ISSN 1472-2739 (on-line) 1472-2747 (printed)

Algebraic & Geometric Topology
Volume 1 (2001) 469–489
Published: 9 September 2001

Lefschetz fibrations, complex structures and Seifert fibrations on $S^1 \times M^3$

Tolga Etgü

Abstract

We consider product 4–manifolds $S^1 \times M$, where M is a closed, connected and oriented 3–manifold. We prove that if $S^1 \times M$ admits a complex structure or a Lefschetz or Seifert fibration, then the following statement is true:

$S^1 \times M$ admits a symplectic structure if and only if M fibers over S^1,

under the additional assumption that M has no fake 3–cells. We also discuss the relationship between the geometry of M and complex structures and Seifert fibrations on $S^1 \times M$.

AMS Classification 57M50, 57R17, 57R57; 53C15, 32Q55

Keywords Product 4–manifold, Lefschetz fibration, symplectic manifold, Seiberg–Witten invariant, complex surface, Seifert fibration

1 Introduction

A closed, oriented, smooth 4–manifold X which fibers over a Riemann surface admits a symplectic structure unless the fiber class is torsion in $H_2(X;\mathbb{Z})$. In particular, a fibration of a closed, oriented 3–manifold M over S^1 induces a symplectic form on $S^1 \times M$.

Conjecture T *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a symplectic structure. Then M fibers over S^1.*

This conjecture was first stated by Taubes [27] and is still open. Recent work of Chen and Matveyev [4] proves that it holds when M has no fake 3–cells, $S^1 \times M$ admits a symplectic structure and a Lefschetz fibration with symplectic fibers.

In this paper, we generalize Chen and Matveyev's result proving that the conjecture holds when $S^1 \times M$ admits an arbitrary Lefschetz fibration (possibly with nonsymplectic fibers). More generally, we prove the following:

Theorem 1.1 *Suppose M is a closed 3–manifold without a fake 3–cell.*

(L) *If $S^1 \times M$ admits a Lefschetz fibration, then Conjecture T holds.*

(S) *If $S^1 \times M$ admits a Seifert fibration, then Conjecture T holds.*

(K) *If $S^1 \times M$ admits a Kähler structure, then Conjecture T holds.*

(C) *If $S^1 \times M$ admits a complex structure, then Conjecture T holds.*

Here, a fake 3–cell means a compact, contractible 3–manifold which is not homeomorphic to D^3. Note that the Poincaré conjecture implies that there is no fake 3–cell.

Remark We'll see that a nonsymplectic Lefschetz fibration on a product 4–manifold has no singular fibers and has fiber a torus. Since a Seifert fibration can be thought of as a T^2–fibration with multiple fibers, (S) is a further generalization of (L). Statement (C) is clearly a generalization of (K). Note that all (symplectic) product 4–manifolds which admit complex structures turn out to be Seifert fibered. This means that all other statements follow from (S) using the result of Chen and Matveyev on symplectic Lefschetz fibrations.

In the remark above and the rest of the paper, by a product 4–manifold we mean the product of S^1 with a (compact, oriented, connected) 3–manifold.

In order to prove Theorem 1.1, besides other techniques, we use classification results on complex surfaces and Lefschetz fibered 4–manifolds and apply them to product manifolds. In particular, we get results on the classification of product 4–manifolds which admit certain structures or fibrations and interesting relations between the geometry of M and complex structures and Seifert fibrations on $S^1 \times M$.

Remark In their paper [9] on taut contact circles on 3–manifolds, Geiges and Gonzalo classified product 4–manifolds carrying complex structures with respect to which the obvious circle action is holomorphic. Since we don't require this action to be holomorphic and we are mainly interested in the symplectic structure on product manifolds, we prove different type of results even though we use similar methods.

Remark As a consequence of Theorem 1.1 we see that when M is a nonhyperbolic geometric 3–manifold Conjecture T holds. On the other hand, assuming Thurston's conjecture on the geometrization of 3–manifolds, if $S^1 \times M$ admits a symplectic structure, then M is prime (see [16] or [32]). So it might be interesting to try to prove Conjecture T (at least up to the geometrization conjecture) by first proving it when M is hyperbolic, then considering geometric 3–manifolds with boundary (disjoint union of tori) and finally using Seiberg–Witten theory of 4–manifolds glued along T^3.

In the next section we recall definitions and some basic theorems on Lefschetz fibrations, complex surfaces, Seiberg–Witten invariants, Seifert fibrations and geometric structures on 3– and 4–manifolds. In Section 3, we discuss nonsymplectic Lefschetz fibrations on $S^1 \times M$. By using the Seiberg–Witten theory of symplectic 4–manifolds and S^1–bundles over surfaces, we prove (L) of Theorem 1.1 in Section 4. In Section 5, product 4–manifolds which admit complex structures are considered and (K) is proved first. As a result of a slightly more careful investigation we prove (C). Finally we consider Seifert fibered 4–manifolds and prove (S). In the last section, we discuss the relation between various structures and fibrations on $S^1 \times M$ and M.

In this paper, by a fiber bundle we mean a locally trivial one and an F–bundle means a (locally trivial) fiber bundle with fiber F. All fibrations (of any kind) are oriented and all manifolds are compact, smooth, oriented and connected, unless stated otherwise.

Acknowledgment The author is grateful to his thesis advisor Rob Kirby for numerous discussions.

2 Background

Let us first state some topological information on $S^1 \times M$.

