{r,s}$ (m times), with an appropriate choice of r, s, to obtain $\mathcal{L}(h; 0) = q$ for any $q \in \mathbb{Q}_{>0}$. $\qquad \square$

What would also be interesting to investigate is the set of possible exponents when the analytic set (germ) A is fixed. Suppose that $0 \in A$ and A is pure k-dimensional in $\mathbb{C}^m$. Put

$$\mathcal{L}(A) := \{q > 0 \mid \exists f \colon (A, 0) \to (\mathbb{C}^k, 0) \text{ } c\text{-holomorphic,}$$
$$\text{non-constant and such that } \mathcal{L}(f; 0) = q\}.$$

We already have the following inclusions:

THEOREM 6.4. In the introduced setting, if $k > 0$, then

$$\mathbb{N} \subset \mathcal{L}(A) \subset \mathbb{Q}_{>0}.$$

PROOF. The second inclusion is a consequence of [**9**], Corollary 3.1 and Theorem 1.5.

The first inclusion follows from lemma (5.4) from [**3**] – since coordinates may be chosen so that $\mathrm{ord}_0(x_j|_A) = 1$, then $\mathrm{ord}_0(x_j^n|_A) = n$ for any positive integer n (x_j^n denotes the n-th power of x_j). Thus for any increasing injection $\lambda\colon \{1,\ldots,k\} \to \{1,\ldots,m\}$ we obtain $\mathcal{L}(f_{\lambda,n};0) \geq n$ for $f_{\lambda,n}(x) := (x_{\lambda(1)}^n,\ldots,x_{\lambda(k)}^n)$ on A (due to [**3**], Theorem (5.5)). To see that in fact n is a good exponent for $f_{\lambda,n}$ we proceed as follows. To simplify notation take $\lambda = \mathrm{id}$. Since coordinates are chosen by means of [**3**], Lemma (5.4), the natural projection onto the first k coordinates realizes $\deg_0 A$. Then in particular

$$A \cap U \subset \{x \in \mathbb{C}^m \mid |(x_{k+1},\ldots,x_m)| \leq \mathrm{const} \cdot |(x_1,\ldots,x_k)|\},$$

where U is a neighbourhood of zero. We may take the norm ℓ_1 (sum of moduli). Thus for $x \in A \cap U$ we have $|x|^n \leq \mathrm{const} \cdot |f_{\lambda,n}(x)|$ and the proof is complete. $\quad\square$

As suggested by A. Płoski, in the one-dimensional case we may say even more. Suppose that Γ is an irreducible curve germ at $0 \in \mathbb{C}^m$. Then let $\gamma\colon (\mathbb{C},0) \to (\mathbb{C}^m,0)$ be its Puiseux parametrization. In particular, $\deg_0 \Gamma = \mathrm{ord}_0\gamma = \min_j \mathrm{ord}_0\gamma_j$. Let us choose neighbourhoods $0 \in U \subset \mathbb{C}$ and $0 \in V \subset \mathbb{C}^m$ such that $\gamma\colon U \to \Gamma \cap V$ is a homeomorphism.

Then for any $a \in \Gamma \cap V$ there is a unique $t_a \in U$ such that $\gamma(t_a) = a$. We define a function $f\colon \Gamma \cap V \to \mathbb{C}$ setting $f(a) := t_a$. Since $f \circ \gamma = \mathrm{id}_U$, by theorem 4.3 f is c-holomorphic on $\Gamma \cap V$. Furthermore, by $(*)$ we obtain

$$\mathcal{L}(f;0) = \frac{\mathrm{ord}_0(f \circ \gamma)}{\mathrm{ord}_0\gamma} = \frac{1}{\deg_0\Gamma}$$

and so $\mathrm{ord}_0 f^n = n/\deg_0\Gamma$. We have thus proved

PROPOSITION 6.5. *If $\Gamma \subset \mathbb{C}^m$ is an irreducible analytic curve germ at zero, then $\mathcal{L}(\Gamma) = \{n/\deg_0\Gamma \mid n \in \mathbb{N}\}$.*

In connection to this we shall give here also a most interesting general example, which we are indebted to A. Płoski for:

EXAMPLE 6.6. Let $\Gamma := \{F(x,y) = 0\} \subset \mathbb{C}^2$ be an irreducible curve germ at zero parametrized by $\gamma(t)$. Let $d := \deg_0\Gamma = \mathrm{ord}_0 F$. By μ, we shall denote the *Milnor number* of F, i.e., for $J_F = \langle \partial F/\partial x, \partial F/\partial y\rangle$,

$$\mu = \dim_{\mathbb{C}}(\mathcal{O}_2/J_F) = m_0(\mathrm{grad}F).$$

Clearly, by Tsikh–Yuzhakov inequality, $\mu \geq (d-1)^2$.

Let $\ell := (\mu - 1)/d$. By the previous proposition, ℓ is the Łojasiewicz exponent of some c-holomorphic function f on Γ. However, such a function f

cannot have a holomorphic extension onto any neighbourhood of zero in $\mathbb{C}^2$. Indeed, it is a classical result that

$$\mathcal{S}(\Gamma) := \{\mathrm{ord}_0(G \circ \gamma) \mid G \in \mathcal{O}_2 \colon G \not\equiv 0 \bmod F\}$$

is a subsemigroup of $(\mathbb{N}, +)$ whose *threshold* (or *seuil*) is equal to μ. That means that any integer $n \geq \mu$ belongs to $\mathcal{S}(\Gamma)$ while $\mu - 1 \notin \mathcal{S}(\Gamma)$. Therefore, were f strongly holomorphic, we would have a holomorphic extension $G \supset f$ such that $\ell = \mathrm{ord}_0(G \circ \gamma)/d$, whence $\mathrm{ord}_0(G \circ \gamma) = \mu - 1$. This in turn implies $G \equiv 0 \bmod F$ and so $G = 0$ on Γ which is contrary to the assumptions ($f \not\equiv 0$).

Note that if $d \geq 3$ we obtain in this way an example of an exponent $\ell \geq 1$ which cannot be the Łojasiewicz exponent of the restriction to Γ of some holomorphic function.

7. Quotients of c-holomorphic functions on curves. Using the c-holomorphicity criterion 4.3 and $(*)$, we prove

THEOREM 7.1. *Let $\Gamma \subset \mathbb{C}^m$ be an irreducible curve germ at zero and let $f, g \in \mathcal{O}_c(\Gamma)$. Then $f/g \in \mathcal{O}_c(\Gamma)$ iff $\mathrm{ord}_0 f \geq \mathrm{ord}_0 g$.*

PROOF. Let φ be the Puiseux parametrization of Γ. Then $f \circ \varphi$ and $g \circ \varphi$ are holomorphic at $0 \in \mathbb{C}$. By $(*)$, $\mathrm{ord}_0 f = \mathrm{ord}_0(f \circ \varphi)/\deg_0 \Gamma$ and so for $\mathrm{ord}_0 g = \mathrm{ord}_0(g \circ \varphi)/\deg_0 \Gamma$.

Therefore, $\mathrm{ord}_0 f \geq \mathrm{ord}_0 g$ iff $\mathrm{ord}_0(f \circ \varphi) \geq \mathrm{ord}_0(g \circ \varphi)$. The latter is equivalent to the holomorphicity of $(f/g) \circ \varphi$. By 4.3, this completes the proof. $\square$

A natural question arises here which is the following: could such a result hold true in the general case? To be more precise, consider an analytic set A in an open set $\Omega \subset \mathbb{C}^m$ and two c-holomorphic functions $f, g \in \mathcal{O}_c(A)$. Suppose that $0 \in A$ and the germ of A at zero is irreducible. If we want the quotient germ (at zero) f/g to be c-holomorphic we must obviously start with the assumption that $g^{-1}(0) \subset f^{-1}(0)$. Note here that by [**4**], if both functions are non-constant and $k = \dim A$, then $f^{-1}(0)$ and $g^{-1}(0)$ are analytic sets of pure dimension $k - 1$. We would like to obtain an analogue of Theorem 7.1.

EXAMPLE 7.2. Suppose first that $A = \Omega$, so f, g are holomorphic. Let $\mathcal{O}_m$ denote the ring of holomorphic germs at $0 \in \mathbb{C}^m$. Then $f/g \in \mathcal{O}_m$ iff there is a constant $C > 0$ such that $|f| \leq C|g|$ holds in a neighbourhood of zero (cf. Riemann Extension Theorem). On the other hand, it is a classical result that the latter is equivalent to $\mathrm{ord}_0(f \circ \gamma) \geq \mathrm{ord}_0(g \circ \gamma)$ for all holomorphic curve germs $\gamma \colon (\mathbb{C}, 0) \to (\mathbb{C}^m, 0)$ (cf. the equivalent conditions for a function germ $f \in \mathcal{O}_m$ to be integral over an ideal $I \subset \mathcal{O}_m$).

Also it is well known that f/g is holomorphic iff $\mathrm{ord}_z f \geq \mathrm{ord}_z g$ for all $z \in \mathrm{Reg}\, g^{-1}(0)$ (see e.g. [**11**]).