Lemma 2.1 *Let M be a closed, oriented and connected 3–manifold. Then $X = S^1 \times M$ is a spin 4–manifold with $\sigma(X) = \chi(X) = 0$, $b_\pm(X) = b_1(M)$ (in particular, $b_2(X)$ is even), where σ, χ and b_* denote the signature, Euler characteristic and the corresponding Betti number, respectively.*

Proof Both S^1 and M are spin, so X is spin. Since $\chi(S^1) = 0$, the Euler characteristic of X vanishes. The boundary of $D^2 \times M$ is X, so $\sigma(X) = 0$. The facts about the Betti numbers follow easily from the definitions of σ and χ in terms of Betti numbers. □

2.1 Lefschetz fibrations and pencils

Definition 2.2 A Lefschetz fibration on a compact, connected, oriented and smooth 4–manifold X is a smooth map $\pi\colon X \longrightarrow \Sigma$, where Σ is a compact, connected, oriented surface and $\pi^{-1}(\partial\Sigma) = \partial X$, such that each critical point of π lies in the interior of X and has an orientation-preserving coordinate chart on which $\pi(z_1, z_2) = z_1^2 + z_2^2$ relative to a suitable smooth chart on Σ.

Definition 2.3 A Lefschetz pencil on a closed, connected, oriented, smooth 4–manifold X is a non-empty finite subset B of X called the base locus, together with a smooth map $\pi\colon X - B \longrightarrow \mathbb{C}P^1$ such that each point $b \in B$ has an orientation-preserving coordinate chart in which π is given by the projectivization $\mathbb{C}^2 - \{0\} \longrightarrow \mathbb{C}P^1$, and each critical point has a local coordinate chart as in the definition of a Lefschetz fibration above.

Definition 2.4 A Lefschetz fibration is called relatively minimal if no fiber contains an exceptional sphere, in other words it cannot be obtained by blowing up another Lefschetz fibration.

Definition 2.5 A Lefschetz fibration is called a symplectic Lefschetz fibration if the total space admits a symplectic structure such that generic fibers are symplectic submanifolds, otherwise it is called nonsymplectic.

Theorem 2.6 (Gompf) *A Lefschetz fibration on a 4–manifold X is symplectic if and only if the homologous class of the fiber is not torsion in $H_2(X;\mathbb{Z})$.*

The close relation between Lefschetz fibrations and symplectic structures is stated in the following theorems.

Theorem 2.7 (Donaldson [5]) *Every symplectic 4–manifold admits a Lefschetz pencil by symplectic surfaces.*

Theorem 2.8 (Gompf [10]) *If a 4–manifold admits a Lefschetz pencil (with non-empty base locus), then it admits a symplectic structure.*

It is necessary that the base locus is non-empty as we have examples of 4–manifolds, e.g. $S^1 \times S^3$, which admit Lefschetz fibrations over S^2 but no symplectic structure.

If a manifold admits a Lefschetz pencil, then one can blow-up the points of the base locus and construct a Lefschetz fibration (over S^2). So Donaldson's theorem implies that every symplectic 4–manifold has a blow-up which admits a Lefschetz fibration. Even though it is always possible to put a Lefschetz pencil on a symplectic $S^1 \times M$ it may not be possible to find a Lefschetz fibration on it. Note that a blow-up of $S^1 \times M$ can never be a product.

For more details on Lefschetz pencils and fibrations see [10].

2.2 Seiberg–Witten invariants

Let X be a closed, oriented, connected and homology oriented 4–manifold with $b_+(X) > 0$. The Seiberg–Witten invariant SW of a $Spin_c$ structure on X was first extracted from monopole equations by Witten in [35]. If $b_+(X) > 1$, then SW is an integer-valued diffeomorphism invariant of X. When $b_+(X) = 1$ it may depend on the chosen metric. The Seiberg–Witten invariant of a $Spin_c$ structure ξ on X is denoted by $SW_X(\xi)$. We call $\alpha \in H^2(X;\mathbb{Z})$ a basic class if there exists a $Spin_c$ structure ξ such that $SW_X(\xi) \neq 0$ with $c_1(\det(\xi)) = \alpha$, where $\det(\xi)$ denotes the determinant (complex) line bundle of ξ. If there is no 2–torsion in $H^2(X;\mathbb{Z})$, then there is a unique $Spin_c$ structure ξ with $c_1(\det(\xi)) = \alpha$ for any characteristic class $\alpha \in H^2(X;\mathbb{Z})$. In general, the set of isomorphism classes of $Spin_c$ structures on X is an affine space modeled on $H^2(X;\mathbb{Z})$.

Seiberg–Witten invariants of 3–dimensional manifolds are defined similarly. As we state in Section 4, Seiberg–Witten invariants of a 3–manifold M carry exactly the same information as those of $S^1 \times M$ at least when $b_1(M) > 1$. The reader is referred to [14] and [23] for the theory of Seiberg–Witten invariants in dimension 3.

We have the following important theorem on the Seiberg–Witten invariants of symplectic manifolds.

Theorem 2.9 (Taubes [25], [26]) *Let X be a closed 4–manifold with $b_+ > 1$ and a symplectic form ω. Then there is a canonical $Spin_c$ structure ξ on X such that $SW_X(\xi) = \pm 1$ and $\det(\xi)$ is the canonical line bundle K of (X,ω).*

Moreover,

$$0 \leq |\alpha \cdot [\omega]| \leq |c_1(K) \cdot [\omega]| \ ,$$

where α is any basic class; $0 = \alpha \cdot [\omega]$ if and only if $\alpha = 0$; $|\alpha \cdot [\omega]| = |c_1(K) \cdot [\omega]|$ if and only if $\alpha = \pm c_1(K)$.

See [10], [18] and [14] for more details on Seiberg–Witten invariants of 4-manifolds.

2.3 Geometric structures and the geometrization conjecture

Definition 2.10 A metric on a manifold is called locally homogeneous if any pair of points can be mapped to each other by isometries of open neighborhoods.

Definition 2.11 A manifold is called geometric if it admits a complete, locally homogeneous metric.

Definition 2.12 A simply connected geometric manifold together with the isometry group corresponding to a complete (locally) homogeneous metric is called a geometry.

Up to isometry, there are eight 3–dimensional and nineteen 4–dimensional geometries with compact quotients. These are classified by Thurston and Filipkiewicz [7] respectively. See [24] and [33] for detailed discussions on 3– and 4–dimensional geometries.

A manifold is called prime if it cannot be written as the connected sum of two manifolds none of which is a sphere. In [17] Milnor showed that, up to homeomorphism and the permutation of the summands, there is a unique way to write a compact, oriented 3–manifold as the connected sum of prime manifolds. There is also a reasonably canonical way to cut compact, prime 3–manifolds along tori into pieces which no longer have embedded tori in them other than their boundary components (up to homology). Thurston's geometrization conjecture asserts that these pieces should all be geometric.

2.4 Seifert fibered spaces

A trivial fibered solid torus is $S^1 \times D^2$ with the product foliation by circles. A fibered solid torus is a solid torus with a foliation by circles that is finitely covered by a trivial fibered solid torus. It can be constructed by gluing two ends $D^2 \times \{0\}$ and $D^2 \times \{1\}$ of $D^2 \times I$ after a q/p rotation.

A Seifert fibered space is a 3–manifold with a decomposition into disjoint circles such that each circle has a neighborhood isomorphic to a fibered solid torus. A circle bundle over a surface is naturally a Seifert fibered space. By identifying each of these circles with a point, we can consider a Seifert fibered space as a fibration over a 2–orbifold base. Such a fibration is called a Seifert fibration. Fibers of a Seifert fibration are obviously circles and singularities of the base orbifold correspond to the fibers without trivial fibered solid torus neighborhoods. A fiber is called regular if it projects to a nonsingular point of the base, otherwise it is called a multiple fiber.

Lemma 2.13 (cf. Lemma 3.2 in [24]) *Suppose M admits a Seifert fibration over a 2–orbifold X. Then there is a short exact sequence*

$$1 \longrightarrow G \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(X) \longrightarrow 1 \ ,$$

where G denotes the cyclic subgroup of $\pi_1(M)$ generated by a regular fiber and $\pi_1^{orb}(X)$ denotes the fundamental group of X as an orbifold. The subgroup G is infinite except in cases where M is covered by S^3.

Note that a presentation for $\pi_1^{orb}(X)$ is

$$\left\langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_n \mid x_i^{p_i} = 1, \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{i=1}^{n} x_i = 1 \right\rangle \ ,$$

where g is the genus of the underlying surface of X, assuming X is closed and orientable with n singular points of multiplicities $p_1, \ldots p_n$. The Euler characteristic $\chi(X)$ of such a 2–orbifold X is defined by

$$\chi(X) = 2 - 2g - \sum_{i=1}^{n} \left(1 - \frac{1}{p_i}\right) \ .$$

An orbifold is called spherical (Euclidean or hyperbolic) if its Euler characteristic is positive (zero ornegative).

For more details on Seifert fibered spaces see [22] and [21]. For geometric structures on Seifert fibered spaces see [24].

2.5 Seifert fibered 4–manifolds

A Seifert fibration on a 4–manifold is analogous to a Seifert fibration on a 3–manifold.

Definition 2.14 A smooth map $\pi\colon X \longrightarrow \Sigma$ from a smooth 4–manifold X to a surface Σ is called a Seifert fibration if there exists a finite set of isolated points B in Σ such that the restriction of π to $\pi^{-1}(\Sigma - B)$ is a torus bundle and for each element $b \in B$, $\pi^{-1}(b)$ has a tubular neighborhood diffeomorphic to the product of a fibered solid torus with a circle.

A Seifert fibration can be thought of as a torus fibration over a 2–orbifold. In the complex category it corresponds to an elliptic fibration without singular fibers (possibly with multiple ones). If a 4–manifold admits a Seifert fibration it is called a Seifert 4–manifold. We have analogous statements for Seifert fibered

4–manifolds to most of the properties of Seifert fibered spaces, e.g. Lemma 2.13. See [33] and [34] for geometric structures on elliptic surfaces without singular fibers, [30] and [31] for a general picture of Seifert 4–manifolds in terms of geometric structures.

3 Nonsymplectic Lefschetz fibrations on $S^1 \times M$

In this section our aim is to show that nonsymplectic Lefschetz fibrations on $S^1 \times M$ are in fact locally trivial torus bundles. We also investigate which of these fibrations have symplectic total spaces and which of them give rise to fibrations of M over S^1.

Theorem 3.1 (Chen-Matveyev [4]) *Let π be a symplectic Lefschetz fibration on $S^1 \times M$, where M is a closed, connected, oriented 3–manifold without any fake 3–cells. Then there exists a fibration p on M over S^1. Moreover, the symplectic structure with which π is compatible is deformation equivalent (up to self-diffeomorphisms of $S^1 \times M$) to the canonical symplectic structure associated to the fibration $Id \times p\colon S^1 \times M \to S^1 \times S^1$.*

The symplectic form (canonical up to deformation equivalence) on the total space of a surface bundle over a compact, oriented surface is obtained by extending a symplectic form on a fiber and adding a (sufficiently large) multiple of the pullback of a symplectic form on the base to it (see [29] and [20] for details and more general cases). The following lemma plays a crucial role in the proof of the theorem above.