The example above is quite optimistic. Consider, however, the following one:

EXAMPLE 7.3. Consider $A := \{(x, y, z) \in \mathbb{C}^3 \mid x^2 y = z^2\}$ and the functions $f(x, y, z) = z$, $g(x, y, z) = x$ restricted to A. Clearly, $g^{-1}(0) \subset f^{-1}(0)$. We will show that for all curve germ $\gamma \colon (\mathbb{C}, 0) \to (A, 0)$ intersecting $g^{-1}(0)$ only at zero the function $(f/g) \circ \gamma$ is holomorphic, but f/g cannot be c-holomorphic in any neighbourhood of zero in A. This is mostly because A is reducible at all points $(0, y, 0) \in A \setminus \{0\}$.
Indeed, any curve germ in A going through 0 has a Puiseux parametrization $\gamma(t)$ which can be written in one of the three following forms:

$$\gamma(t) = (t, 0, 0),$$
$$\text{or } \gamma(t) = (0, t, 0),$$
$$\text{or } \gamma(t) = (\xi(t), \upsilon(t), \zeta(t)) \text{ with } \xi^{-1}(0) = \upsilon^{-1}(0) = \zeta^{-1}(0) = \{0\}.$$

The first one gives $(f/g) \circ \gamma \equiv 0$, the second one parametrizes $g^{-1}(0)$ and so we do not take it into account. It is the third one which is of interest. Its components satisfy the relation $\xi(t)^2 \upsilon(t) = \zeta(t)^2$. This means that $(\zeta(t)/\xi(t))^2$ is holomorphic. Therefore, $\mathrm{ord}_0 \zeta^2 \geq \mathrm{ord}_0 \xi^2$, but that yields $\mathrm{ord}_0 \zeta \geq \mathrm{ord}_0 \xi$ and so $\zeta/\xi = (f/g) \circ \gamma$ is holomorphic.
On the other hand, if we consider a line $\{(x, \varepsilon, 0) \mid x \in \mathbb{C}\}$ with a fixed $\varepsilon > 0$ arbitrarily small, there are only two curves in A lying over it, namely $\{(x, \varepsilon, \sqrt{\varepsilon}x) \mid x \in \mathbb{C}\}$ and $\{(x, \varepsilon, -\sqrt{\varepsilon}x) \mid x \in \mathbb{C}\}$. Thus f/g restricted to the first one of these curves is $\sqrt{\varepsilon}$, while its restriction to the second one is $-\sqrt{\varepsilon}$ and so the value $(f/g)(0, \varepsilon, 0)$ is undefined (see also [**11**]).

Note that in this case the 'simplest' universal denominator for A is the function $Q(x, y, z) = z$ (cf. e.g. [**11**]). It is clearly a minimal one and if we denote by F and G the holomorphic extensions of f and g, respectively, then along $G^{-1}(0) \cap A$ the order of vanishing of F is not smaller than the order of G. Nonetheless $f/g = F/G$ is not c-holomorphic, as we saw above.

It is thence clear that we shall restrict ourselves to locally irreducible sets. As a matter of fact, reducibility is an obstacle even in the one-dimensional case:

EXAMPLE 7.4. Let $\Gamma := \Gamma_1 \cup \Gamma_2 \subset \mathbb{E}^2$, where $\Gamma_1 := \{y^2 = x^3\}$, $\Gamma_2 := \{y^2 = x^5\}$ and $\mathbb{E}$ is the unit disc in $\mathbb{C}$. Consider the c-holomorphic functions

$$f(x, y) = \begin{cases} y, & \text{if } (x, y) \in \Gamma_1, \\ \frac{y}{x}, & \text{if } (x, y) \in \Gamma_2 \setminus \{(0, 0)\}; \end{cases} \qquad g(x, y) = \begin{cases} \frac{y}{x}, & (x, y) \in \Gamma_1 \setminus \{(0, 0)\}, \\ y, & \text{if } (x, y) \in \Gamma_2. \end{cases}$$

Calculating the orders according to $(*)$, we see that

$$\mathrm{ord}_0 g|_{\Gamma_2} > \mathrm{ord}_0 f|_{\Gamma_2} = \mathrm{ord}_0 f = \mathrm{ord}_0 f|_{\Gamma_1} > \mathrm{ord}_0 g|_{\Gamma_1} = \mathrm{ord}_0 g.$$

Therefore, by Theorem 7.1, f/g is c-holomorphic on Γ_1 but not on Γ_2, hence not on the whole of Γ. However, $\mathrm{ord}_0 f > \mathrm{ord}_0 g$.

It remains an open question what kind of assumption involving the orders of vanishing could make possible the division of two given c-holomorphic functions. A first step towards an analytic solution through a duality theorem (not yet established however) was made in [**5**].

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