Lemma 3.2 [4] *Let π be a symplectic Lefschetz fibration on $S^1 \times M$, where M is a closed, connected, oriented 3–manifold. Then π doesn't have any critical points.*

First of all, we give the following generalization of this lemma.

Lemma 3.3 *Let π be a Lefschetz fibration on $S^1 \times M$, where M is a closed, connected, oriented 3–manifold. Then π is a fiber bundle. If π is not symplectic, then it is a torus bundle.*

Proof We only need to consider the case where π is not symplectic, i.e. fibers are not symplectic submanifolds of $X = S^1 \times M$. By Theorem 2.6 the fiber class $[F]$ is torsion in $H_2(X;\mathbb{Z})$. This is possible only if F is a torus since otherwise

$$0 \neq \chi(F) = \langle e(TF), [F] \rangle \ .$$

Note that $e(TF)$ extends to $H^2(X;\mathbb{Z})$ since TF is the pull-back (by the inclusion $F \hookrightarrow X$) of the vertical (with respect to π) subbundle of TX. On the other hand, the Euler characteristic of the total space of a Lefschetz fibration is equal to the product of the those of the base and the fiber plus the number of vanishing cycles (assuming there is a unique singular point on each fiber). In our case this leads to

$$0 = \chi(S^1 \times M) = \chi(T^2) \cdot \chi(B) + \#\{\text{vanishing cycles}\} \ .$$

Hence there are no vanishing cycles. Therefore π is a torus bundle. □

This lemma shows that nonsymplectic Lefschetz fibrations on $S^1 \times M$ are all torus bundles over Riemann surfaces. We investigate these bundles in three groups according to the genera of their bases.

Lemma 3.4 *Let $S^1 \times M$ be the total space of a nontrivial T^2–bundle over S^2. Then $S^1 \times M$ carries no symplectic form.*

Proof Since the torus bundle is nontrivial, $b_1(S^1 \times M) < 2$ and therefore $b_2(S^1 \times M) = 2 \cdot b_1(M) = 0$. Hence all closed 2–forms on $S^1 \times M$ are degenerate. □

Remark As we mentioned before, a fibration of M over S^1 induces a symplectic form on $S^1 \times M$. Therefore, when $S^1 \times M$ is as in the lemma M doesn't fiber over the circle.

We have a totally different picture for T^2–bundles over T^2.

Theorem 3.5 (Geiges [8]) *Let X be the total space of an oriented T^2–bundle over T^2. Then X admits a symplectic structure. Moreover, there exists a symplectic T^2–bundle over T^2 with total space X unless X is the total space of a nontrivial S^1–bundle over the total space of a nontrivial S^1–bundle over T^2.*

Let X be an exception, i.e. a twisted circle bundle over a twisted circle bundle over the torus. Then $b_1(X) = b_2(X) = 2$. Moreover, $H^1_{DR}(X;\mathbb{R})$ is generated by $[\alpha]$ and $[\beta]$, where α and β are closed 1–forms on X such that $n \cdot \alpha \wedge \beta = d\gamma$, where n is the Euler number of the (nontrivial) S^1–bundle over T^2 and γ is a 1–form on X (see [6] for details). In particular, $(H^1(X;\mathbb{R}))^{\cup 2} = 0$, where $(H^1(X;\mathbb{R}))^{\cup 2}$ denotes the image of the cup product of $H^1(X;\mathbb{R})$ with itself. On the other hand, $H^1(S^1 \times M;\mathbb{R}) \cong H^1(S^1;\mathbb{R}) \oplus H^1(M;\mathbb{R})$ and obviously $(H^1(S^1 \times M;\mathbb{R}))^{\cup 2} \neq 0$. Therefore we have the following corollary.

Corollary 3.6 *If $S^1 \times M$ is the total space of a T^2–bundle over T^2, then $S^1 \times M$ admits a symplectic Lefschetz fibration.*

For T^2–bundles over higher genus surfaces we have

Lemma 3.7 *Let $S^1 \times M$ be the total space of a T^2–bundle over B, where B is a closed, oriented surface of genus ≥ 2. Also assume that M has no fake 3–cells. Then M fibers over the circle if and only if the torus bundle is trivial.*

We are going to use the following lemma to prove the one above.

Lemma 3.8 (cf. [22] Theorem 7.2.4) *Let M be a closed,oriented 3–manifold which is the total space of a circle bundle over a closed, oriented surface B of genus ≥ 2. Then M fibers over the circle if and only if $M = S^1 \times B$.*

Proof Recall that $\pi_1(M)$ has the presentation

$$\left\langle a_1, b_1, \ldots, a_g, b_g, \alpha \;\middle|\; [a_i,\alpha] = [b_i,\alpha] = 1, [a_1,b_1]\cdots[a_g,b_g] = \alpha^k \right\rangle ,$$

where $g = genus(B)$ and k is the Euler number of the S^1–bundle. In particular, $H_1(M) \cong \mathbb{Z}^{2g+1}$ if $k = 0$ and $H_1(M) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}_{|k|}$ otherwise.

We also have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_1(S^1) & \xrightarrow{j_\#} & \pi_1(M) & \longrightarrow & \pi_1(B) & \longrightarrow & 1 \\
 & & \downarrow \cong & & \downarrow & & \downarrow & & \\
 & & H_1(S^1) & \xrightarrow{j_*} & H_1(M) & \longrightarrow & H_1(B) & \longrightarrow & 0
\end{array}$$

where vertical maps are Hurewicz epimorphisms. Note that the homomorphism j_* is injective if and only if $Im(j_\#) \cap [\pi_1(M):\pi_1(M)] = \{1\}$. Now suppose that $F \longrightarrow M \longrightarrow S^1$ is a fibration. There exists a normal subgroup $N \cong \pi_1(F)$ in $\pi_1(M)$ such that $\pi_1(M)/N \cong \mathbb{Z}$. Assume that there exists an element $u \in N \backslash \{1\}$ such that $u = j_\#(v)$. Then there is a normal infinite cyclic subgroup (generated by u) in N and this implies that F is a torus, but M cannot be the total space of a torus bundle over the circle since $b_1(M) \geq 2g \geq 4 > 3$. Therefore $Im(j_\#) \cap N = \{1\}$. On the other hand, $[\pi_1(M):\pi_1(M)] \subset N$

because $\pi_1(M)/N \cong \mathbb{Z}$. So $Im(j_\#) \cap [\pi_1(M) : \pi_1(M)] = \{1\}$, j_* is injective and we have the short exact sequence

$$0 \longrightarrow H_1(S^1) \longrightarrow H_1(M) \longrightarrow H_1(B) \longrightarrow 0$$

which clearly splits. Hence $b_1(M) = 2g + 1$ and M is the product $S^1 \times B$. □

Proof of Lemma 3.7 We have the homotopy sequence of the T^2–bundle

$$0 \longrightarrow \pi_1(T^2) \xrightarrow{j_\#} \pi_1(S^1 \times M) \xrightarrow{\pi_\#} \pi_1(B) \longrightarrow 1 \ . \tag{1}$$

Let u be a generator of $\pi_1(S^1 \times pt)$. Assume that $\pi_\#(u) = v \neq 1 \in \pi_1(B)$. Then v generates a normal cyclic subgroup in $\pi_1(B)$ and this contradicts the fact that $genus(B) \geq 2$. Therefore $u \in ker(\pi_\#) = im(j_\#)$, where j is the inclusion map. Let a be $j_\#^{-1}(u)$. We can find another element $b \in \pi_1(T^2)$ such that the restriction of $j_\#$ to the subgroup $\langle b \rangle$ generated by b gives the short exact sequence

$$0 \longrightarrow \langle b \rangle \longrightarrow \pi_1(M) \longrightarrow \pi_1(B) \longrightarrow 1 \ . \tag{2}$$

By Theorem 11.10 in [11] M admits an S^1–bundle over B (we use the assumption that M has no fake 3–cells). Lemma 3.8 finishes the proof. □

We should note that the idea of extracting (2) from (1) was first used in [4].

Proposition 3.9 *Suppose $S^1 \times M$ admits a nonsymplectic Lefschetz fibration, where M is a closed, oriented 3–manifold. If the base space of the fibration is a torus, then $S^1 \times M$ admits a symplectic form and a symplectic Lefschetz fibration. Otherwise M doesn't fiber over S^1 or it has a fake 3–cell.*

Proof Let π be a nonsymplectic Lefschetz fibration on $X = S^1 \times M$. By Lemma 3.3, π is relatively minimal, has no critical points and the fibers are tori. It is a nontrivial bundle since otherwise it would be symplectic. If the base space B is a torus, then X admits a symplectic Lefschetz fibration by Corollary 3.6. If $B = S^2$, then X doesn't admit a symplectic structure by Lemma 3.4 and in particular, M doesn't fiber over S^1 since such a fibration would induce a symplectic form on X. Finally, if $genus(B) \geq 2$ and M has no fake 3–cells, then Lemma 3.7 implies that M doesn't fiber over S^1. □

4 Seiberg–Witten invariants of symplectic manifolds and S^1–bundles over surfaces

In this section we use Seiberg–Witten theory of symplectic manifolds and S^1–bundles over closed, oriented surfaces to prove the following theorem which in turn implies that the existence of a symplectic form and a Lefschetz fibration on $S^1 \times M$ is possible only if there is a symplectic Lefschetz fibration on $S^1 \times M$ (Theorem 4.5). Statement (L) of Theorem 1.1 is a consequence of this.

Theorem 4.1 *Let M be the total space of an oriented S^1–bundle over a Riemann surface B. Then $X = S^1 \times M$ admits a symplectic structure if and only if the bundle is trivial or B is a torus.*

The following theorem follows from the work of Mrowka, Ozsváth and Yu on the SW invariants of Seifert fibered spaces [19]. See [1] for a different (and more elementary) approach.

Theorem 4.2 *Let M be the S^1–bundle over a Riemann surface B of genus $g \geq 1$ with Euler class $n\lambda$, where λ is the (positive) generator of $H^2(B;\mathbb{Z})$. If $n \neq 0$, then all basic classes of M are in $\{k \cdot \pi^*(\lambda) \mid 0 \leq k \leq |n|-1\}$, where π is the bundle projection. Moreover, we have*

$$SW_M(k \cdot \pi^*(\lambda)) = \sum_{s \equiv k \ (mod\ n)} SW_{S^1 \times B}(s \cdot pr_2^*(\lambda)) \ , \qquad (3)$$

where pr_2 is the projection $S^1 \times B \to B$.

It is well-known that the Seiberg–Witten invariants of $S^1 \times B$ are given by

$$SW_{S^1 \times B}(t) = (t - t^{-1})^{2g-2} \ ,$$

where g is the genus of B and the coefficient of t^p on the right hand side corresponds to the Seiberg–Witten invariant of the $Spin_c$ structure with determinant line bundle L with $c_1(L) = p \cdot pr_2^*(\lambda)$. Therefore the sum of all Seiberg–Witten invariants of $S^1 \times B$ is 0 if $g > 1$. This sum is preserved under twisting of the S^1–bundle as can be seen from (3).

Corollary 4.3 *Let M be as in the previous theorem and $g > 1$. Then*

$$\sum_{\alpha} SW_M(\alpha) = 0 \ ,$$

where the sum is over all $Spin_c$ structures on M.

The following is also well-known and relates the Seiberg–Witten invariants of $S^1 \times M$ with those of M. For a proof see [23].

Theorem 4.4 *If M is a closed, oriented 3–manifold, then*

$$SW_M(\alpha) = SW_{S^1\times M}(pr_2^*(\alpha))$$

for any $\alpha \in H^2(M;\mathbb{Z})$, where pr_2 is the projection $S^1 \times M \to M$. Moreover, if $b_+(S^1 \times M) = b_1(M) > 1$, then all basic classes of $S^1 \times M$ are pull-backs of basic classes of M.

Proof of Theorem 4.1 If the bundle is trivial then $X = T^2 \times B$ and there is a symplectic form on X which is simply the sum of symplectic forms on T^2 and B.

If B is a torus, then X is a torus bundle over a torus and by Theorem 3.5 it admits a symplectic structure.

If the bundle is nontrivial and B is a sphere, then X is a nontrivial T^2–bundle over S^2 and cannot be symplectic as we proved in Lemma 3.4.

From now on we will assume that the bundle is nontrivial and the genus of B is at least 2.

By Corollary 4.3 and Theorem 4.4 (as $b_1(M) \geq 2b_1(B) \geq 4$)

$$\sum_\alpha SW_M(\alpha) = \sum_\beta SW_X(\beta) = 0 \ , \tag{4}$$

where sums are over all $Spin_c$ structures on M and X respectively.

Assume that X admits a symplectic form ω. First of all, by the conditions on equality in Theorem 2.9, the canonical class $K = c_1(X,\omega)$ cannot be a nonzero torsion class. On the other hand, Theorem 4.4 and the first part of Theorem 4.2 imply that all basic classes of X are torsion. Therefore the only basic class of X is $K = 0$ and $SW_X(0) = \pm 1$, in particular,

$$\sum_\beta SW_X(\beta) = \pm 1 \ ,$$

where the sum is over all $Spin_c$ structures on X. This contradicts (4) hence X does not admit a symplectic structure. □

Theorem 4.5 *Let M be a closed, oriented 3–manifold such that $S^1 \times M$ admits a Lefschetz fibration and a symplectic form. Then $S^1 \times M$ admits a symplectic Lefschetz fibration or M has a fake 3–cell.*

Proof Let $X = S^1 \times M$ admit a Lefschetz fibration and a symplectic form. Assume that there is no symplectic Lefschetz fibration on it. Then by Lemma 3.3 it admits a torus bundle over a Riemann surface B. Any such bundle should be nontrivial since otherwise it would be symplectic. By Theorem 3.9, B is not a torus, and it cannot be a sphere by Lemma 3.4. So $genus(B) \geq 2$. If M has no fake 3–cells, then as we have seen in the proof of Lemma 3.7, M is the total space of an S^1–bundle over B and this contradicts Theorem 4.1. □

This theorem (together with Theorem 3.1) finishes the proof of statement (L) of Theorem 1.1.

Remark Symplectic Lefschetz fibrations on product 4–manifolds were classified in [4]. As a result of our discussion, we see that nonsymplectic Lefschetz fibrations on nonsymplectic $S^1 \times M$ are nontrivial torus bundles over spherical or hyperbolic surfaces. On the other hand, nonsymplectic Lefschetz fibrations on a symplectic $S^1 \times M$ are torus bundles over tori and by Proposition 3.9 any such manifold admits a symplectic Lefschetz fibration.

5 Complex structures and Seifert fibrations on the product four–manifolds

In this section, we use the classification of complex surfaces to prove statements (K) and (C) of Theorem 1.1. To prove the latter, we also use an interesting result in Seiberg–Witten theory of complex surfaces due to Biquard. Then we consider Seifert fibered product 4–manifolds and prove that those which admit symplectic structures also admit either Kähler structures or torus bundles over tori. This observation finishes the proof of Theorem 1.1.

At this point we know exactly when the existence of a Lefschetz fibration on $S^1 \times M$ is sufficient for M to fiber over the circle. Since our motivation is to determine whether the existence of a symplectic structure on $S^1 \times M$ is sufficient for M to fiber over the circle, it is quite natural to ask which symplectic (product) 4–manifolds admit Lefschetz fibrations. This question doesn't seem to be any easier than Conjecture T itself even though Donaldson proved that every symplectic 4–manifold admits a Lefschetz pencil. In fact, statement (L) of Theorem 1.1 implies that they are equivalent when M has no fake 3–cells. On the other hand, allowing multiple fibers and considering Seifert fibrations, one can still get interesting results on Conjecture T. Seifert fibered product

4–manifolds turn out to be closely related to complex surfaces and this is the main reason of our discussion on complex structures on product 4–manifolds.

Now suppose that $S^1 \times M$ is a closed complex surface. Since it is a spin 4–manifold its intersection form is even, so there is no exceptional sphere to blow-down, thus it is a minimal complex surface. We are going to use the Enriques–Kodaira classification of compact complex surfaces (see [10] or [3]) to prove the following theorem.

Theorem 5.1 (cf. Theorem 4.1 in [9]) *Let* $S^1 \times M$ *be a closed 4–manifold.*

If $S^1 \times M$ *admits a complex structure, then it is either an elliptic surface or of* $Class$ VII_0.

If $S^1 \times M$ *is also symplectic, then the only possibilities are the following:*

(i) $S^1 \times M \cong S^2 \times T^2$.

(ii) $S^1 \times M$ *admits a* T^2*–bundle over* T^2.

(iii) $S^1 \times M$ *admits a Seifert fibration over a hyperbolic orbifold.*

Proof Let $\kappa(X)$ be the Kodaira dimension of $X = S^1 \times M$ as a complex surface.

Case 1: $\kappa(X) = -\infty$ In this case X is either $\mathbb{C}P^2$ or geometrically ruled or of $Class$ VII_0. The complex projective plane $\mathbb{C}P^2$ is simply-connected, but X is not. If X is a complex surface of $Class$ VII_0, then $0 = b_1(X) - 1 = b_+(X)$ hence it cannot be symplectic. If it is geometrically ruled, then it is the total space of a $\mathbb{C}P^1$–bundle over a Riemann surface B and $0 = \chi(X) = \chi(\mathbb{C}P^1) \cdot \chi(B)$, hence B is a torus. Moreover, X is diffeomorphic to $S^2 \times T^2$ since the total space of the nontrivial S^2–bundle over T^2 is not spin.

Case 2: $\kappa(X) = 0$ Any minimal complex surface of Kodaira dimension 0 is a $K3$ surface, an Enriques surface, a primary Kodaira surface, a secondary Kodaira surface, a hyperelliptic surface or a complex torus. Since $b_1(X) \geq 1$ X cannot be a $K3$ or an Enriques surface. In three of the other four cases, X is diffeomorphic to the total space of a T^2–bundle over T^2. When X is a secondary Kodaira surface it admits an elliptic fibration over $\mathbb{C}P^1$ (without singular fibers) and $b_1(X) = 1$. So in this case, X cannot be symplectic because $b_+(X) = b_1(X) - 1 = 0$.

Case 3: $\kappa(X) = 1$ In this case X is a (properly) elliptic surface. An elliptic fibration on X cannot have singular fibers but only multiple fibers since the Euler characteristic of X vanishes. In particular, X is a Seifert 4–manifold.

While investigating geometric structures on elliptic surfaces Wall (see [33] or [34]) proves that the base orbifold of such a fibration must be hyperbolic if $\kappa(X) = 1$.

These are the only possibilities since every minimal surface of general type has positive Euler characteristic, but $\chi(X) = 0$. □

Remark By a well-known result of Bogomolov [28] a complex surface of *Class* VII_0 with vanishing second Betti number is either a Hopf surface or an Inoue surface. Since the center of the fundamental group of an Inoue surface is trivial (cf. Proposition 4.2 in [9]) no Inoue surface is a product. On the other hand, Kato's work on Hopf surfaces [12] implies that if a Hopf surface is diffeomorphic to a product, then it must be elliptic. In particular, it is Seifert fibered since vanishing of the Euler characteristic implies that an elliptic fibration on a product can have no singular fibers (but only multiple ones).

Recall that a closed complex surface is Kähler if and only if its first Betti number is even. Therefore statement (K) of Theorem 1.1 is a consequence of the following theorem.

Theorem 5.2 *Let $S^1 \times M$ be a closed, connected complex surface. If $b_1(M)$ is odd and M has no fake 3–cells, then M is a Seifert fibered space which fibers over S^1.*

Proof Since $b_1(X) = b_1(M) + 1$ is even, $X = S^1 \times M$ admits a Kähler structure. By Theorem 5.1, X is diffeomorphic to $S^2 \times T^2$ or admits a T^2–bundle over T^2 or a properly elliptic fibration without any singular (possibly with multiple) fibers.

If X is diffeomorphic to $S^2 \times T^2$, then M fibers over S^1 by Theorem 3.1. Moreover, the diffeomorphism between $S^1 \times M$ and $S^1 \times (S^2 \times S^1)$ gives a homotopy equivalence between M and $S^2 \times S^1$ and as they both fiber over S^1 this homotopy equivalence must be a homeomorphism, in particular, M is a Seifert fibered space.

If X admits a T^2–bundle over T^2, then by Corollary 3.6 and Theorem 3.1 M fibers over S^1 with fiber a torus and in particular it is geometric. On the other hand, by Theorem 3 in [8] the geometric type of M is $\mathbb{E}^3$, where $\mathbb{E}^n$ is $\mathbb{R}^n$ with its standard metric. This implies that $M = T^3$ (see p.446 in [24]). In particular, M is Seifert fibered.

If X admits a Seifert fibration over a hyperbolic orbifold B, then it is geometric and the geometric type of it must be $\mathbb{E}^2 \times \mathbb{H}^2$ by Theorem 4.5 in [34] as X

admits a Kähler structure, where $\mathbb{H}^2$ is the hyperbolic plane. It should be noted that there is a mistake in [34] which was later corrected by Kotschick in [13]; since it concerns manifolds with nonvanishing Euler characteristic, it doesn't effect our discussion on product 4–manifolds. On the other hand, we get the following exact sequence from the Seifert fibration

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(S^1 \times M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1 \ ,$$

where F is a regular fiber and $\pi_1^{orb}(B)$ denotes the fundamental group of B as an orbifold. This exact sequence leads to another one

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1 \ ,$$

just as in the proof of Lemma 3.7, since B is hyperbolic and its orbifold fundamental group doesn't contain an infinite cyclic normal subgroup. So there exists an infinite cyclic normal subgroup in $\pi_1(M)$ and M is a Seifert 3–manifold by Corollary 12.8 in [11]. (Note that as $b_1(M)$ is odd it is nonzero and M is sufficiently large.) In particular, M is geometric. Since $S^1 \times M$ is type $\mathbb{E}^2 \times \mathbb{H}^2$, M must be type $\mathbb{E}^1 \times \mathbb{H}^2$, in other words the rational Euler class of a Seifert fibration on M is 0. A generalization of Lemma 3.8 (e.g. Theorem 8.1 in [21]) implies that M fibers over S^1. □

In order to prove statement (C) of Theorem 1.1 we use the following result of Biquard (cf. Théorème 8.2 in [2]):

Theorem 5.3 *A properly elliptic non–Kähler surface admits no symplectic structure.*

Proof of Statement (C) in Theorem 1.1 We have seen in Theorem 5.1 that if $X = S^1 \times M$ admits a complex and a symplectic structure, then there are three possibilities. The product $S^2 \times T^2$ admits a Kähler structure hence if $X = S^2 \times T^2$, then M fibers over S^1 by Theorem 5.2. If X admits a T^2–bundle over T^2, then M fibers over S^1 by Corollary 3.6 and Theorem 3.1. If X is a properly elliptic surface, then it has to be Kähler by Theorem 5.3 hence M fibers over S^1 by Theorem 5.2. □

The following is a well-known theorem. For a nice proof see [36].

Theorem 5.4 *If M is a closed, oriented Seifert fibered space, then $S^1 \times M$ admits a complex structure.*

Proposition 5.5 *Let M be a closed, oriented 3–manifold with no fake 3–cells. Suppose $S^1 \times M$ admits a symplectic structure and a Seifert fibration. Then $S^1 \times M$ admits a Kähler structure or a T^2–bundle over T^2.*

Proof We have the following short exact sequence coming from the Seifert fibration

$$1 \longrightarrow \pi_1(F) \longrightarrow \pi_1(S^1 \times M) \xrightarrow{\pi_\#} \pi_1^{orb}(B) \longrightarrow 1 \ ,$$

where F is a generic fiber, $\pi_1^{orb}(B)$ denotes the fundamental group of B as an orbifold and π is the projection map of the fibration. Let u be a generator of $\pi_1(S^1 \times \{pt\})$ in $\pi_1(S^1 \times M)$ as in the proof of Lemma 3.7.

First assume that $\pi_\#(u)$ is nontrivial in $\pi_1^{orb}(B)$. Then it generates an infinite, cyclic, normal subgroup (cf. proof of Lemma 3.7). Existence of such a subgroup in $\pi_1^{orb}(B)$ is possible only if B is a nonsingular orbifold of genus 1, i.e. a torus. So the Seifert fibration we have is in fact a T^2–bundle over T^2.

Now assume $u \in ker(\pi_\#)$. Then as in the proof of Theorem 5.2 we have

$$1 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1^{orb}(B) \longrightarrow 1 \ .$$

In particular, there is an infinite cyclic normal subgroup of $\pi_1(M)$. Since X admits a symplectic structure $b_+(X) \geq 1$ and so is $b_1(M)$. This implies that M is sufficiently large. Therefore we can use Corollary 12.8 in [11] to conclude that M is a Seifert fibered space. So $S^1 \times M$ admits a complex structure by Theorem 5.4, hence it admits a Kähler structure or a T^2–bundle over T^2 as in the proof of statement (C). □

This proposition (together with Theorem 5.2 and Corollary 3.6) finishes the proof of Theorem 1.1.

6 Geometry of M and structures on $S^1 \times M$

During the course of our proof of Theorem 1.1 we made observations on the interaction between various structures and fibrations on M and $S^1 \times M$. In this section, we recall some of those observations and use them to prove a couple of theorems on the relation between the geometry of M and $S^1 \times M$.

Throughout this section we will assume that M is a closed, connected and oriented 3–manifold with no fake 3–cells.

In the proof of Proposition 5.5 we used the existence of a symplectic structure on $S^1 \times M$ to conclude that $b_+(S^1 \times M) = b_1(M) > 0$. Note that $b_1(M) > 0$ implies that M is sufficiently large.

Theorem 6.1 *If $S^1 \times M$ is Seifert fibered and M is sufficiently large, then M admits a nonhyperbolic geometric structure.*

Proof As in the proof of Proposition 5.5 we look at the homotopy sequence of the Seifert fibration. There are two different cases depending on the image of a generator u of $\pi_1(S^1 \times \{pt\}) \subset \pi_1(S^1 \times M)$:

If u is in the kernel, then we have an infinite cyclic normal subgroup in $\pi_1(M)$. Since M is sufficiently large, Corollary 12.8 in [11] implies that M is a Seifert fibered space.

If u is not in the kernel, then $S^1 \times M$ admits a T^2–bundle over T^2, in particular it is symplectic. Hence (e.g. by (L) of Theorem 1.1) M fibers over the circle with fiber a torus. By Theorem 5.5 in [24] M is geometric of type $\mathbb{E}^3$, Nil^3 or Sol^3.

It is now clear that in any case M is geometric but not hyperbolic. □

As we mentioned before if M is Seifert fibered, then $S^1 \times M$ admits a complex structure. If M is geometric of type Sol^3, then $S^1 \times M$ is obviously geometric of type $\mathbb{E}^1 \times Sol^3$ and as a consequence $S^1 \times M$ doesn't admit any complex structure [33].

On the other hand, Theorem 5.1 says that if $S^1 \times M$ admits a complex structure, then it is either of *Class* VII_0 or an elliptic surface and in any case, by the remark following Theorem 5.1 $S^1 \times M$ is Seifert fibered.

This discussion leads us to the following conclusion which is a partial converse of the well-known Theorem 5.4.

Theorem 6.2 *If $S^1 \times M$ admits a complex structure and M is sufficiently large, then M is a Seifert fibered space.*

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Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720, USA

Email: `tolga@math.berkeley.edu`

Received: 7 August 2001